An introduction to Lorenzen's "Algebraic and logistic investigations on free lattices" (1951)

Thierry Coquand   Stefan Neuwirth

Contents

Lorenzen's "Algebraische und logistische Untersuchungen über freie Verbände" appeared in 1951 in *The Journal of Symbolic Logic*. These "Investigations" have immediately been recognised as a landmark in the history of infinitary proof theory. Their approach and method of proof have not been incorporated into the corpus of proof theory.[1] We propose a translation and this introduction with the intent of giving a new impetus to their reception.

The "Investigations" are best known for providing a constructive proof of consistency for ramified type theory without axiom of reducibility. They do so by showing that it is a part of a trivially consistent "inductive calculus" that describes our knowledge of arithmetic without detour. The proof resorts only to the inductive definition of formulas and theorems.

They propose furthermore a definition of a semilattice, of a distributive lattice, of a pseudocomplemented semilattice, and of a countably complete boolean lattice as deductive calculuses, and show how to present them for constructing conservatively the respective free object over a given preordered set. They illustrate that lattice theory is a bridge between algebra and logic for which the construction of an element corresponds to a step in a proof.

We shall describe the history of their reception, which focusses mainly on the ω-rule. The fruitfulness of this device is immediately recognised by Kurt Schütte. It triggers the analysis by Ackermann (1951) of the concept of accessibility predicate as defined by an infinitary inductive definition in Gentzen 1936, with the goal of proving transfinite induction up to ordinal terms beyond ε_0, which is also taken over by Schütte (1952).

This article is an elaboration of the second half of Coquand and Neuwirth 2017, and it is a sequel to the elaboration Coquand and Neuwirth 2020 of its first half; we have tried to avoid repetitions and to keep this article self-contained; we invite the reader to look up our 2020 paper for more details about the genesis of Lorenzen's work.

1. The beginnings. Lorenzen will later recall a talk by Gerhard Gentzen on the consistency of elementary number theory in 1937 or 1938 as a trigger for his discovery that the reformulation of ideal theory in lattice-theoretic terms reveals that his "algebraic works [...] were concerned with a problem that had *formally* the same structure as the problem of freedom from contradiction of the classical calculus of logic" (letter to Carl Friedrich Gethmann, see Gethmann 1991, page 76). This explains the title of Lorenzen's article.

1. More precisely, Lorenzen proves the admissibility of cut by double induction, on the cut formula and on the complexity of the derivations, without using any ordinal assignment, contrary to the presentation of cut elimination in most standard texts on proof theory.

2. The 1944 manuscript. The preliminary manuscript "Ein halbordnungstheoretischer Widerspruchsfreiheitsbeweis", published as Lorenzen 2020 [1944], contains already the main ideas, the ω-rule, the connection with free constructions in lattice theory, the admissibility of the cut rule, and applies them to a constructive proof of consistency for elementary number theory.

Note that Lorenzen expresses the concept of an admissible rule only in Lorenzen 1948a (see also his self-review Lorenzen 1949, both are posterior to Lorenzen 1951) and names it "erlaubt [permitted]" and then (in Lorenzen 1950a) "eliminierbar [eliminable]" before adopting today's terminology "zulässig" in Lorenzen 1955. Petr Sergeevich Novikov expresses this concept contemporaneously (see Citkin 2016).

3. The 1945 manuscript. In a letter dated 11 December 1945, Heinrich Scholz submits Lorenzen's manuscript "Die Widerspruchsfreiheit der klassischen Logik mit verzweigter Typentheorie" [2] to Paul Bernays (ETH-Bibliothek, Hochschularchiv, Hs. 975:4111) with a "request for judgement".

3.1. Stripping away lattice theory. The manuscript begins as follows.

> The proof of consistency undertaken in the sequel originated as an application of a purely algebraic theorem of existence about "free" complete boolean lattices. In the present work, though, I limit myself exclusively to the logistic application and use no algebraic conceptions whatsoever. [3]

The choice of stripping away lattice theory may be motivated by targeting a public of logicians. In this way, the affinity with the strategy of Gentzen (1935, IV, § 3) becomes more visible: the deductive calculus of ramified type theory with the axioms of comprehension, extensionality and infinity, but without the axiom of reducibility, is compared to an inductive calculus that proceeds "without detour"; with respect to Gentzen's calculus; it features an induction rule (compare rule [4] on page 39) with infinitely many premisses, i.e. it is an ω-rule in today's terminology.

3.2. Formula inductions and theorem inductions. Lorenzen emphasises as follows.

> This proof uses as auxiliary devices only formula inductions vs. theorem inductions, i.e. the fact that the concept of formula and the concept of theorem is defined inductively. The harmlessness of these auxiliary means seems to me to be even more perspicuous than the harmlessness of explicit transfinite inductions. [4]

2. "The consistency of classical logic with ramified type theory". A copy of this manuscript can be found in Niedersächsische Staats- und Universitätsbibliothek Göttingen, Cod. Ms. G. Köthe M 10.

3. "Der im folgenden durchgeführte Widerspruchsfreiheitsbeweis ist als eine Anwendung eines rein algebraischen Existenztheorems über 'freie' vollständige Boole'sche Verbände entstanden. In dieser Arbeit beschränke ich mich jedoch ausschließlich auf die logistische Anwendung und benutze keinerlei algebraische Begriffsbildungen."

4. "Dieser Beweis benutzt als Hilfsmittel nur Formelinduktionen bzw. Satzinduktionen, d. h. die Tatsache, daß der Formelbegriff und der Satzbegriff induktiv definiert ist. Die Unbedenklichkeit dieser Hilfsmittel scheint mir noch einleuchtender zu sein, als die Unbedenklichkeit expliziter transfiniter Induktionen."

These inductions establish that the deductive calculus is a part of the inductive calculus in section 7 on pages 41–45:

(1a) the "logical axiom" $c \preccurlyeq c$ is proved by formula induction;

(1b) the axiom of comprehension follows from the construction of a λ-calculus and a rule of constants;

(1c) the axiom of extensionality results from a formula induction by the aid of two auxiliary rules proved by theorem induction;

(1d) the axiom of infinity follows from the properties of the order on numbers;

(2a) the admissibility of the cut rule is proved by a formula induction on the cut formula: if it is a numerical formula, a double theorem induction on the premisses is used; the only difficulties in the induction step result from the copresence of constants and free and bound variables in rules like [3d] on page 39; as usual, contraction plays an important rôle.

3.3. Bernays' judgement. Bernays is able to appreciate its content on the spot and replies with detailed comments to Scholz on 24 April 1946 (carbon copy, Hs. 975:4112). On 17 April 1946, Lorenzen writes directly to Bernays (Hs. 975:2947); he gets an answer on 22 May 1946 with the following appreciation.

> It seems to me that your argumentation accomplishes in effect the desired and that thereby at the same time also a new, methodically more transparent proof of consistency for the number-theoretic formalism, as well as for Gentzen's subformula theorem [5] is provided.
>
> In the circumstance that all this is included in your result shows at the same time the methodical superiority of your method of proof with respect to a proof (that probably did not come to your knowledge) that F. B. Fitch [...] gave in 1938, and that also bears on the comparison of the deductive formalism with a system of formulas which is not delimited in a purely operative way; namely, this delimitation is carried out there according to a definition of truth in which the "tertium non datur" (indeed only with respect to the species of natural numbers) is made use of. [7] By determining your system of comparison according to the idea of a generalisation of Gentzen's thought of "deduction without detour", you gain the possibility

5. In the letter of 24 April 1946, Bernays writes more precisely to Scholz "that one also gets a proof for the main theorem of Gentzen's 'Investigations into logical deduction' out of it, if on the one hand one omits the higher axioms [(1b, c, d) on page 36] in the deductive calculus, on the other hand one retains from the rules of the inductive calculus (for determining the concept of theorem) only [[2], [3a–d] on page 39], while one takes also the formula pairs [$c \preccurlyeq c$] as starting theorems for this calculus." [6]

6. "dass man aus ihm auch einen Beweis für den Hauptsatz von Gentzen's 'Untersuchungen über das logische Schliessen' erhält, indem man einerseits beim deduktiven Kalkul die höheren Axiome 1.) b), c), d) weglässt, andererseits von den Regeln des induktiven Kalkuls (zur Bestimmung des Satzbegriffes) nur 2) a)–d) beibehält, während man als Ausgangssätze auch für diesen Kalkul die Formelpaare $\mathfrak{c} \subset \mathfrak{c}$ nimmt."

7. See Fitch 1938 and its review Bernays 1939.

of applying the constructive proof-theoretic view also in the case of your "inductive calculus", i.e. of such an inference system that does not comply with the recursiveness conditions that the customary formalisms fulfil. [8]

3.4. Independence of the axiom of reducibility. Lorenzen learns about Fitch's proof of consistency only by this letter. In his answer (dated 7 June 1946, Hs. 975:2949), he explains the lattice-theoretic background of his proof and encloses a manuscript, "Über das Reduzibilitätsaxiom", [9] which is a preliminary version of the last section of the published article, in which the axiom of reducibility is shown to be independent. However, Bernays seems to already have received this manuscript with Scholz's letter of 11 December 1945 (see his letter of 24 April 1946).

More precisely, he proves the consistency of the calculus obtained by adding an axiom that expresses that all infinite sets are countable and then shows that the axiom of reducibility is false in this calculus. In fact, as Lorenzen notes, Fitch (1939) proves this in his framework. These results answer questions raised by Whitehead and Russell (1925, pages xiv, xlii–xxliii) in the introduction to the second edition of their *Principia mathematica* after Leon Chwistek (1925): without the axiom of reducibility, "Cantor's proof that $2^n > n$ breaks down unless n is finite".

4. The 1947 manuscript.

4.1. Restoring the lattice theory part. By a letter dated 21 February 1947, Lorenzen writes to Bernays:

> After a revision of my proof of consistency according to your precious remarks and after addition of an algebraic part, I would like to allow myself to ask you for your intercession for a publication in the Journal of Symbolic Logic. [10]

This new draft is a kind of synthesis of "Ein halbordnungstheoretischer Widerspruchsfreiheitsbeweis" and "Die Widerspruchsfreiheit der klassischen Logik mit verzweigter

8. "Es scheint mir, dass Ihre Beweisführung in der Tat das Gewünschte leistet und dass damit zugleich auch ein neuer, methodisch durchsichtigerer Wf.-Beweis für den zahlentheoretischen Formalismus wie auch ein solcher für Gentzen's Teilformelsatz geliefert wird.

"In dem Umstande, dass alles dies in Ihrem Ergebnis eingeschlossen ist, zeigt sich zugleich die methodische Überlegenheit Ihres Beweisverfahrens gegenüber einem (Ihnen wohl nicht zur Kenntnis gelangten) Beweis, den F. B. Fitch 1938 für die Widerspruchsfreiheit der verzweigten Typentheorie gegeben hat (im Journal of symb. logic, vol. 3, S. 140-149), und der auch auf dem Vergleich des deduktiven Formalismus mit einem Formelsystem beruht, das auf eine nicht rein operative Art abgegrenzt ist; diese Abgrenzung erfolgt nämlich dort im Sinne einer Wahrheitsdefinition, wobei von dem 'tertium non datur' (allerdings nur demjenigen in Bezug auf die Gattung der natürlichen Zahlen) Gebrauch gemacht wird. Indem Sie Ihr Vergleichssystem gemäss der Idee einer Verallgemeinerung von Gentzen's Gedanken der 'umweglosen Herleitung' bestimmen, gewinnen Sie die Möglichkeit, die konstruktive beweistheoretische Betrachtung auch im Falle Ihres 'induktiven Kalküls' anzuwenden, d. h. eines solchen Folgerungssystems, welches nicht den durch die üblichen Formalismen erfüllten Rekursivitätsbedingungen genügt." (Hs. 975:2948.)

9. "On the axiom of reducibility", Hs. 974:149. Another copy of this manuscript is in the Universitätsarchiv Bonn.

10. "nach einer Überarbeitung meines Wf.beweises nach Ihren wertvollen Bemerkungen und nach Hinzufügung eines algebraischen Teiles möchte ich mir erlauben, Sie um Ihre Fürsprache zu bitten für eine Veröffentlichung im Journal of symbolic logic" (Hs. 975:2949.)

Typentheorie", or rather a juxtaposition of two parts: the seams remain apparent. However, the introduction now takes into account the added algebraic part. Its first paragraph (see page 23) emphasises that lattice theory is relevant for ideal theory with a reference to the reshaping of Krull's Fundamentalsatz for integral domains in lattice-theoretic terms provided by his habilitation (Lorenzen 1950b, see Neuwirth 2021).

4.2. Semilattices. In the new algebraic part, the construction of free semilattices and free distributive lattices stems in fact from ideal theory. Theorems 1–4 in section 2 (page 25) introduce a semilattice as a "single statement entailment relation" and construct the free semilattice over a preordered set. This approach is parallelled in Lorenzen 1952 by the definition of a system of ideals for an arbitrary preordered set M on which a monoid G acts by order-preserving operators x: it is a relation satisfying items 1–4 of theorem 1 and furthermore

$$\text{if } a_1, \ldots, a_n \vdash b, \text{ then } xa_1, \ldots, xa_n \vdash xb$$

(compare Coquand, Lombardi, and Neuwirth 2019, § 1C).

4.3. Distributive lattices. In the same way, theorems 5–8 provide the description of a distributive lattice as a deductive system that has been called since Scott (1971) an "entailment relation". This description strikes Bernays as new to him (letter of 3 April 1947, Paul-Lorenzen-Nachlass, Philosophisches Archiv, Universität Konstanz, PL 1-1-118). His theorem 7 on page 27 corresponds in fact to theorem 1 in Cederquist and Coquand 2000, obtained independently. This construction is used in Lorenzen 1953 for embedding a preordered group endowed with a system of ideals into a lattice group containing this system (compare Coquand, Lombardi, and Neuwirth 2019, § 2C).

4.4. Lorenzen algebras. Section 3 deals with (finite) pseudocomplemented semilattices while his 1944 manuscript deals with countably complete ones. We propose to call them "Lorenzen algebras". [11]

Definition 1. A *Lorenzen algebra* is a meet-semilattice with a *negation*, i.e. a least element 0 and an operation $\bar{a}$ such that $b \leqslant \bar{a}$ if, and only if, $b \wedge a = 0$. (This semilattice has then automatically a greatest element $1 = \bar{0}$.) Such an algebra is *countably complete* if it is endowed with an infinitary meet operation over any sequence of elements.

He constructs the free Lorenzen algebra generated by a preordered set as a cut-free sequent calculus, while his 1944 manuscript deals with the countably complete case. [12]

In section 4, he shows how to apply the construction of the free Lorenzen algebra to a simple intuitionistic logical calculus. He emphasises that the decision problem has a positive answer.

We shall use Lorenzen algebras in our explanation of an impredicative system in terms of inductive definitions: see Section 7.

11. The theory of Lorenzen algebras continues to develop: one can find an account of it by Grätzer (2011, pages 99–101) and Chajda, Halaš, and Kühr (2007, chapter 3).

12. The existence of the free Lorenzen algebra over a preordered set seems to be unknown in the literature, which considers only the case where the preorder is trivial; in the latter case, the decision problem was solved by Tamura (1974). Neuwirth 2015 proposes a streamlined presentation.

4.5. Boolean algebras. Lorenzen proceeds with describing the free countably complete boolean lattice [13] generated by a preordered set as a cut-free infinitary sequent calculus with ω-rules [3.9] and [3.10] on page 33. The main step in the construction is again to prove that the cut rule (which states on page 33 that $a_1 \leqslant c \vee b_1$ and $a_2 \wedge c \leqslant b_2$ implies $a_1 \wedge a_2 \leqslant b_1 \vee b_2$) is admissible.

He sketches this construction, which goes along the same lines as the construction of the free Lorenzen algebra, with one significant difference: in the latter case, he is able to prove contraction (see lemma (8) on page 30), whereas he has to put it into the definition in the former setting (he provides a counterexample on page 34). Compare with his 1944 manuscript, where a contraction rule is present (see the comment in our 2020, end of § 2), and with the calculus defined by Per Martin-Löf (1970, § 30) for Borel sets, where the problem of contraction is eluded by "identify[ing] sequents which are equal considered as finite sets".

Let us comment on two aspects of this construction.

- — Lorenzen works systematically with a preorder $\leqslant$ instead of an order and does not quotient with respect to the equivalence relation $a \equiv b$ defined by $a \leqslant b$ and $b \leqslant a$. If he did so, he would need to resort to the axiom of choice for defining meets.
- — The universal property corresponding to freeness is proved by parallelling the construction of the free object with the construction of the sought-after morphism: see items (i)–(iii) on page 30. The Univalent Foundations Program (2013) indicates a way to avoid the use of choice in a constructive setting exactly by defining inductively the free Boolean algebra so that its objects and their equalities are defined simultaneously.

Compare Section 7 for the relevance of these two aspects.

Then Lorenzen shows how to deduce consistency for the logic of ramified type by an iterated construction of free countably complete boolean lattices, starting from a calculus without free variables, along the hierarchy of types.

5. Toward publication.

5.1. Finitary vs. constructive logic. At the end of his letter of 21 February 1947, Lorenzen asks:

> I beg once again to ask you for your advice—namely, it is not clear to me whether I rightly call the logic used here "finite" logic. [14]

Bernays provides the following answer in his letter of 3 April 1947:

---

13. In the second paragraph of the introduction, he addresses complete boolean algebras over a preordered set as studied by MacNeille (1937). The question about the existence of the free complete boolean algebra is usually attributed to Rieger (1951) and has led to the works of Gaifman (1964) and Hales (1964) that provide a negative answer; it may be seen as an anomaly of the set-theoretic approach to actual infinity, because such infinities cause that free complete boolean algebras would be too big. See also the proof by Solovay (1966) inspired by forcing.

14. "Ich bitte noch einmal Sie um Ihren Rat fragen zu dürfen – es ist mir nämlich nicht klar, ob ich die hier benutzte Logik mit Recht 'finite' Logik nenne." (Hs. 975:2950.)

> When it comes to the methodical standpoint and to the terminology to be used in relation, then it seems advisable to me to keep with the mode chosen by Mister Gentzen, that one speaks of "finite" reflections only in the narrower sense, i.e. relating to considerations that may be formalised in the framework of recursive number theory (possibly with extension of the domain of functions to arbitrary computable functions), that one uses in contrast the expression "constructive" for the appropriate extension of the standpoint of the intuitive self-evidence; by the way, this is employed also by many an American logician in the corresponding sense.
>
> Your proof of consistency cannot, I deem, be a finite one in the narrower sense. Of course, this would conflict with the Gödel theorem. Actually, a nonfinite element of your reflection lies in the induction rule of the inductive calculus, which contains indeed a premiss of a more general form. [15]

In other words, the ω-rule does not fit into a formal system, and this explains why Gödel's theorem does not apply here. But Hilbert (1931, page 491) describes the ω-rule as a "finite deduction rule" and this is probably why Lorenzen qualifies his deductions as "finite"; note also the emphasis of Gentzen (1936, § 16.1 1) that accessibility is a finitary concept (see section 6.7 below). Lorenzen answers as follows on 4 May 1947.

> Your proposal to call the means of proof not "finite" but "constructive" acted on me as a sort of redemption. I was sticking so far to the word finite only to emphasise that these are hilbertian ideas that I am trying to pursue. [16]

5.2. Publication. Lorenzen prepares another final draft that is very close to the published version. [17] Bernays sends a first series of comments on 1 September 1947 (PL 1-1-112) and a second series (on a version including the final section on the axiom of reducibility) on 6 February 1949 (PL 1-1-107); the article is submitted to *The Journal of Symbolic Logic* soon afterwards [18] and published as Lorenzen 1951 with date of reception 17

15. "Was den methodischen Standpunkt und die in Bezug darauf zu verwendende Terminologie betrifft, so erscheint es mir als empfehlenswert, den von Herrn Gentzen gewählten Modus beizubehalten, dass man von 'finiten' Betrachtungen nur im engeren Sinne spricht, d. h. mit Bezug auf Überlegungen, die sich im Rahmen der rekursiven Zahlentheorie (eventuell mit Erweiterung des Funktionenbereiches auf beliebige berechenbare Funktionen) formalisieren lassen, dass man dagegen für die sachgemässe Erweiterung des Standpunktes der anschaulichen Evidenz den Ausdruck 'konstruktiv' verwendet; dieser wird übrigens auch von manchen amerikanischen Logikern im entsprechenden Sinn gebraucht.

"Ihr Wf-Beweis kann, so meine ich, kein finiter in dem genannten engeren Sinne sein. Das würde doch dem Gödelschen Theorem widerstreiten. Tatsächlich liegt, so viel ich sehe, ein nicht-finites Element Ihrer Betrachtung in der Induktionsregel des induktiven Kalkuls, welche ja eine Prämisse von allgemeinerer Form enthält." (PL 1-1-118.)

16. "Ihr Vorschlag, die Beweismittel nicht 'finit', sondern 'konstruktiv' zu nennen, hat wie eine Art Erlösung auf mich gewirkt. Ich habe bisher an dem Wort finit nur festgehalten, um zu betonen, dass es Hilbertsche Ideen sind, die ich fortzuführen versuche." (Hs. 975:2953.)

17. Two pages of this draft may be found in the file OB 5-3b-5; Cod. Ms. G. Köthe M 10 contains an excerpt of Part I.

18. See the letter of 27 April 1949 to Alonzo Church, in which Lorenzen thanks him for acknowledging receipt of the manuscript, writes a few words on its history, and proposes Bernays as a referee (Alonzo Church Papers, box 26 folder 4, Manuscripts division, Department of rare books and special collections,

March 1950. In fact, in 1947, Lorenzen already starts his project of layers of language which will lead to his operative logic (see Neuwirth 2022 on the circumstances of this switch).

6. Reception.

6.1. Early accounts. For early accounts of the manuscripts, see Lorenzen 1948b, Köthe 1948, Schmidt 1950, § 11.

Let us present the reactions to Lorenzen's article by subject.

6.2. The difference with Fitch's proof. Hao Wang (1951) writes the review for *The Journal of Symbolic Logic* and tries to compare Lorenzen's approach with Fitch's; see Coquand 2014 for a discussion of this review. Wang 1954 (page 252) provides a more accurate comparison.

6.3. Induction rules instead of transfinite inductions. Gottfried Köthe and Lorenzen have worked together on lattice theory before World War II. In Spring 1947, they correspond on foundations of mathematics and physics. Köthe is preparing lectures on proofs of consistency up to Lorenzen's to be given in Fall 1947 at Mainz (see Cod. Ms. G. Köthe G 3).[19] In an answer to a letter by Köthe dated 8 June 1947 (PL 1-1-114), Lorenzen writes on 17 June 1947 about his work: "The formalisation of proofs of freedom from contradiction that I am striving for is not at all intent on staying inside a transfinite theory of numbers, but uses instead of 'transfinite inductions' induction rules like the 'formula induction' and 'theorem induction' of my proof of freedom from contradiction—these could also be formalised in a theory of numbers with sufficiently large constructible ordinal numbers, but nothing is gained from this".[20]

6.4. The logical status of the ω-rule. In his latter dated 9 June 1947, Ackermann would find appropriate that one "would describe somehow the constructive, contentful thinking" in Lorenzen's rule of induction. Four years later, Ackermann reflects upon the logical status of the ω-rule in the following terms.

> Indeed, the expression "derivation rule" does not seem entirely appropriate to us. For on the one hand it is a derivation rule with infinitely many premisses we are dealing with, so that no rigorous formalisation of thought is carried out, as the fact that a formal derivation can be given for each of the infinitely many premisses is the result of considerations in terms of content. [...] Here it matters only to me to show that one can add certain basic formulas that have been obtained according to certain principles [...] without loosing freedom from contradiction. (Ackermann 1952, pages 368–369.)

This analysis is the same as that of Hilbert who writes the following about his proof theory and formalisation twenty years earlier:

Princeton university library).

19. See Köthe's letter of 3 September 1947 (PL 1-1-113) and Cod. Ms. G. Köthe M 10.

20. "Die Formalisierung der Wf beweise, die ich anstrebe, geht nun gar nicht darauf aus, innerhalb einer transfiniten Zahlentheorie zu bleiben, sondern benutzt statt der 'transfiniten Induktionen' Induktionsregeln wie die 'Formelinduktion' und 'Satzinduktion' meines Wf beweises – – diese ließen sich zwar auch in einer Zahlentheorie mit genügend großen konstruierbaren Ordinalzahlen formalisieren, dadurch wird aber nichts gewonnen." (Cod. Ms. G. Köthe M 10.)

> Deducing in terms of content is superseded by an exterior acting according to rules, namely the use of the deduction scheme and substitution. [...].
>
> To the proper thus formalised mathematics comes an in a way new mathematics, a metamathematics, that is necessary to secure the former, in which—contrary to the purely formal ways of deducing of proper mathematics—deducing in terms of content is applied, but only for verifying that the axioms are free from contradiction.
>
> The axioms and provable assertions, i.e. the formulas that arise in this interplay, are the images of the thoughts that have been making up the habitual proceeding of mathematics up to now.
>
> ....
>
> If it is verified that the formula
>
> $$\mathfrak{A}(\mathfrak{z})$$
>
> becomes a correct numerical formula every time $\mathfrak{z}$ is a presented numeral, then the formula
>
> $$(x)\mathfrak{A}(x)$$
>
> may be put on as starting formula. (Hilbert 1931, pages 489, 491.)

Thus the ω-rule is considered as a rule of metamathematics applied in terms of content and it is ancillary in studying formal systems.

Ackermann discusses a version of Lorenzen's work, most probably the 1945 manuscript, in a letter to Lorenzen dated 31 April 1950 (sic, PL 1-1-95). Ackermann 1953 provides a version of the consistency of ramified type theory in the context of his type-free logic (see also Schütte 1954a).

6.5. The ω-rule combined with transfinite induction. In an answer dated 4 November 1948 to a letter by Bernays that informs him about Lorenzen's work, Schütte states that Arnold Schmidt has acquainted him with it and writes: "As means of proof going beyond the narrower finite standpoint, Mister Lorenzen uses inferences with infinitely many premisses, while I (as Gentzen) draw on beginning cases of the transfinite induction." [21]

In a letter to Bernays dated 26 August 1949, Schütte writes: "I believe that my investigations are not superfluous besides those of Lorenzen because with them the required metamathematical means of proof and the connections with the derivability of the formalised transfinite induction are uncovered." [22]

Schütte writes to Lorenzen on 1 May 1950 in order to acknowledge the latter's priority in implementing the ω-rule into proofs of consistency. [23]

21. "Als Beweismittel, die über den engeren finiten Standpunkt hinausgehen, benutzt Herr Lorenzen Schlüsse mit unendlich viel Prämissen, während ich (wie Gentzen) Anfangsfälle der transfiniten Induktion heranziehe." (Hs. 975:4228, note that Szabo translates "Anfangsfälle der" by the epithet "restricted".)

22. "Ich glaube, daß meine Untersuchungen neben denen von Lorenzen deshalb nicht überflüssig sind, weil die benötigten metamathematischen Beweismittel und die Zusammenhänge mit der Herleitbarkeit der formalisierten transfiniten Induktion dabei aufgedeckt werden" (Hs. 975:4230.)

23. Both are not aware of the work of Novikov (1939, 1943) in this respect. See Mints 1991, 1.2.

> [...] I came to know that you had provided already before a proof of consistency for a still more general domain, and had arrived at the following result: the cut-eliminability, that in Gentzen had been carried out only in pure logic, may also be transferred to mathematical formalisms, if instead of the inference of complete induction more general schemes of inference with infinitely many premisses are drawn on by extending the concept of derivation so that it may contain infinitely many formulas. This insight gained by you, that appears to me exceptionally important for fundamental research, I have now taken up.[24]

6.6. Semi-formal systems. In fact, the reception of the logistic part of Lorenzen's article takes place mostly through reading the articles Schütte 1951, 1952 and the book Schütte 1960, not directly:[25] see e.g. Mendelson 1964, Appendix, Tait 1968, Girard 1987, Chapter 6 and Girard 2000, § 2.1.

In his book, Schütte (1960) introduces the ω-rule as "rule UJ*", where "UJ" stands for "infinite induction", with a description of its meaning in terms of content by a reference to constructiveness: "For the application of rule UJ* requires a metalogical investigation. It will be requested that infinitely many formulas $F(z_1, \ldots, z_n)$ have been proved to be derivable on the basis of general considerations before it is allowed to infer the derivability of the formula $F(a_1, \ldots, a_n)$." Here the z_i's are numerals while the a_i's are free variables. He writes further: "We request of a metalogical proof needed for the application of the infinite induction that it is led as all metalogical investigations in a constructive way. That is: the metalogical proof is to consist in the specification of a general procedure after which the requested derivations resp. derivation parts may be immediately exhibited" (compare also Schütte 1951, page 369.) He coins the expression "semi-formal system" for a formal system extended with the ω-rule. In contrast, in the second edition of his book, Schütte (1977) keeps silent about the meaning of the ω-rule

24. "[...] erfuhr ich, daß Sie schon vorher einen Widerspruchsfreiheitsbeweis für einen noch allgemeineren Bereich erbracht hatten und dabei zu folgendem Ergebnis gekommen waren: Die Schnitt-Eliminierbarkeit, die bei Gentzen nur in der reinen Logik durchgeführt wurde, läßt sich auch auf mathematische Formalismen übertragen, wenn statt des Schlusses der vollständigen Induktion allgemeinere Schlußschemata mit unendlich vielen Prämissen herangezogen werden, indem der Begriff der Herleitung so erweitert wird, dass er unendlich viele Formeln enthalten darf. Diese von Ihnen gewonnene Erkenntnis, die mir außerordentlich wichtig für die Grundlagenforschung zu sein scheint, habe ich nun aufgegriffen." (PL 1-1-95.)

25. See § 18 and chapter IX. In a letter to Bernays dated 7 March 1957, Schütte gives the following foremost reason to write his book: "After Mister Lorenzen has published a book from his point of view [Lorenzen 1955], it seems necessary to me that the axiomatic direction also has its say. At any rate, I have the impression that the Americans and also the Münster school do not rightly take notice of the results of Mister Ackermann and myself."[26] Note that Schütte 1977 is not a mere translation of Schütte 1960, as the author abandons the treatment of ramified type theory in this second edition; in doing so, he conceals Lorenzen's contributions to proof theory but for a spurious mention of Lorenzen 1951 in the bibliography. Also the survey article Schütte and Schwichtenberg 1990 records Lorenzen's contribution to logic in an elusive way.

26. "Nachdem Herr Lorenzen ein Buch von seinem Standpunkt aus herausgebracht hat, erscheint es mir nötig, daß auch die axiomatische Richtung zu Wort kommt. Ich habe jedenfalls den Eindruck, daß die Amerikaner und auch die Münstersche Schule die Ergebnisse von Herrn Ackermann und mir nicht recht beachten." (Hs. 975:4234.)

and states only this: "We call the system DA* *semi-formal* since, as opposed to formal systems, it has basic inferences (S2.0*) with infinitely many premises" (page 174). This silence is in stark opposition to the introduction which insists in the same terms as the first edition on the "constructive character" and the "constructive standpoint" as the framework of metamathematics. Thus the way of dealing with infinitely many premisses is considered as a private business of the proof theorist: he should not need to express on the record the meaning of infinitary proof objects and might e.g. resort to set theory for these. The metalogical investigation is eluded.

Schütte 1962, written for a general audience, compares three methods for proving the consistency of arithmetic: Gentzen's use of transfinite induction; Lorenzen's "semi-formal" use of the ω-rule; Gödel's use of computable functionals of finite type. The last is the "most direct" as it possesses "a character of immediate evidence" whereas "ordinal inductions appear as admissible only after a corresponding foundation"; "semi-formal systems permit analyses the logico-mathematical deduction that suggest themselves and are particularly transparent"; "transfinite ordinals [. . .] give us the possibility to characterise the different induction principles used metamathematically with respect to logical strength by equivalent ordinal inductions" (pages 106–107).

The detour via Schütte's reception may have contributed to proof theory continuing to focus on measuring logical strength by ordinal numbers, whereas the fact that Lorenzen does not resort to ordinals in his proof of consistency should be considered as a feature of his approach.

Tait (1968) provides a very clear presentation of Schütte's approach with a mention of Lorenzen; see also Feferman and Sieg 1981, § 3.2, for an account of Tait 1968.

6.7. The generalised inductive definition of accessibility. Wilhelm Ackermann hears about Lorenzen's proof of consistency in 1946 through Bernays (see his letter to Lorenzen dated 11 November 1946, PL 1-1-125). As he writes in a subsequent letter dated 21 May 1947 (PL-1-1-117), Ackermann is working at the time at setting up "mathematics out of a type-free logical system that is demonstrably free from contradiction" [27] and for this he needs a "considerably higher ordinal number" than in the "transfinite inductions up to the first ε-number" that "Gentzen and [he] use in [their] proofs of freedom from contradiction of arithmetic". He is therefore interested in "constructively recordable numbers of the second number class" following Church (1938). [28]

In their correspondence, they also are interested in a setup of ordinal numbers by Lorenzen: see Ackermann's letters of 21 May and of 9 June 1947 (PL 1-1-115). In the latter he writes: "As far as I understand the train of thought of your setup from your hints, your setup of the ordinal numbers of the 2. number class appears to me as constructively usable".

Recall that Gentzen (1936, § 15.4) proves the finiteness of his reduction procedure

27. "aus einem nachweislich widerspruchsfreien typenfreien logischen Axiomensystem die Mathematik aufzubauen".

28. "So benutzen Gentzen und ich bei unseren Widerspruchsfreiheitsbeweisen für die Arithmetik transfinite Induktionen bis zur ersten ε-Zahl. Bei den Untersuchungen, an denen ich augenblicklich arbeite, gehe ich bis zu einer wesentlich höheren Ordinalzahl. Unter einer konstruktiv erfassbaren Zahl der II. Zahlenklasse verstehe ich dabei im Anschluss an A. Church [. . .]"

by a transfinite induction described as the generalised inductive definition of the "accessibility [Erreichbarkeit]" of the first ε-number. He discusses the constructiveness of this concept in § 16.1 1, which may be seen as the very heart of his proof of the consistency of elementary number theory: "[Accessibility] acquires a *sense* merely by being predicated of a definite [ordinal] number for which *its validity* is simultaneously *proved*."

Ackermann (1951) sets up the segment of ordinal numbers that he needs and gives a remarkably precise account of the constructiveness of accessibility, relying in particular on Lorenzen's ω-rule. He provides in fact a description for the generalised inductive definition of the concept of accessibility, and we guess that it has benefitted from his exchanges with Lorenzen. The "o-numbers" below are given as a certain recursively defined totally ordered system of symbols.

> Now we have to show that the o-numbers are really ordinal numbers, or, in other words, that we are allowed to apply *deductions by transfinite induction*.
>
> In order to lay out the text of the following considerations with less difficulty, it is advisable to introduce the following symbols, but with which we only express concisely contentful states of affairs. These symbols are: $\mathfrak{A}(\alpha)$, "the property $\mathfrak{A}$ applies to α"; $\mathfrak{K}_x(\alpha, \mathfrak{A}(x))$, "the property $\mathfrak{A}$ applies to all o-numbers which are less than α"; $\mathfrak{V}_x(\alpha, \mathfrak{A}(x))$, "with $\mathfrak{K}_x(\beta, \mathfrak{A}(x))$ also $\mathfrak{A}(\beta)$ has always to be the case, provided $\beta \leqq \alpha$". It may now be that $\mathfrak{K}_x(\alpha + 1, \mathfrak{A}(x))$ is derivable from the assumptions $\mathfrak{A}(1)$ and $\mathfrak{V}_x(\alpha, \mathfrak{A}(x))$ by constructive deductions without assuming anything else about $\mathfrak{A}$. We would then first say that the number α is accessible through $\mathfrak{A}$. Now, if an o-number is accessible through $\mathfrak{A}$, then it is also accessible through any other property $\mathfrak{B}$. For, as we had not assumed anything about $\mathfrak{A}$ in the derivation of $\mathfrak{K}_x(\alpha + 1, \mathfrak{A}(x))$ but that $\mathfrak{A}(1)$ and $\mathfrak{V}_x(\alpha, \mathfrak{A}(x))$ has to be the case, so we need only to replace everywhere $\mathfrak{A}$ by $\mathfrak{B}$ in the deductions that lead from $\mathfrak{A}(1)$ and $\mathfrak{V}_x(\alpha, \mathfrak{A}(x))$ to $\mathfrak{K}_x(\alpha + 1, \mathfrak{A}(x))$. We may therefore, instead of saying that α is accessible through $\mathfrak{A}$, simply say that α is accessible. It might now seem that accessibility is defined by a claim of generality over predicates. But this is not our conception. We want to conceive the accessibility of a number α as a certain intuitive fact, viz. precisely as the presence of a certain system of deductions that leads from $\mathfrak{A}(1)$ and $\mathfrak{V}_x(\alpha, \mathfrak{A}(x))$ to $\mathfrak{K}_x(\alpha + 1, \mathfrak{A}(x))$. All deductions of the so-called positive logic are to belong to these deductions, further also the number-theoretic induction and the corresponding operating with the universal sign for o-numbers as it would e.g. fit intuitionistic number theory. We do not want to specify here these deductions in detail because the following proof shows which are needed. We remark only that the following deduction is also needed: if a claim may be derived for each concrete o-number, then also the corresponding universal claim is to be derivable. [2] For the use of the universal sign we remark that the o-numbers represent a countable set that is precisely defined and set up constructively, so that the use of a universal sign for o-numbers is legitimated in the same way as that of the universal sign for natural num-

bers.

2. This deduction appears self-evident from the point of view of content. It has been applied first by P. Lorenzen inside a logistic system. Cf. P. Lorenzen, The freedom from contradiction of the logic of ramified type (Journal of Symbolic Logic, to appear shortly).

In a letter to Ackermann dated 3 March 1951, Lorenzen writes:

> Thank you very much for your work "Constructive setup of a segment of the 2. number class"—your construction impresses me very much, I have tried earlier a similar one but not as far-reaching. I wholly share your views on the constructiveness of your definitions and proofs. (Ackermann 1983, page 197.)

Schütte (1954b, § 4) takes over the argument of Ackermann 1951 and extends the latter's system of o-numbers into so-called "Klammersymbole" that generalise also the system of ordinal fixed points by Veblen (1908). This argument may also be found in Schütte 1960, § 11–12, that presents an intermediate system inspired by the coding as integers of Hilbert and Bernays 1939, § 5.3.c. In contrast, it is absent from Schütte 1977, as is any description of the nature of a constructive argument; as the beginning of §24 introducing higher ordinals states, the reading, constructive or axiomatic, is up to the reader: "we use the notions *map* (*function*) and *set* in a naive way. But these may also be regarded as being determined axiomatically (in the context of a general axiom system for set theory)" (page 221).

See also the account of accessibility by Gödel (1990 [1972], note c on page 272).

6.8. Are infinitary inductive definitions predicative? In 1962, Schütte dissociates himself from the analysis of Ackermann 1951 and qualifies the generalised inductive definition of accessibility as impredicative:

> Hereby one proceeds in an impredicative way by including the concept of accessibility itself, defined with reference to the totality of all properties of certain ordinals, into these properties. (Schütte 1962, page 110).

In 1965, he defines "an ordering relation $\prec$ of equivalence classes of natural numbers representing a sufficiently large segment of the second number class in a constructive way" (page 280). He writes: "The relation $\prec$ can be proved by *impredicative methods* to be a *well-ordering* using a proof similar to that for the related $\prec$-relation in § 12 of Schütte 1960" (page 286).

In particular, Lorenzen's theorem induction is impredicative if one follows Schütte's qualification.

6.9. Further accounts. Lorenzen (1955) expands on the rôle of lattices in mathematics. Lorenzen (1958, 1987) provide a proof of Gentzen's subformula theorem by the method of his article. Lorenzen (1962, § 7) returns to the subject of proofs of consistency.

Beth (1959, page 253) gives a short and precise account of Lorenzen's article.

Haskell B. Curry (1963, Chapter 4, Theorem B9) follows Lorenzen in characterising a distributive lattice as a lattice satisfying cut.

Manfred E. Szabo (1969, pages 12–13) comments on the relationship of Lorenzen's article with Gentzen's work.

Oskar Becker (1954) refers to Lorenzen's article in the last pages of his book *Grundlagen der Mathematik in geschichtlicher Entwicklung*.

Its philosophical significance is addressed by Matthias Wille (2013, 2016).

Coquand (2021) describes Lorenzen's standpoint in a broader context than given here.

7. Our reception: impredicative quantification and inductive definitions. Whitehead and Russell (1925, page 57) see the axiom of reducibility as a generalisation of Leibniz's identity of indiscernibles. For instance, the formula $\forall_X\, X(3) \to X(x)$, which seems impredicative since it contains a quantification over all predicates, is actually equivalent to the predicative formula $3 = x$. Using in a crucial way ideas from Lorenzen 2020 [1944], we extend this remark to a predicative interpretation of some formulae involving a seemingly impredicative universal quantification over all predicates.

7.1. Free Lorenzen algebras and countable choice. Look up Definition 1 of Lorenzen algebras on page 6. The main result of Lorenzen 2020 [1944] is to essentially build the free countably complete Lorenzen algebra K over a given preordered set P and to show that the canonical map $P \to K$ is an embedding. This is a purely semilattice-theoretic reformulation of Gentzen's 1936 consistency proof of arithmetic. We write "essentially" since the exact statement is a little more complex if one wants to avoid the use of the axiom of countable choice. We provide such a refined statement below and note that a possible constructive way to avoid the axiom of countable choice is provided by the setting of the Univalent Foundations Program (2013).

7.2. An impredicative formal system: syntax. We consider the following language. The terms are of the form $S^k(0)$ and $S^k(x)$. A closed term t represents such a natural number, that we will also write t. The atomic formulae are of the form $X(t)$ or $P(t_1, \ldots, t_n)$ where P represents some $n$ary boolean-valued function.

We have formulae built from $\top$, $\neg\psi$, $\varphi \wedge \psi$ and $\forall_x \psi$ and $\forall_X \psi$.

A formula is arithmetical if it does not contain any quantification over predicates.

A formula is *strict* Π^1_1 if it is of the form $\forall_X \psi$ where ψ is arithmetical and uses only X as a predicate variable.

We consider the fragment of the language where we only form strict Π^1_1 universal quantifications $\forall_X \psi$.

We also have terms for predicates $T, U, \ldots$. They are of the form $T = \lambda_x \varphi$. We define the substitution $\psi(T/X)$, for a closed predicate T by induction on ψ

$$\begin{aligned}
(X(t))(T/X) &= \varphi(t/x) & (\neg\psi)(T/X) &= \neg(\psi(T/X))\\
(Y(t))(T/X) &= Y(t) & (\psi_0 \wedge \psi_1)(T/X) &= \psi_0(T/X) \wedge \psi_1(T/X)\\
P(t_1, \ldots, t_n)(T/X) &= P(t_1, \ldots, t_n) & (\forall_x \psi)(T/X) &= \forall_x \psi(T/X)
\end{aligned}$$

7.3. A semilattice defined in a predicative metatheory. We use Lorenzen 2020 [1944] to build, in a predicative metatheory, a special semilattice: the countably complete Lorenzen algebra Ω.

We let $n, m, \dots$ range over natural numbers.

We start from an infinite set of symbols $X, Y, \dots$, representing predicate variables on numbers.

We consider then the symbolic objects S built inductively by the rules

$$a, b, \dots \ ::= \ X(n) \mid \bigwedge_n a_n \mid a \wedge b \mid \neg a \mid 0$$

If Ω is a countably complete Lorenzen algebra, and $\rho(X)$ associates to any predicate variable X occurring in a a function $\mathbb{N} \to \Omega$, we can compute $a\rho$ in Ω by induction on a, by the following laws

$$X(n)\rho = \rho(X)(n) \quad 0\rho = 0 \quad \neg a\rho = \overline{a\rho} \quad (a \wedge b)\rho = a\rho \wedge b\rho \quad (\bigwedge_n a_n)\rho = \bigwedge_n (a_n\rho)$$

One way to interpret Lorenzen 2020 [1944] is that he defines a *preorder* relation $a \leqslant b$ on the set S such that $a \leqslant \neg b$ if, and only if, $a \wedge b \leqslant 0$; $0 \leqslant a$ for all a; $b \leqslant \bigwedge_n a_n$ if, and only if, $b \leqslant a_n$ for all n; $c \leqslant a \wedge b$ if, and only if, $c \leqslant a$ and $c \leqslant b$. Furthermore, for any countably complete Lorenzen algebra Ω and any assignment $\rho(X)$ in $\mathbb{N} \to \Omega$, we have $a\rho \leqslant b\rho$ in Ω whenever $a \leqslant b$.

If we wanted to build the free countably complete Lorenzen algebra on the atoms $X(n)$, we would have to quotient by the equivalence relation defined by $a \leqslant b$ and $b \leqslant a$, and countable choice would be needed to show that we get a countably complete Lorenzen algebra.

We let S_{f} be the subset of S of elements which depend on *finitely many* predicate symbols.

An *ideal* A is a subset of S_{f} containing 0 and such that $a \in A$ whenever $b \in A$ and $a \leqslant b$. In particular any element a in S_{f} defines the principal ideal $\downarrow a$ of all elements in S_{f} such that $b \leqslant a$.

We let Ω be the set of all ideals.

Ω has a structure of countably complete Lorenzen algebra, with meet as intersection, and a least element $\{0\}$. The negation operation defines $\bar{A}$ as the ideal of elements b in S_{f} such that $b \wedge a \leqslant 0$ whenever a is in A.

In particular, if $\rho(X)$ assigns an element in $\mathbb{N} \to \Omega$ for each predicate symbol X, we can consider the interpretation $a\rho$ in Ω of an element a in S_{f}.

By induction on a, we can show the following result.

Lemma 2. *If a is in S_{f} and we have $\rho(X)(n) = \downarrow X(n)$ for X free in a then $a\rho = \downarrow a$.*

Let $c(X)$ be an element of S_{f} with at most X as free variable. We write $c(X)(X = f)$ for the element $c(X)\rho$ computed in Ω for ρ assigning the function $f : \mathbb{N} \to \Omega$ to X.

The key result which provides a predicative analysis of (strict) impredicative comprehension is then the following.

Theorem 3. *The family $c(X)(X = f)$ for $f : \mathbb{N} \to \Omega$ has a g.l.b. $\bigwedge_f c(X)(X = f)$ in the semilattice Ω.*

Proof. Define A to be the set of all elements a such that $a \leqslant c(X)$ for X not free in a.

Note that if X and Y are not free in a then $a \leqslant c(X)$ is equivalent to $a \leqslant c(Y)$. It follows from this remark that a is in A if, and only if, $a \leqslant c(X)$ for *all* X not free in a if, and only if, $a \leqslant c(X)$ for *some* X not free in a. If then a is in A and $b \leqslant a$, then $a \leqslant c(X)$ for some X not free in a. We can then find Y not free both in a *and* b. We then have $a \leqslant c(Y)$ and hence $b \leqslant c(Y)$, and so b is in A. The set A is thus an ideal, i.e. an element of Ω.

We claim that a is in A if, and only if, it belongs to all $c(X)(X = f)$, which will show that A is the g.l.b. of the family $c(X)(X = f)$.

If a is in A then $a \leqslant c(X)$ for X not free in a. Define $\rho(Z)(n) = {\downarrow}Z(n)$ for Z free in a and $\rho(X) = f$. We then have $a\rho \leqslant c(X)\rho$. But $a\rho = {\downarrow}a$ by Lemma 2 and $c(X)\rho = c(X)(X = f)$. Hence ${\downarrow}a \leqslant c(X)(X = f)$ and a is in $c(X)(X = f)$.

Conversely if a is in all $c(X)(X = f)$, with X not free in a, for all f in $\mathbb{N} \to \Omega$, then in particular it is in $c(X)(X = g)$ for $g(X)(n) = {\downarrow}X(n)$. In this case $c(X)(X = g)$ is ${\downarrow}c(X)$ by Lemma 2 and so $a \leqslant c(X)$, i.e. a is in A. □

7.4. Interpretation of strict Π^1_1 comprehension: semantics. Any arithmetical formula ψ defines an element $[\psi]$ in S_{f} by the rules

$$[X(t)] = X(t) \qquad [P(t_1, \ldots, t_n)] = \delta_P(t_1, \ldots, t_n)$$
$$[\neg\psi] = \neg[\psi] \qquad [\psi_0 \wedge \psi_1] = [\psi_0] \wedge [\psi_1] \qquad [\forall_x \psi] = \bigwedge\nolimits_n [\psi(n/x)]$$

where δ_P is the $n$ary $0, 1$-valued function representing P.

If ρ assigns a value $\rho(X)$ in $\mathbb{N} \to \Omega$ for X free in ψ we can define the semantics $[\![\psi]\!]\rho$ as an element of Ω.

We define it first for an arithmetical formula by the clauses

$$[\![X(t)]\!]\rho = \rho(X)(t) \qquad [\![P(t_1, \ldots, t_n)]\!] = \delta_P(t_1, \ldots, t_n)$$
$$[\![\neg\psi]\!]\rho = \neg([\![\psi]\!]\rho) \qquad [\![\psi_0 \wedge \psi_1]\!]\rho = [\![\psi_0]\!]\rho \cap [\![\psi_1]\!]\rho \qquad [\![\forall_x \psi]\!]\rho = \bigcap\nolimits_n [\![\psi(n/x)]\!]\rho$$

We can now define $[\![T]\!]\rho$ as a function $\mathbb{N} \to \Omega$ by $[\![\lambda_x \psi]\!]\rho(n) = [\![\psi(n/x)]\!]\rho$.

If ψ is an arithmetical formula with at most one free variable X, then $[\![\psi]\!](X = f)$ is equal to $[\psi](X = f)$. Hence by Theorem 3, the family $[\![\psi]\!](X = f)$ has a g.l.b. and we can define

$$[\![\forall_X \psi]\!]\rho = \bigwedge\nolimits_f [\![\psi]\!](\rho, X = f)$$

In particular, we get

$$[\![\forall_X \psi]\!]\rho \leqslant [\![\psi]\!](\rho, X = [\![T]\!]\rho) = [\![\psi(T/X)]\!]\rho$$

and our semantics, which we have built in a predicative metatheory, justifies comprehension for strict Π^1_1-formulae.

References

Hans Richard Ackermann. Aus dem Briefwechsel Wilhelm Ackermanns. *Hist. Philos. Logic*, 4(2), 181–202, 1983. doi:10.1080/01445348308837054.

Wilhelm Ackermann. Konstruktiver Aufbau eines Abschnitts der zweiten Cantorschen Zahlenklasse. *Math. Z.*, 53, 403–413, 1951. `http://eudml.org/doc/169197`.

Wilhelm Ackermann. Widerspruchsfreier Aufbau einer typenfreien Logik. (Erweitertes System). *Math. Z.*, 55, 364–384, 1952. `http://eudml.org/doc/169263`.

Wilhelm Ackermann. Widerspruchsfreier Aufbau einer typenfreien Logik. II. *Math. Z.*, 57, 155–166, 1953. `http://eudml.org/doc/169311`.

Oskar Becker. *Grundlagen der Mathematik in geschichtlicher Entwicklung.* Karl Alber, Freiburg, 1954.

Paul Bernays. Review of Fitch 1938. *J. Symb. Log.*, 4(2), 97–98, 1939. `http://www.jstor.org/stable/2269079`.

Evert W. Beth. *The foundations of mathematics: a study in the philosophy of science*. Studies in Logic and the Foundations of Mathematics, North-Holland, Amsterdam, 1959.

Wilfried Buchholz, Solomon Feferman, Wolfram Pohlers, and Wilfried Sieg. *Iterated inductive definitions and subsystems of analysis: recent proof-theoretical studies.* Lecture Notes in Mathematics, 897, Springer-Verlag, Berlin, 1981.

Jan Cederquist and Thierry Coquand. Entailment relations and distributive lattices. In *Logic Colloquium '98: proceedings of the annual European summer meeting of the Association for symbolic logic, held in Prague, Czech Republic, August 9-15, 1998*, edited by Samuel R. Buss, Petr Hájek, and Pavel Pudlák, 127–139. Lecture Notes in Logic, 13, Association for Symbolic Logic, Urbana, 2000.

Ivan Chajda, Radomír Halaš, and Jan Kühr. *Semilattice structures*. Research and Exposition in Mathematics, 30, Heldermann Verlag, Lemgo, 2007.

Alonzo Church. The constructive second number class. *Bull. Amer. Math. Soc.*, 44(4), 224–232, 1938. doi:10.1090/S0002-9904-1938-06720-1.

Leon Chwistek. The theory of constructive types. (Principles of logic and mathematics). II. Cardinal arithmetic. *Ann. Soc. Polon. Math.*, 3, 92–141, 1925. `http://name.umdl.umich.edu/AAS7985.0001.001`.

Alex Citkin. Multiple conclusion rules in logics with the disjunction property. In *Logical foundations of computer science: international symposium, LFCS 2016: Deerfield Beach, FL, USA, January 4–7, 2016: proceedings*, edited by Sergei Artemov and Anil Nerode, 76–89. Lecture Notes in Comput. Sci., 9537, Springer, Cham, 2016. doi:10.1007/978-3-319-27683-0_6.

Thierry Coquand. Recursive functions and constructive mathematics. In *Constructivity and computability in historical and philosophical perspective*, edited by Jacques Dubucs and Michel Bourdeau, 159–167. Logic, Epistemology, and the Unity of Science, 34, Springer, Dordrecht, 2014.

Thierry Coquand. Lorenzen and constructive mathematics. In *Heinzmann and Wolters 2021*, 47–61, 2021.

Thierry Coquand, Henri Lombardi, and Stefan Neuwirth. Lattice-ordered groups generated by an ordered group and regular systems of ideals. *Rocky Mountain J. Math.*, 49, 1449–1489, 2019. doi:10.1216/rmj-2019-49-5-1449.

Thierry Coquand and Stefan Neuwirth. An introduction to Lorenzen's "Algebraic and logistic investigations on free lattices" (1951), 2017. doi:10.48550/arXiv.1711.06139.

Thierry Coquand and Stefan Neuwirth. Lorenzen's proof of consistency for elementary number theory. *Hist. Philos. Logic*, 41(3), 281–290, 2020. doi:10.1080/01445340.2020.1752034.

Haskell B. Curry. *Foundations of mathematical logic.* McGraw-Hill, New York, 1963.

Solomon Feferman, John W. Dawson, Jr., Stephen C. Kleene, Gregory H. Moore, Robert M. Solovay, and Jean van Heijenoort (eds.). *Kurt Gödel: Collected works, II: Publications 1938–1974.* Clarendon Press, Oxford, 1990.

Solomon Feferman and Wilfried Sieg. Iterated inductive definitions and subsystems of analysis. In *Buchholz, Feferman, Pohlers, and Sieg 1981*, chap. I, 16–77, 1981.

Frederic B. Fitch. The consistency of the ramified *Principia*. *J. Symb. Log.*, 3(4), 140–149, 1938. http://www.jstor.org/stable/2267777.

Frederic B. Fitch. The hypothesis that infinite classes are similar. *J. Symb. Log.*, 4(4), 159–162, 1939. http://www.jstor.org/stable/2268716.

Haim Gaifman. Infinite Boolean polynomials. I. *Fund. Math.*, 54, 229–250, 1964. http://eudml.org/doc/213758. Errata in *Fund. Math.*, 57, 117, 1965, http://eudml.org/doc/213847.

Gerhard Gentzen. Untersuchungen über das logische Schließen. II. *Math. Z.*, 39(1), 405–431, 1935. http://eudml.org/doc/168556. Translation by Manfred E. Szabo: Investigations into logical deduction, sections IV and V, in Szabo 1969, 103–131.

Gerhard Gentzen. Die Widerspruchsfreiheit der reinen Zahlentheorie. *Math. Ann.*, 112, 493–565, 1936. http://eudml.org/doc/159839. Translation by Manfred E. Szabo: The consistency of elementary number theory, in Szabo 1969, 132–201.

Carl Friedrich Gethmann. Phänomenologie, Lebensphilosophie und konstruktive Wissenschaftstheorie: eine historische Skizze zur Vorgeschichte der Erlanger Schule. In *Lebenswelt und Wissenschaft: Studien zum Verhältnis von Phänomenologie und Wissenschaftstheorie*, edited by Carl Friedrich Gethmann, 28–77. Neuzeit und Gegenwart: philosophische Studien, 1, Bouvier, Bonn, 1991.

Jean-Yves Girard. *Proof theory and logical complexity.* Studies in Proof Theory, 1, Bibliopolis, Naples, 1987.

Jean-Yves Girard. Du pourquoi au comment: la théorie de la démonstration de 1950 à nos jours. In *Development of mathematics 1950–2000*, edited by Jean-Paul Pier, 515–545. Birkhäuser, Basel, 2000.

Kurt Gödel. On an extension of finitary mathematics which has not yet been used. In *Feferman, Dawson, Kleene, Moore* et al. *1990*, 271–280, 1990 [1972].

George Grätzer. *Lattice theory: foundation.* Birkhäuser, Basel, 2011. doi:10.1007/978-3-0348-0018-1.

Alfred W. Hales. On the non-existence of free complete Boolean algebras. *Fund. Math.*, 54, 45–66, 1964. http://eudml.org/doc/213773.

Gerhard Heinzmann and Gereon Wolters (eds.). *Paul Lorenzen—mathematician and logician.* Logic, Epistemology, and the Unity of Science, 51, Springer, Cham, 2021. doi:10.1007/978-3-030-65824-3.

David Hilbert. Die Grundlegung der elementaren Zahlenlehre. *Math. Ann.*, 104(1), 485–494, 1931. http://eudml.org/doc/159480.

David Hilbert and Paul Bernays. *Grundlagen der Mathematik. II*. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, 50, Springer, Berlin, 1939.

Reinhard Kahle and Michael Rathjen (eds.). *The legacy of Kurt Schütte*. Springer, Cham, 2020.

Gottfried Köthe. Verbände. In *Süss 1948*, 81–95, 1948.

Paul Lorenzen. *Einführung in die Logik*. Mathematisches Seminar der Universität, Bonn, 1948a.

Paul Lorenzen. Grundlagen der Mathematik. In *Süss 1948*, 11–22, 1948b.

Paul Lorenzen. Einführung in die Logik. *Arch. Math. (Basel)*, 2, 60–65, 1949. doi:10.1007/BF02036755.

Paul Lorenzen. Konstruktive Begründung der Mathematik. *Math. Z.*, 53, 162–202, 1950a. http://eudml.org/doc/169180.

Paul Lorenzen. Über halbgeordnete Gruppen. *Math. Z.*, 52, 483–526, 1950b. http://eudml.org/doc/169131.

Paul Lorenzen. Algebraische und logistische Untersuchungen über freie Verbände. *J. Symb. Log.*, 16(2), 81–106, 1951. http://www.jstor.org/stable/2266681.

Paul Lorenzen. Teilbarkeitstheorie in Bereichen. *Math. Z.*, 55, 269–275, 1952. http://eudml.org/doc/169251.

Paul Lorenzen. Die Erweiterung halbgeordneter Gruppen zu Verbandsgruppen. *Math. Z.*, 58, 15–24, 1953. http://eudml.org/doc/169331.

Paul Lorenzen. *Einführung in die operative Logik und Mathematik*. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, 28, Springer, Berlin, 1955.

Paul Lorenzen. *Formale Logik*. Sammlung Göschen, 1176-1176a, Walter de Gruyter, Berlin, 1958. Translation by Frederick J. Crosson: *Formal logic*, D. Reidel, Dordrecht, 1965.

Paul Lorenzen. *Metamathematik*. B·I-Hochschultaschenbücher, 25, Bibliographisches Institut, Mannheim, 1962. Translation by Jean-Blaise Grize: *Métamathématique*, Gauthier-Villars, Paris, 1967. Translation by Jacobo Muñoz: *Metamatemática*, Tecnos, Madrid, 1971.

Paul Lorenzen. *Lehrbuch der konstruktiven Wissenschaftstheorie*. Bibliographisches Institut, Mannheim, 1987.

Paul Lorenzen. Ein halbordnungstheoretischer Widerspruchsfreiheitsbeweis. *Hist. Philos. Logic*, 41(3), 265–280, 2020 [1944]. doi:10.1080/01445340.2020.1752040. Dual German-English text, edited and translated by Stefan Neuwirth.

Holbrook M. MacNeille. Partially ordered sets. *Trans. Amer. Math. Soc.*, 42(3), 416–460, 1937. doi:10.2307/1989739.

Per Martin-Löf. *Notes on constructive mathematics*. Almqvist & Wiksell, Stockholm, 1970.

Elliott Mendelson. *Introduction to mathematical logic*. D. Van Nostrand Co., Princeton, 1964.

Grigori Mints. Proof theory in the USSR 1925–1969. *J. Symb. Log.*, 56(2), 385–424, 1991. http://www.jstor.org/stable/2274689.

Stefan Neuwirth. The free Lorenzen algebra generated by a preordered set, 2015. http://lmb.univ-fcomte.fr/IMG/pdf/the_free_lorenzen_algebra_generated_by_a_preordered_set.pdf. Technical note.

Stefan Neuwirth. Lorenzen's reshaping of Krull's Fundamentalsatz for integral domains (1938–1953). In *Paul Lorenzen—Mathematician and Logician: Proceedings of the conference (Constance, 8–9 March 2018)*, edited by Gerhard Heinzmann and Gereon Wolters, 141–180. Springer, 2021.

Stefan Neuwirth. "C'est comme œuf": Paul Lorenzen et l'effondrement des preuves de non-contradiction. In *Eurêka ! Récits savants de découverte et d'invention*, edited by Laurence Dahan-Gaida. Cahiers d'histoire et de philosophie des sciences, Hermann, Paris, 2022. http://hal.archives-ouvertes.fr/hal-03104374.

Petr Sergeevich Novikov. Sur quelques théorèmes d'existence. *C. R. (Dokl.) Acad. Sci. URSS (N.S.)*, 23, 438–440, 1939.

Petr Sergeevich Novikov. On the consistency of certain logical calculus. *Mat. Sb. (N. S.)*, 12(54)(2), 231–261, 1943. http://mi.mathnet.ru/eng/msb6158. Announced in Novikov 1939.

Ladislav Rieger. On free $\aleph_\xi$-complete Boolean algebras (with an application to logic). *Fund. Math.*, 38, 35–52, 1951. http://eudml.org/doc/213251. Review by Lorenzen: Zbl. 0044.26103.

Arnold Schmidt. Mathematische Grundlagenforschung. In *Enzyklopädie der mathematischen Wissenschaften mit Einschluß ihrer Anwendungen, Band I: Algebra und Zahlentheorie, 1. Teil: A. Grundlagen B. Algebra*, edited by Helmut Hasse and Erich Hecke, chap. 2. B. G. Teubner, Leipzig, second ed., 1950.

Kurt Schütte. Beweistheoretische Erfassung der unendlichen Induktion in der Zahlentheorie. *Math. Ann.*, 122, 369–389, 1951. http://eudml.org/doc/160227.

Kurt Schütte. Beweistheoretische Untersuchung der verzweigten Analysis. *Math. Ann.*, 124, 123–147, 1952. http://eudml.org/doc/160263.

Kurt Schütte. Ein widerspruchsloses System der Analysis auf typenfreier Grundlage. *Math. Z.*, 61, 160–179, 1954a. http://eudml.org/doc/169456.

Kurt Schütte. Kennzeichnung von Ordnungszahlen durch rekursiv erklärte Funktionen. *Math. Ann.*, 127, 15–32, 1954b. http://eudml.org/doc/160361.

Kurt Schütte. *Beweistheorie*. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, 103, Springer, Berlin, 1960.

Kurt Schütte. Logische Abgrenzungen des Transfiniten. In *Logik und Logikkalkül*, edited by Max Käsbauer and Franz von Kutschera, 105–114. K. Alber, Freiburg, 1962.

Kurt Schütte. Predicative well-orderings. In *Formal systems and recursive functions: proceedings of the eighth logic colloquium: Oxford, July 1963*, edited by John N. Crossley and Michael Dummett, 280–303. Studies in logic and the foundations of mathematics, 40, North-Holland, Amsterdam, 1965.

Kurt Schütte. *Proof theory*. Grundlehren der mathematischen Wissenschaften, 225, Springer, Berlin, 1977. Translation from the German by John N. Crossley.

Kurt Schütte and Helmut Schwichtenberg. Mathematische Logik. In *Ein Jahrhundert Mathematik: 1890–1990*, edited by Gerd Fischer, Friedrich Hirzebruch, Winfried Scharlau, and Willi Törnig, 717–740. Dokumente zur Geschichte der Mathematik, 6, Friedr. Vieweg, Braunschweig, 1990. Reprinted in Kahle and Rathjen 2020, 71–91.

Dana Scott. On engendering an illusion of understanding. *J. Philos.*, 68(21), 787–807, 1971. http://www.jstor.org/stable/2024952.

Robert M. Solovay. New proof of a theorem of Gaifman and Hales. *Bull. Amer. Math. Soc.*, 72, 282–284, 1966. doi:10.1090/S0002-9904-1966-11493-3.

Wilhelm Süss (ed.). *Naturforschung und Medizin in Deutschland 1939–1946, 1: Reine Mathematik, I.* Fiat Review of German Science, Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948.

Manfred E. Szabo (ed.). *The collected papers of Gerhard Gentzen.* Studies in Logic and the Foundations of Mathematics, North-Holland, Amsterdam, 1969.

William W. Tait. Normal derivability in classical logic. In *The syntax and semantics of infinitary languages*, edited by Jon Barwise, 204–236. Lecture Notes in Mathematics, 72, Springer, Berlin, 1968.

Saburo Tamura. A solution of the decision problem for free pseudo-complemented semilattices. *J. Fac. Liberal Arts Yamaguchi Univ.*, 8, 235–238, 1974.

The Univalent Foundations Program. *Homotopy type theory: univalent foundations of mathematics.* https://homotopytypetheory.org/book, Institute for Advanced Study, 2013.

Oswald Veblen. Continuous increasing functions of finite and transfinite ordinals. *Trans. Amer. Math. Soc.*, 9(3), 280–292, 1908. doi:10.2307/1988605.

Hao Wang. Review of Lorenzen 1951. *J. Symb. Log.*, 16(4), 269–272, 1951. http://www.jstor.org/stable/2267912.

Hao Wang. The formalization of mathematics. *J. Symb. Log.*, 19(4), 241–266, 1954. http://www.jstor.org/stable/2267732. Reprinted in *A survey of mathematical logic*, North-Holland, Amsterdam, 1963, 559–584.

Alfred North Whitehead and Bertrand Russell. *Principia mathematica*, vol. I. University Press, Cambridge, 1925.

Matthias Wille. Zwischen Algebra und Erlanger Schule: Paul Lorenzens Beiträge zur Beweistheorie. *Siegener Beiträge zur Geschichte und Philosophie der Mathematik*, 1, 79–108, 2013. http://nbn-resolving.de/urn:nbn:de:hbz:467-8245.

Matthias Wille. Verzweigte Typentheorie, relative Konsistenz und Fitch-Beweis: wie Lorenzen (nach eigener Auskunft) Hilberts Forderungen für die Analysis erfüllte. In *Paul Lorenzen und die konstruktive Philosophie*, edited by Jürgen Mittelstraß, 163–174. mentis, Münster, 2016.

Algebraic and logistic investigations on free lattices

Paul Lorenzen

It is well known that lattice theory was founded by Dedekind by means of his ideal-theoretic investigations. It has turned out lately that the essential property of Dedekind's system of ideals lies in the fact that ideals form a semilattice (§1). Ideal theory leads in this way to the question about all semilattices over an arbitrary preordered set M. The simple answer is contained in §2. The question about all distributive lattices over M may be answered just as simply. In both cases, among the semilattices vs. distributive lattices over M, one is distinguished by the fact that all others are homomorphic to it. We call this distinguished semilattice vs. distributive lattice the free semilattice vs. free distributive lattice over M.

One comes to new questions at investigating more special semilattices over M. It is known e.g. that M can always be extended into a complete boolean lattice. Is there, among these extensions, always the free complete boolean lattice over M, which is distinguished by the fact that all others are homomorphic to it?

In §3, first the existence of the free orthocomplemented semilattice over M is proved. The method used here may also be followed to lead the proof of existence for the free countably complete boolean lattice over M.

The significance of the proofs of existence is not exhausted in pure lattice theory, but finds an important application in logistics. It is well known that the formalisation of logic has been—beside ideal theory—a further impulse for the development of lattice theory. Nevertheless, logistics were only able to exploit a modicum of lattice-theoretic results.

In §4 we however show on a simple calculus of propositions how the question of consistency and the decision problem is answered immediately by the proof of existence for free orthocomplemented semilattices.

In part II (§§5–8), the consistency of ramified type logic including the axiom of infinity is being proved by the method of the proof of existence for the free countably complete boolean lattice. Knowledge of part I (§§1–4) is however not assumed.

By the fact that the basic thought of lattice theory is being used only implicitly, the proof of consistency appears somehow as the continuation of the original approach by which Gentzen proved the consistency of arithmetic without complete induction in his Ph.D. thesis. Consistency results in fact as an immediate conclusion therefrom, that each theorem of the calculus may be deduced "without detour". The proof described

Translator's note: in order to enhance the readability of the text for a public of lattice theorists as well as of logicians, we have replaced the signs $<$, —, $\subset$, and $\prec$ with the signs $\leqslant$, $\vdash$, $\subseteq$, and $\preccurlyeq$, respectively; thirty misprints have been tacitly corrected.

here goes however beyond Gentzen's proof, as the calculus whose consistency is proved contains arithmetic including complete induction as part. This calculus is equivalent to the one used by Russell and Whitehead in the *Principia mathematica* if the axiom of reducibility is removed there. As this axiom is not comprised, our calculus does not contain classical analysis, although the analytic modes of inference may still be represented in this calculus—with the restrictions required by ramified type theory.

The extension of Gentzen's approach to a so much richer calculus succeeds without addition of new means. Only the concept of deducibility without detour is extended by allowing certain induction rules in which a conclusion is inferred from infinitely many premisses.

The progress with regard to the work of Fitch[1] lies in the constructive character of all inferences used. Only hereby does our proof fulfil the demands that have been addressed since Hilbert to a proof of consistency.

In §5 the calculus whose consistency is to be proved is presented. It will be called shortly the deductive calculus. It will be confronted in §6 to an inductive calculus that may be thought of as a specification of the concept of deducibility without detour. The inductive calculus is consistent in a trivial way, so that for the consistency of the deductive calculus one has to show that the inductive calculus is stronger than the deductive. This proof uses only inductions on formulae vs. theorems as auxiliary means, i.e. the fact that the concept of formula vs. theorem is defined constructively. In contrast, the so-called transfinite induction is not used.

By a little modification of the proof, it is established over and above in §8 that the axiom of reducibility is independent from the remaining axioms of the deductive calculus. In fact, the deductive calculus remains consistent if countability of all sets is requested in addition. Cantor's diagonal procedure yields then in the extended calculus the refutability of the axiom of reducibility.

1. Basic concepts. We gather first the basic concepts of the theory of semilattices.

Let M be a set and $\leqslant$ a binary relation in M. Let $a, b, \cdots$ be the elements of M.

(A) M is called "preordered" (w.r.t. $\leqslant$) if holds:

1. $\quad a \leqslant a.$
2. $\quad a \leqslant b \,\textbf{,}\, b \leqslant c \quad \rightarrow \quad a \leqslant c.$[2]

Instead of $b \leqslant a$ we also write $a \geqslant b$. If $a \leqslant b$ and $a \geqslant b$ hold, then we write $a \equiv b$.[3] $\equiv$ is an equivalence relation.

(B) M is called a "semilattice" (w.r.t. $\leqslant$) if M is preordered (w.r.t. $\leqslant$) and if for each a, b there is a c with:

3.1 $\quad c \leqslant a.$

3.2 $\quad c \leqslant b.$

3.3 $\quad x \leqslant a \,\textbf{,}\, x \leqslant b \quad \rightarrow \quad x \leqslant c.$

c is uniquely determined (w.r.t. $\equiv$). We write $c \equiv a \wedge b$.

[1]F. B. Fitch, *The consistency of the ramified* Principia, J. Symbolic Logic, vol. 3 (1938), pp. 140–149, and *The hypothesis that infinite classes are similar*, ibid., vol. 4 (1939), pp. 159–162.

[2]I.e. $a \leqslant b$ and $b \leqslant c$ implies $a \leqslant c$.

[3]We do not assume $a \equiv b \rightarrow a = b$. The concepts under (E) and (F) must hereby be defined somewhat differently than usual.

(C) M is called a "lattice" (w.r.t. $\leqslant$) if M is a semilattice (w.r.t. $\leqslant$) and simultaneously a semilattice (w.r.t. $\geqslant$). If c fulfils the conditions 3.1–3.3 with $\geqslant$ instead of $\leqslant$, then we write $c \equiv a \vee b$.

(D) M is called a "distributive lattice" (w.r.t. $\leqslant$) if M is a lattice (w.r.t. $\leqslant$) and if holds

4. $$a \wedge c \leqslant b \,,\, a \leqslant b \vee c \quad \rightarrow \quad a \leqslant b.$$

(E) If M vs. M' is a preordered set (w.r.t. $\leqslant$) vs. (w.r.t. $\leqslant'$), then M' is called a "part" of M if M' is a subset of M and if for each $a', b' \in M'$ holds $a' \leqslant' b' \rightleftarrows a' \leqslant b'$.[4] If M is a semilattice vs. lattice, then M is called a semilattice vs. lattice "over M'" if M' is a part of M.

If M is a semilattice vs. lattice over M', then M is called a "minimal" semilattice vs. lattice over M' if M does not contain a proper subset M_0 for which holds:

(1) $$M' \subseteq M_0.$$
(2) $$a_0, b_0 \in M_0 \,,\, c \equiv a_0 \wedge b_0 \quad \rightarrow \quad c \in M_0.$$
(3) $$a_0, b_0 \in M_0 \,,\, c \equiv a_0 \vee b_0 \quad \rightarrow \quad c \in M_0.$$

(F) If M and M' are preordered sets (w.r.t. $\leqslant$), then a relation ρ between M and M' is called a "homomorphism" from M into M' if holds:

[1] To each $a \in M$ there is an $a' \in M'$ with $a \,\rho\, a'$.
[2] $$a \,\rho\, a_1' \,,\, a_1' \equiv a_2' \quad \rightarrow \quad a \,\rho\, a_2'.$$
[3] $$a \,\rho\, a' \,,\, b \,\rho\, b' \,,\, a \leqslant b \quad \rightarrow \quad a' \leqslant b'.$$

If M and M' are semilattices vs. lattices, then a "homomorphism" from M into M' is called a "semilattice homomorphism" vs. "lattice homomorphism" if holds:

[3.1] $$a \,\rho\, a' \,,\, b \,\rho\, b' \quad \rightarrow \quad a \wedge b \,\rho\, a' \wedge b'.$$
[3.2] $$a \,\rho\, a' \,,\, b \,\rho\, b' \quad \rightarrow \quad a \vee b \,\rho\, a' \vee b'.$$

($a \wedge b \,\rho\, a' \wedge b'$ means $c \equiv a \wedge b \,,\, c' \equiv a' \wedge b' \rightarrow c \,\rho\, c'$.)

An homomorphism ρ from M into M' is called an "isomorphism" from M into M' if holds:

[4] $$a \,\rho\, a' \,,\, b \,\rho\, b' \,,\, a' \leqslant b' \quad \rightarrow \quad a \leqslant b.$$

An homomorphism vs. isomorphism from M into M' is called an homomorphism vs. isomorphism from M "onto M'" if holds:

[5] To each $a' \in M'$ there is an $a \in M$ with $a \,\rho\, a'$.

M' is called "homomorphic" vs. "isomorphic" to M if there is an homomorphism vs. isomorphism from M onto M'. If M' is homomorphic vs. isomorphic to M, and if M_0 is a part of M and M', then M' is called homomorphic vs. isomorphic to M "over M_0" if there is an homomorphism from M onto M' such that for each $a_0 \in M_0$ holds $a_0 \,\rho\, a_0$.

2. Free semilattices and distributive lattices. Let M be a preordered set. The minimal semilattices over M may be characterised by relations in M with a finite number of places, as show the following theorems.

THEOREM 1. *If H is a semilattice over M, then for the relation in M defined by*

$$a_1, \cdots, a_n \vdash b \rightleftarrows a_1 \wedge \cdots \wedge a_n \leqslant b$$

holds:

[4] I.e. $a' \leqslant' b'$ equivalent $a' \leqslant b'$.

1. $a \vdash a.$
2. $a_1, \cdots, a_n \vdash b \rightarrow a_1, \cdots, a_n, c \vdash b.$
3. $a_1, \cdots, a_n \vdash b \rightarrow a_1, \cdots, a_{i+1}, a_i, \cdots, a_n \vdash b.$
4. $a_1, \cdots, a_n \vdash c\,;\, a_1, \cdots, a_n, c \vdash b \rightarrow a_1, \cdots, a_n \vdash b.$

Theorem 2. *If M is preordered, then the relation defined by*

$$a_1, \cdots, a_n \vdash b \rightleftarrows (\textit{there is an } a_i \textit{ with } a_i \leqslant b)$$

fulfils the conditions 1.–4. *of theorem* 1.

Theorem 3. *To each relation $a_1, \cdots, a_n \vdash b$ in M,*[5] *that fulfils the conditions* 1.–4. *of theorem* 1*, there is an (up to isomorphy over M) uniquely determined minimal semilattice over M for which holds*

$$a_1, \cdots, a_n \vdash b \rightleftarrows a_1 \wedge \cdots \wedge a_n \leqslant b.$$

We call the semilattice associated to the relation of theorem 2 according to theorem 3 the "free" semilattice over M.

Theorem 4. *If H is the free semilattice over M, then each minimal semilattice over M is homomorphic*[6] *to H over M.*

The proofs of these theorems are so simple that we omit them.

For the proof of theorem 3, one forms the set H of all finite sequences $a_1 \wedge \cdots \wedge a_n$ out of elements of M and defines in H a preorder $\leqslant$ by

$$a_1 \wedge \cdots \wedge a_n \leqslant b_1 \wedge \cdots \wedge b_m \rightleftarrows (\text{for each } b_i,\ \ a_1, \cdots, a_n \vdash b_i).$$

H is the sought-after semilattice.

Minimal distributive lattices over M may be characterised just as simply as semilattices.

Theorem 5. *If V is a distributive lattice over M, then for the relation in M defined by*

$$a_1, \cdots, a_m \vdash b_1, \cdots, b_n \rightleftarrows a_1 \wedge \cdots \wedge a_m \leqslant b_1 \vee \cdots \vee b_n$$

holds:

1. $a \vdash a.$
2. $a_1, \cdots, a_m \vdash b_1, \cdots, b_n \rightarrow a_1, \cdots, a_m, c \vdash b_1, \cdots, b_n.$
   $a_1, \cdots, a_m \vdash b_1, \cdots, b_n \rightarrow a_1, \cdots, a_m \vdash c, b_1, \cdots, b_n.$
3. $a_1, \cdots, a_m \vdash b_1, \cdots, b_n \rightarrow a_1, \cdots, a_{i+1}, a_i, \cdots, a_m \vdash b_1, \cdots, b_n.$
   $a_1, \cdots, a_m \vdash b_1, \cdots, b_n \rightarrow a_1, \cdots, a_m \vdash b_1, \cdots, b_{i+1}, b_i, \cdots, b_n.$
4. $a_1, \cdots, a_m, c \vdash b_1, \cdots, b_n\,;\, a_1, \cdots, a_m \vdash c, b_1, \cdots, b_n$
   $\rightarrow a_1, \cdots, a_m \vdash b_1, \cdots, b_n.$

Theorem 6. *If M is preordered, then the relation defined by*

$$a_1, \cdots, a_m \vdash b_1, \cdots, b_n \rightleftarrows (\textit{there is an } a_i \textit{ and a } b_j \textit{ with } a_i \leqslant b_j)$$

fulfils the conditions 1.–4. *of theorem* 5.

[5] M is preordered by the relation $a \vdash b$.

[6] I.e. there is a semilattice homomorphism.

THEOREM 7. *To each relation $a_1, \cdots, a_m \vdash b_1, \cdots, b_n$ in M,*[5] *that fulfils the conditions* 1.–4. *of theorem* 5*, there is an (up to isomorphy over M) uniquely determined minimal distributive lattice over M for which holds*

$$a_1, \cdots, a_m \vdash b_1, \cdots, b_n \rightleftarrows a_1 \wedge \cdots \wedge a_m \leqslant b_1 \vee \cdots \vee b_n.$$

We call the distributive lattice associated to the relation of theorem 6 according to theorem 7 the "free" distributive lattice over M.

THEOREM 8. *If V is the free distributive lattice over M, then each minimal distributive lattice over M is homomorphic*[7] *to V over M.*

We may again omit the proofs.

For the proof of theorem 7, one forms first—as at proving theorem 3—the set H of finite sequences $a_1 \wedge \cdots \wedge a_n$ out of elements of M. If $\alpha = a_1 \wedge \cdots \wedge a_m$ and $\beta_i = b_{i1} \wedge \cdots \wedge b_{in_i}$ are elements of H, then one sets $\alpha \vdash \beta_1, \cdots, \beta_n$ if $a_1, \cdots, a_m \vdash b_{1j_1}, \cdots, b_{nj_n}$ holds for each $j_1, \cdots, j_n$. This relation fulfils conditions that correspond to 1.–4. of theorem 1. One forms then—correspondingly as at proving theorem 3—the set V of finite sequences $\alpha_1 \vee \cdots \vee \alpha_n$ out of elements of H and defines a preorder $\leqslant$ in H by

$$\alpha_1 \vee \cdots \vee \alpha_m \leqslant \beta_1 \vee \cdots \vee \beta_n \rightleftarrows (\text{for each } \alpha_i,\ \ \alpha_i \vdash \beta_1, \cdots, \beta_n).$$

V is the sought-after lattice.

3. Free orthocomplemented semilattices. After the general semilattices and the distributive lattices we investigate now more special semilattices.

A preordered set (w.r.t. $\leqslant$) is called "bounded" if there are elements 0 and 1 in M with $0 \leqslant a \leqslant 1$ for each $a \in M$. 0 vs. 1 is called "zero element" vs. "unit element" of M.

A bounded semilattice is called "orthocomplemented" if to each c there is a d with

$$a \wedge c \leqslant 0 \rightleftarrows a \leqslant d \text{ (for all } a).$$

We write $d \equiv \bar{c}$.

If H is an orthocomplemented semilattice over M, then H is called a "minimal" orthocomplemented semilattice over M if H contains no proper subset H' for which holds:

i. $\quad M \subseteq H'$

ii. $\quad a' \in H' \,,\, b' \in H' \,,\, c \equiv a' \wedge b' \quad \to \quad c \in H'.$

iii. $\quad c' \in H' \,,\, d \equiv \overline{c'} \quad \to \quad d \in H'.$

If H and H' are orthocomplemented semilattices, then a semilattice homomorphism ρ from H into H' is called an "orthocomplemented" semilattice homomorphism if holds

$$a\, \rho\, a' \to \bar{a}\, \rho\, \overline{a'}.$$

Correspondingly to theorem 4 and theorem 8 we define now: an orthocomplemented semilattice over M is called a "free" orthocomplemented semilattice over M if each minimal orthocomplemented semilattice over M is homomorphic[8] to H. The free orthocomplemented semilattice over M is uniquely determined up to isomorphy over M.

[7] I.e. there is a lattice homomorphism.

[8] I.e. there is an orthocomplemented semilattice homomorphism.

We prove below the existence of the free orthocomplemented semilattice over an arbitrary preordered set by construction. We prove more precisely:

THEOREM 9. *If M is a bounded preordered set, and $a_1, \cdots, a_n \vdash b$ a relation that fulfils the conditions*

1. $$a \vdash b \rightleftarrows a \leqslant b$$

and 2.–4. of theorem 1, then there is an orthocomplemented semilattice H over M for which holds that for arbitrary elements $a_1, \cdots, a_n, b$ out of M,

$$a_1, \cdots, a_n \vdash b \rightleftarrows a_1 \wedge \cdots \wedge a_n \leqslant b,$$

and to which each minimal orthocomplemented semilattice that fulfils these conditions is homomorphic[8] *over M.*

Let H_0 be the set M. Let H'_i be set of two-term sequences $(a \wedge b)$ out of elements a, b of H_i, and let $H_{i+1} = H_i \cup \overline{H_i} \cup H'_i$. Let H be the union of the sets H_i $(i = 0, 1, \cdots)$.

The elements of H we call shortly "formulae," the elements of M "prime formulae". Then the following "formula induction" holds: if a claim holds

1. for each prime formula,
2. for $a \wedge b$, $\bar{a}$ if for a, b,

then it holds for each formula.

In H we define constructively a relation $\leqslant$ by:

(1) For prime formulae $a_1, \cdots, a_n, b$, $a_1 \wedge \cdots \wedge a_n \leqslant b$ if $a_1, \cdots, a_n \vdash b$.

(2) Structure rules. If $a_1 \wedge \cdots \wedge a_n \leqslant b$ holds, then a valid relation arises again by the following structure changes to the left formula: association, i.e. the grouping by brackets may be changed;[9] transposition, i.e. the sequential arrangement may be changed.

(3.1) $a \leqslant b \to a \wedge c \leqslant b.$

(3.2) $c \leqslant a \,,\, c \leqslant b \to c \leqslant a \wedge b.$

(3.3) $a \wedge b \leqslant 0 \to a \leqslant \bar{b}.$

(3.4) $a \leqslant b \to a \wedge \bar{b} \leqslant c.$

In order to achieve that the unit element 1 of M also becomes the unit element of H, we moreover set that these rules are also to hold if a formula $1 \wedge x$ is replaced by x. Thus the rules (3.1), (3.3), and (3.4) include:

$$1 \leqslant b \to c \leqslant b.$$

$$b \leqslant 0 \to 1 \leqslant \bar{b}.$$

$$1 \leqslant b \to \bar{b} \leqslant c.$$

If $a \leqslant b$ holds, then we call the formula pair a, b a "theorem." The theorems $a_1 \wedge \cdots \wedge a_n \leqslant b$ out of prime elements $a_1, \cdots, a_n$, and b we call "prime theorems." In the rules (3) (and correspondingly in (2)) we call the formula pairs to the left of $\to$ the "premisses", and the formula pair to the right of $\to$ the "conclusion." The following "theorem induction" holds: if a claim holds

1. for each prime theorem,
2. for the conclusion of each rule whose premisses are theorems if for these premisses,

[9] We have hence omitted brackets beforehand.

then it holds for each theorem.

We now show that H is an orthocomplemented semilattice. For this we have to prove

(4.1) $$c \leqslant c.$$

(4.2) $$0 \leqslant c \leqslant 1.$$

(4.3) $$a \leqslant c \,,\, c \leqslant b \rightarrow a \leqslant b.$$

(5.1) $$a \leqslant b_1 \wedge b_2 \rightarrow a \leqslant b_1.$$

(5.2) $$a \leqslant b_1 \wedge b_2 \rightarrow a \leqslant b_2.$$

(5.3) $$a \leqslant \bar{b} \rightarrow a \wedge b \leqslant 0.$$

We first prove (4.1) by formula induction. For prime formulae c holds $c \leqslant c$ because of $c \vdash c$. From $c_1 \leqslant c_1$ and $c_2 \leqslant c_2$ follows $c_1 \wedge c_2 \leqslant c_1$ and $c_1 \wedge c_2 \leqslant c_2$, thus $c_1 \wedge c_2 \leqslant c_1 \wedge c_2$. From $c \leqslant c$ follows moreover $c \wedge \bar{c} \leqslant 0$, and then $\bar{c} \leqslant \bar{c}$.

(4.2) follows as well by formula induction. For prime formulae c holds $0 \leqslant c$ because of $0 \vdash c$. From $0 \leqslant c_1$ and $0 \leqslant c_2$ follows $0 \leqslant c_1 \wedge c_2$. $0 \leqslant \bar{c}$ holds because of $0 \leqslant 0 \rightarrow 0 \wedge c \leqslant 0$, and $c \leqslant 1$ follows directly from (3.1) $1 \leqslant 1 \rightarrow c \leqslant 1$.

(4.3) we prove—because of the difficulty—last.

For the proof of (5.1) and (5.2) we use theorem induction. The induction claim states that for each theorem that has the form $a \leqslant b_1 \wedge b_2$, also $a \leqslant b_1$ and $a \leqslant b_2$ are theorems. For prime theorems there is nothing to prove as there are no prime theorems of the form $a \leqslant b_1 \wedge b_2$. Let now $a \leqslant b_1 \wedge b_2$ be conclusion of a rule whose premisses are theorems, and let the claim hold for the premisses. There are then only the following possibilities:

(a) $a \leqslant b_1 \wedge b_2$ is conclusion of a structure rule with a premiss $a' \leqslant b_1 \wedge b_2$ in which a arises by a structure change out of a'.

(b) One has $a = a_1 \wedge a_2$, and $a \leqslant b_1 \wedge b_2$ is conclusion of rule (3.1) with the premiss $a_1 \leqslant b_1 \wedge b_2$.

(c) $a \leqslant b_1 \wedge b_2$ is conclusion of rule (3.2) with the premisses $a \leqslant b_1$ and $a \leqslant b_2$.

(d) One has $a = a_1 \wedge \bar{a}_2$, and $a \leqslant b_1 \wedge b_2$ is conclusion of rule (3.4) with the premiss $a_1 \leqslant a_2$.

In case (a) holds according to the induction hypothesis $a' \leqslant b_1$ and $a' \leqslant b_2$, from which $a \leqslant b_1$ and $a \leqslant b_2$ arise by a structure rule.

In case (b) holds according to induction hypothesis $a_1 \leqslant b_1$ and $a_1 \leqslant b_2$, from which according to (3.1) follows $a_1 \wedge a_2 \leqslant b_1$ and $a_1 \wedge a_2 \leqslant b_2$.

In case (c) holds $a \leqslant b_1$ and $a \leqslant b_2$, as the premisses are theorems.

In case (d) follows from $a_1 \leqslant a_2$ according to (3.4) $a_1 \wedge \bar{a}_2 \leqslant b_1$ and $a_1 \wedge \bar{a}_2 \leqslant b_2$.

Thereby (5.1) and (5.2) are proved by theorem induction. The single proof steps are all trivial, and may therefore be skipped in later similar cases.

Next (5.3) results from such a trivial theorem induction.

Of (4.3) the special case $a \leqslant 0 \rightarrow a \leqslant b$ may be proved immediately by formula induction on b and theorem induction on a.

For the general case we need three lemmas:

(6) $$a \wedge \overline{\bar{c} \wedge d} \leqslant b \rightarrow a \wedge c \leqslant b.$$ [10]

[10]By our agreement on the replacement of $1 \wedge x$ by x, (6) includes: $a \wedge \bar{\bar{c}} \leqslant b \rightarrow a \wedge c \leqslant b$, $\bar{\bar{c}} \leqslant b \rightarrow c \leqslant b$. The rules arising by structure changes are—in order to abbreviate—not specified explicitly.

Proof by theorem induction.

(7) If $a \wedge \bar{c} \leqslant p$ holds for a prime formula p, then holds $a \leqslant p$ or $a \wedge \bar{c} \leqslant 0$.

Proof by theorem induction.

(8) $$a \wedge c \wedge c \leqslant b \to a \wedge c \leqslant b.$$[11]

We use a formula induction on c. For prime formulae c, (8) follows by theorem induction. If (8) holds for c_1 and c_2, then naturally also for $c_1 \wedge c_2$. For formulae $\bar{c}$ we prove

$$a \wedge \bar{c} \wedge \bar{c} \leqslant b \to a \wedge \bar{c} \leqslant b$$

by theorem induction. All steps are trivial, except in the case in which $a \wedge \bar{c} \wedge \bar{c} \leqslant b$ is conclusion of rule (3.4) with the premiss $a \wedge \bar{c} \leqslant c$. Let $c = p_1 \wedge \cdots \wedge p_m \wedge \bar{c}_1 \wedge \cdots \wedge \bar{c}_n$ with prime formulae p_μ. We have as induction hypothesis of our formula induction the validity of (8) for each c_ν $(\nu = 1, \cdots, n)$. From $a \wedge \bar{c} \leqslant p_\mu$ follows according to (7) $a \wedge \bar{c} \leqslant 0$ or $a \leqslant p_\mu$ for each p_μ. From $a \wedge \bar{c} \leqslant \bar{c}_\nu$ follows according to (6) $a \wedge c_\nu \leqslant \bar{c}_\nu$; thus according to (5.3) $a \wedge c_\nu \wedge c_\nu \leqslant 0$, and therefore $a \wedge c_\nu \leqslant 0$, thus $a \leqslant \bar{c}_\nu$. Together follows $a \wedge \bar{c} \leqslant 0$ or $a \leqslant p_1 \wedge \cdots \wedge p_m \wedge \bar{c}_1 \wedge \cdots \wedge \bar{c}_n$; thus in each case $a \wedge \bar{c} \leqslant 0$ and $a \wedge \bar{c} \leqslant b$.

By the aid of (8) we can now instead of (4.3) even prove

(9) $$a_1 \leqslant c \,\boldsymbol{,}\, a_2 \wedge c \leqslant b \to a_1 \wedge a_2 \leqslant b$$

by formula induction for each formula c.

For prime formulae c proof by theorem induction.

If (9) holds for c_1 and c_2, then also for $c_1 \wedge c_2$, for from $a_1 \leqslant c_1 \wedge c_2$ and $a_2 \wedge c_1 \wedge c_2 \leqslant b$ follows according to (5.1), (5.2) $a_1 \leqslant c_1$ and $a_1 \leqslant c_2$; thus $a_1 \wedge a_1 \wedge a_2 \leqslant b$, i.e. $a_1 \wedge a_2 \leqslant b$.

Now let (9) hold for c. We prove the validity for $\bar{c}$ by theorem induction for all theorems $a_2 \wedge \bar{c} \leqslant b$. All steps are trivial, except in the case in which $a_2 \wedge \bar{c} \leqslant b$ is conclusion of rule (3.4) with the premiss $a_2 \leqslant c$. From $a \leqslant \bar{c}$ follows according to (5.3) and the already proved special case of (4.3), $a_1 \wedge c \leqslant b$; thus $a_1 \wedge a_2 \leqslant b$ according to induction hypothesis because of $a_2 \leqslant c$.

Thereby (9) is proved and especially (4.3).

For the proof of theorem 9 we have to note now that for elements $a_1, \cdots, a_n, b$ out of M, $a_1, \cdots, a_n \vdash b$ holds exactly if $a_1 \wedge \cdots \wedge a_n \leqslant b$; for out of prime theorems do only through structure rules and rule (3.1) arise again prime theorems. But these rules follow from conditions 1.–4. of theorem 1. Now only the verification that each minimal orthocomplemented semilattice H' over M for which

$$a_1, \cdots, a_n \vdash b \rightleftarrows a_1 \wedge \cdots \wedge a_n \leqslant b$$

holds is homomorphic to H over M is lacking. We define for this inductively a relation ρ between H and H' by:

(i) $a \;\rho\; a$ for $a \in M$.

(ii) $a \;\rho\; a' \,\boldsymbol{,}\, b \;\rho\; b' \to a \wedge b \;\rho\; a' \wedge b'$.

(iii) $a \;\rho\; a' \to \bar{a} \;\rho\; \bar{a}'$.

ρ is an homomorphism, for holds

$$a \;\rho\; a' \,\boldsymbol{,}\, b \;\rho\; b' \,\boldsymbol{,}\, a \leqslant b \to a' \leqslant b',$$

[11] These claims include: $a \wedge \bar{c} \leqslant c \to a \leqslant c$, $\bar{c} \leqslant c \to 1 \leqslant c$, $c \wedge c \leqslant b \to c \leqslant b$.

as follows at once by theorem induction. The proof relies simply on the fact that rules (2) and (3) are valid for each orthocomplemented semilattice.

4. Logistic application and complete lattices. The fact that the logic calculuses are semilattices or lattices permits a simple logistic application of free lattices.

A calculus whose formulae form an orthocomplemented semilattice arises in the following way.

We start with propositional variables and add all formulae $a, b, \cdots$ that can be formed out of them by use of the conjunction sign $\wedge$ and the negation sign $^{-}$. Let $a \wedge b$ mean the proposition "a and b," $\bar{a}$ the proposition "not a." To the formulae we add moreover 0 and 1. Let 0 mean the "false," 1 the "true." In the set H of formulae we define a relation $\leqslant$. Let $a \leqslant b$ mean "a implies b." Let hold

$$a \leqslant a,$$
$$0 \leqslant a \leqslant 1.$$

Let moreover hold each relation that may be derived from this on the basis of the following rules:

$$a \leqslant c\,,\, c \leqslant b \rightarrow a \leqslant b.$$
$$c \leqslant a\,,\, c \leqslant b \rightarrow c \leqslant a \wedge b.$$
$$c \leqslant a \wedge b \rightarrow c \leqslant a.$$
$$c \leqslant a \wedge b \rightarrow c \leqslant b.$$
$$a \wedge b \leqslant 0 \rightarrow a \leqslant \bar{b}.$$
$$a \leqslant \bar{b} \rightarrow a \wedge b \leqslant 0.$$

According to this definition, the relation $\leqslant$ is a preorder, and H is an orthocomplemented semilattice with respect to $\leqslant$. For the set M of propositional variables including 0 and 1, a preorder is defined through the relations

$$a \leqslant a,$$
$$0 \leqslant a \leqslant 1.$$

According to the result of §3, there exists the free orthocomplemented semilattice H_0 over M. H is isomorphic to H_0 over M, for H and H_0 consist of the same formulae, and each relation $a \leqslant b$ that holds for H holds also for H_0, as well as the other way round. Thus M is a part of H. From this follows immediately the consistency[12] of the calculus, for from $1 \leqslant a$ and $a \leqslant 0$ would follow $1 \leqslant 0$; but this relation does not hold in M.

Apart from the consistency, a decision procedure for H follows from the construction of H_0. In fact, the validity of a relation $a \leqslant b$ in H_0 is decidable, as obviously for each theorem only finitely many premisses are possible and a chain of premisses always stops after finitely many terms (the maximal number of steps is easy to estimate).

Also the consistency of formalised theories in which one does not work with this propositional calculus but with the classical calculus results by this means, if one con-

[12]The consistency means that no proposition is simultaneously true and false, i.e. for no formula a do $1 \leqslant a$ and $a \leqslant 0$ hold simultaneously.

siders instead of the orthocomplemented semilattices the countably complete boolean lattices.

A "boolean lattice" is a distributive lattice in which to each element c there is an element $\bar{c}$ with $c \wedge \bar{c} \leqslant 0$ and $c \vee \bar{c} \geqslant 1$.

A semilattice H (with respect to $\leqslant$) is called "countably complete" if for each countable subset N of H there is an element c of H such that holds:

$$a \in N \rightarrow c \leqslant a.$$

$$(\text{for each } a \in N,\, x \leqslant a) \rightarrow x \leqslant c.$$

We then write $c = \bigwedge N$.

A lattice V is called "countably complete" if V is a countably complete semilattice with respect to $\leqslant$ and w.r.t. $\geqslant$.

Classical number theory e.g., which for each formula $a(x)$ in which occurs a free individual variable x also contains the formulae $\bigwedge_x a(x)$ and $\bigvee_x a(x)$ ($\bigwedge_x a(x)$ means "for each x, $a(x)$," $\bigvee_x a(x)$ means "for at least one x, $a(x)$") is indeed not a countably complete lattice; it does not contain for each countable subset N of formulae e.g. the conjunction $\bigwedge N$, but only for the sets $N = \{a(1), a(2), \cdots\}$, and then one has $\bigwedge N \equiv \bigwedge_x a(x)$.

Nevertheless, the proof of existence for the free countably complete boolean lattice over any preordered set and the proof of consistency for the classical calculus are so alike that, to avoid repetitions, we only sketch here the proof of existence. All details may be extracted from the proof of consistency undertaken in part II (§§5–8).

Theorem 10. *If M is a bounded preordered set and $a_1, \cdots, a_m \vdash b_1, \cdots, b_n$ a relation that satisfies the conditions*

1. $$a \vdash b \rightleftarrows a \leqslant b$$

and 2.–4. of theorem 5, then there is a countably complete boolean lattice V over M for which holds that for elements $a_1, \cdots, a_m, b_1, \cdots, b_n \in M$

$$a_1, \cdots, a_m \vdash b_1, \cdots, b_n \rightleftarrows a_1 \wedge \cdots \wedge a_m \leqslant b_1 \vee \cdots \vee b_n,$$

and that each minimal[13] *countably complete boolean lattice V' over M that fulfils these conditions is homomorphic*[14] *to V over M.*

---

[13] A countably complete boolean lattice V is called minimal over M if V does not contain a proper subset V_0 for which holds:
1. $M \subseteq V_0$.
2. $a, b \in V_0 \,,\, c \equiv a \wedge b \rightarrow c \in V_0$.
3. $a, b \in V_0 \,,\, c \equiv a \vee b \rightarrow c \in V_0$.
4. $a \in V_0 \,,\, c \equiv \bar{a} \rightarrow c \in V_0$.
5. $N \subseteq V_0$, N countable, $c \equiv \bigwedge N \rightarrow c \in V_0$.
6. $N \subseteq V_0$, N countable, $c \equiv \bigvee N \rightarrow c \in V_0$.

[14] I.e. there is a lattice homomorphism ρ for which holds:
1. $a \,\rho\, a' \rightarrow \bar{a} \,\rho\, \bar{a}'$.
2. If, for a countable subset N, ρ is an homomorphism from N onto N', then holds $\bigwedge N \,\rho \bigwedge N'$ and $\bigvee N \,\rho \bigvee N'$.

First a set V will be defined constructively, for which holds:

$$M \subseteq V.$$
$$a, b \in V \rightarrow a \wedge b \in V.$$
$$a, b \in V \rightarrow a \vee b \in V.$$
$$a \in V \rightarrow \bar{a} \in V.$$
$$N \subseteq V \text{ , } N \text{ countable} \rightarrow \bigwedge N \in V.$$
$$N \subseteq V \text{ , } N \text{ countable} \rightarrow \bigvee N \in V.$$

In doing so, let differently designated elements always be different.

In V a relation is defined constructively by:

[1] $a_1, \cdots, a_m \vdash b_1, \cdots, b_n \rightarrow a_1 \wedge \cdots \wedge a_m \leqslant b_1 \vee \cdots \vee b_n$.

[2] For the left and right formula of $a_1 \wedge \cdots \wedge a_m \leqslant b_1 \vee \cdots \vee b_n$ will be allowed as structure change, apart from association and transposition, also contraction, i.e. of two equal elements one may be omitted.

[3] $1 \wedge x$ and $0 \vee x$ may always be replaced by x.

[3.1] $a \leqslant b \rightarrow a \leqslant b \vee c$.

[3.2] $a \leqslant b \rightarrow a \wedge c \leqslant b$.

[3.3] $a_1 \wedge c \leqslant b \text{ , } a_2 \wedge c \leqslant b \rightarrow (a_1 \vee a_2) \wedge c \leqslant b$.

[3.4] $a \leqslant b_1 \vee c \text{ , } a \leqslant b_2 \vee c \rightarrow a \leqslant (b_1 \wedge b_2) \vee c$.

[3.5] $a \leqslant b \vee c \rightarrow a \wedge \bar{c} \leqslant b$.

[3.6] $a \wedge c \leqslant b \rightarrow a \leqslant b \vee \bar{c}$.

For $N \subseteq V$, N countable:

[3.7] $c \in N \text{ , } a \wedge c \leqslant b \rightarrow a \wedge \bigwedge N \leqslant b$.

[3.8] $c \in N \text{ , } a \leqslant b \vee c \rightarrow a \leqslant b \vee \bigvee N$.

[3.9] (For each $x \in N$, $a \wedge x \leqslant b$) $\rightarrow a \wedge \bigvee N \leqslant b$.

[3.10] (For each $x \in N$, $a \leqslant b \vee x$) $\rightarrow a \leqslant b \vee \bigwedge N$.

By the aid of formula and theorem inductions one has then to prove that V is w.r.t. $\leqslant$ a countably complete boolean lattice.

Instead of the relations (5.1), (5.2) to be proved steps in now:

$$\begin{array}{l|l}
a \leqslant (b_1 \wedge b_2) \vee c \rightarrow a \leqslant b_1 \vee c. & x \in N \text{ , } a \leqslant b \vee \bigwedge N \rightarrow a \leqslant b \vee x. \\
a \leqslant (b_1 \wedge b_2) \vee c \rightarrow a \leqslant b_2 \vee c. & \\
(a_1 \vee a_2) \wedge c \leqslant b \rightarrow a_1 \wedge c \leqslant b. & x \in N \text{ , } a \wedge \bigvee N \leqslant b \rightarrow a \wedge x \leqslant b. \\
(a_1 \vee a_2) \wedge c \leqslant b \rightarrow a_2 \wedge c \leqslant b. &
\end{array}$$

Instead of (5.3) steps in:

$$a \leqslant \bar{b} \vee c \rightarrow a \wedge b \leqslant c.$$
$$a \wedge \bar{b} \leqslant c \rightarrow a \leqslant b \vee c.$$

Instead of (4.3) vs. (9) steps in

$$a_1 \leqslant c \vee b_1 \text{ , } a_2 \wedge c \leqslant b_2 \quad \rightarrow \quad a_1 \wedge a_2 \leqslant b_1 \vee b_2.$$

The special case $a \leqslant 0 \rightarrow a \leqslant b$ results immediately, $1 \leqslant b \rightarrow a \leqslant b$ as well. (8) (and therefore also (6) and (7)) is dispensable here.

The essential difference with regard to the construction in theorem 9 lies in the fact that in V, the validity of a relation $a \leqslant b$ cannot be decided in general, for in the rules [3.9] and [3.10] occur infinitely many premisses.

The structure rule of contraction, that was provable in §3, must be assumed here, as shows the following example. Let $M = \{1, \frac{1}{2}, \cdots, 1/n, \cdots, 0\}$, and let $\leqslant$ be the order according to magnitude. V contains for $N = \{1, \frac{1}{2}, \cdots, 1/n, \cdots\}$ the element $a = \overline{\bigwedge N}$; moreover for $N' = N \cup \{a\}$ the element $\bigwedge N'$. For each $c \in N$ holds then $c \leqslant c$, from which follows

$$\bigwedge N' \leqslant c, \qquad \bigwedge N' \leqslant \bigwedge N,$$
$$\bigwedge N' \wedge a \leqslant 0,$$
$$\bigwedge N' \wedge \bigwedge N' \leqslant 0.$$

In contrast—without assuming contraction—$\bigwedge N' \leqslant 0$ does not hold, for $c \leqslant 0$ holds for no $c \in N$; and also

$$\overline{\bigwedge N} \leqslant 0$$

does not hold, as $1 \leqslant \bigwedge N$ does not hold. The example shows further that contraction also does not become provable if the rules [3.7] and [3.8] are replaced by

$$c_1, \cdots, c_n \in N \text{ , } a \wedge c_1 \wedge \cdots \wedge c_n \leqslant b \qquad \rightarrow \qquad a \wedge \bigwedge N \leqslant b,$$
$$c_1, \cdots, c_n \in N \text{ , } a \leqslant b \vee c_1 \vee \cdots \vee c_n \qquad \rightarrow \qquad a \leqslant b \vee \bigvee N.$$

By the hypothesis of contraction, the proof of theorem 10 simplifies considerably with regard to the proof of theorem 9. The proof of consistency undertaken in part II (§§5–8) requires in contrast again additional considerations because of the use of free variables in the logic calculus.

From theorem 10 the consistency of ramified type logic (incl. arithmetic) may be derived immediately in the following way:

We define a calculus Z_0.

Numbers: $\alpha, \beta, \cdots$.

(N1) $\quad 1.$

(N2) $\quad \alpha \rightarrow \alpha + 1.$

Formulae: $a, b, \cdots$.

(F1) $\quad \bigvee$. (Interpretation: the true.)
$\bigwedge$. (Interpretation: the false.)

(F2) $\quad \alpha > \beta$.

Expressions: $A, B, \cdots$.

$$a \leqslant b \text{ (if } a\text{, then } b\text{)}.$$

Theorems:

(T1) $\quad a \leqslant b$ if interpretation true.

(T2) $\quad a \leqslant a.$
$\bigwedge \leqslant a \leqslant \bigvee.$

(T3) $\quad a \leqslant c \text{ , } c \leqslant b \rightarrow a \leqslant b.$

Z_0 is obviously consistent in the sense that $\bigvee \leqslant \bigwedge$ is not a theorem.

It is to be shown that this consistency is conserved if free vs. bound number variables $x, y, \cdots$ are added, the definition of formulae is extended by

(F3) $\quad x > y, \qquad x > \alpha, \qquad \alpha > x,$

(F4) $\quad$ if a, b, then $a \wedge b, a \vee b$ (and, or),

(F5) $\quad$ if c, then $\bar{c}$ (not), (F6) if $a(x)$, then $\bigwedge_x a(x)$, $\bigvee_x a(x)$ (for all, for some),

and the definition of theorems by

(T4) $a \leqslant b_1 \mathbin{,} a \leqslant b_2 \rightleftarrows a \leqslant b_1 \wedge b_2$,

(T5) $a_1 \leqslant b \mathbin{,} a_2 \leqslant b \rightleftarrows a_1 \vee a_2 \leqslant b$,

(T6) $a \wedge c \leqslant b \rightarrow a \leqslant b \vee \bar{c}$,

(T7) $a \leqslant b \vee c \rightarrow a \wedge \bar{c} \leqslant b$,

(T8) $a \leqslant b(x) \rightleftarrows a \leqslant \bigwedge_x b(x)$ (x not in a),

(T9) $a(x) \leqslant b \rightleftarrows \bigvee_x a(x) \leqslant b$ (x not in b),

(T10) (for each α, $A(\alpha)$) $\rightleftarrows A(x)$.

We call the arising calculus K_0.

The expressions of K_0 without free variables form a partial calculus K_0' that contains Z_0. In the definition of theorems of K_0' step in, instead of (T8)–(T10):

$$\text{(for each } \gamma,\ a \leqslant b(\gamma)) \rightleftarrows a \leqslant \textstyle\bigwedge_x b(x),$$
$$\text{(for each } \gamma,\ a(\gamma) \leqslant b) \rightleftarrows \textstyle\bigvee_x a(x) \leqslant b.$$

Therefore K_0' is contained isomorphically in the free countably complete boolean lattice over Z_0, i.e. Z_0 is a partial calculus of K_0'—in particular K_0' and K_0 are thus consistent.

The consistency follows as well if in Z_0 one admits as formulae apart from $\alpha > \beta$ in addition $\alpha + \beta = \gamma$, $\alpha \cdot \beta = \gamma$ and the like.

In order to obtain the consistency of ramified type logic (incl. arithmetic), we extend K_0 by adding for each formula $a(x)$ out of K_0 the "set" $A = \hat{x}a(x)$, by extending the definition of formulae by

(F2*) $x \in A, \qquad \alpha \in A$,

and the definition of theorems by

(T1*) $a(x) \leqslant x \in A, \qquad x \in A \leqslant a(x)$,

and (T2), (T3), and (T10) also for the formulae (F2*). The calculus Z_1 arising in this way is consistent as K_0 is consistent.

This consistency is conserved by reason of the existence of the free countably complete boolean lattice over Z_1 if free and bound set variables $X, Y, \cdots$ are added, the definition of formulae is extended by

(F3*) $x \in X, \qquad \alpha \in X$,

(F6*) if $a(X)$, then $\bigwedge_X a(X)$, $\bigvee_X a(X)$,

and (F4)–(F6) also for the formulae (F2*), (F3*), (F6*), and the definition of theorems by

(T8*) $a \leqslant b(X) \rightleftarrows a \leqslant \bigwedge_X b(X)$ (X not in a),

(T9*) $a(X) \leqslant b \rightleftarrows \bigvee_X a(X) \leqslant b$ (X not in b),

(T10*) (for each A, $\mathfrak{A}(A)$) $\rightleftarrows \mathfrak{A}(X)$.

Iteration of this extension procedure yields the sought-after consistency of ramified type logic incl. arithmetic.

5. The deductive calculus of ramified type logic. Let λ be a constructible ordinal number (e.g. ω). By "ordinal number" we are always understanding below only the ordinal numbers $\nu < \lambda$. We assume the knowledge of the signs for these ordinal numbers $0, 1, 2, \cdots$, and are building up our calculus out of them by adding finitely many further individual signs:

$$(,), \wedge, \vee, {}^{-}, \bigwedge, \bigvee .$$

By "sign" we understand not only the individual signs, but also their compositions. For communication we use for signs mostly $a, b, \cdots$. $a = b$ means that a and b are signs of the same form. If a is an individual sign, then $c(a)$ means a sign in which a occurs. $c(b)$ means then the sign arising by substitution of a by b. (At this, each occurrence of a is to be substituted.) Also if a is composite, we use this notation in the cases in which no misunderstanding is possible. We communicate sign pairs a, b through the letters $A, B, \cdots$. Let $C(a)$ mean a sign pair in which a occurs in at least one sign. Let $C(b)$ mean the sign pair arising by substitution of a by b.

For the set-up of the deductive calculus we first define which signs we want to call "types". As "type of 0th order" we only take: 0. As "types of νth order" ($\nu > 0$) we take: $\nu\,(\tau_1\tau_2\cdots\tau_n)$ if $\tau_1, \cdots, \tau_n$ are types of orders $\mu_1, \cdots, \mu_n$ with $\mu_i < \nu$ $(i = 1, \cdots, n)$.

For each type τ we then form "free variables" $(\tau)_0, (\tau)_1, \cdots$, and "bound variables" $((\tau))_0, ((\tau))_1, \cdots$. As signs of communication for free vs. bound variables we use $p, q, \cdots$ vs. $x, y, \cdots$. These letters we use possibly with indices. $p^\tau, q^\tau, \cdots$ vs. $x^\tau, y^\tau, \cdots$ mean always variables of type τ. Variables of type 0 we also call "individual variables", the variables of types of higher order also "relation variables."

Next we define which signs are to be called "formulae". As formula of νth order ($\nu > 0$) we take:

(F1) $\quad 0, 1.$

(F2) $\quad (p^\tau p_1^{\tau_1}\cdots p_n^{\tau_n})$ if $\tau = \mu(\tau_1\cdots\tau_n)$ and $\mu \leq \nu$.

These formulae are called the "prime formulae."

(F3) With a, b, $c(p^\tau)$ also $(a \wedge b)$, $(a \vee b)$, $\bar{a}$, $\bigwedge_{x^\tau} c(x^\tau)$, $\bigvee_{x^\tau} c(x^\tau)$ if x^τ does not occur in $c(p^\tau)$ and for the order μ of τ holds $\mu < \nu$.

For the logical interpretation, $p^\tau p_1^{\tau_1}\cdots p_n^{\tau_n}$ is to be read as "the relation p^τ is fulfilled by $p_1^{\tau_1}, \cdots, p_n^{\tau_n}$." $\wedge$, $\vee$, ${}^{-}$, $\bigwedge$, $\bigvee$ is to be read as "et," "vel," "non," "omnes," "existit."

At communicating formulae we omit the brackets as soon as this is possible without misunderstanding. For the communication of formulae $(\bar{a} \vee b)$ vs. $(\bar{a} \vee b) \wedge (\bar{b} \vee a)$ we also use $a \to b$ vs. $a \leftrightarrow b$. We define last the concept of theorem for our calculus. In an only formal opposition to the classical calculus we are not distinguishing certain formulae as theorems, but formula pairs. We write $a \preccurlyeq b$ for communicating that the formula pair a, b is a theorem. For the logical interpretation, this is to be read as "the proposition a implies the proposition b." 0 is the "false," 1 is the "true."

The concept of theorem is defined by:

(1a) $\quad c \preccurlyeq c, \qquad 0 \preccurlyeq c, \qquad c \preccurlyeq 1.$

(1b) For $\tau = \nu\,(\tau_1\cdots\tau_n)$ and formulae of νth order $c(p_1^{\tau_1}, \cdots, p_n^{\tau_n})$,

$$1 \preccurlyeq \bigvee_{x^\tau} \bigwedge_{x_1^{\tau_1}} \cdots \bigwedge_{x_n^{\tau_n}} (x^\tau x_1^{\tau_1}\cdots x_n^{\tau_n} \leftrightarrow c(x_1^{\tau_1}, \cdots, x_n^{\tau_n})).$$

(1c) For $\tau = \nu\,(\tau_1\cdots\tau_n)$ and formulae $c(p^\tau)$,

$$\bigwedge_{x_1^{\tau_1}} \cdots \bigwedge_{x_n^{\tau_n}} (p^\tau x_1^{\tau_1}\cdots x_n^{\tau_n} \leftrightarrow q^\tau x_1^{\tau_1}\cdots x_n^{\tau_n}) \preccurlyeq c(p^\tau) \to c(q^\tau).$$

(1d) For $\tau = 1(00)$,

$$1 \preccurlyeq \bigvee_{x^\tau} (\bigwedge_{x^0} \overline{x^\tau x^0 x^0} \wedge \bigwedge_{x^0} \bigvee_{y^0} x^\tau x^0 y^0 \wedge \bigwedge_{x^0} \bigwedge_{y^0} \bigwedge_{z^0} ((x^\tau x^0 y^0 \wedge x^\tau y^0 z^0) \to x^\tau x^0 z^0)).$$

These theorems are called "axioms."

(2a) $$a \preccurlyeq c \,,\, c \preccurlyeq b \rightarrow a \preccurlyeq b.$$

(2b) $$a \preccurlyeq b \rightarrow a \wedge c \preccurlyeq b.$$
$$a \preccurlyeq b \rightarrow c \wedge a \preccurlyeq b.$$
$$a \preccurlyeq b \rightarrow a \preccurlyeq b \vee c.$$
$$a \preccurlyeq b \rightarrow a \preccurlyeq c \vee b.$$
$$a \preccurlyeq b_1 \,,\, a \preccurlyeq b_2 \rightarrow a \preccurlyeq b_1 \wedge b_2.$$
$$a_1 \preccurlyeq b \,,\, a_2 \preccurlyeq b \rightarrow a_1 \vee a_2 \preccurlyeq b.$$

(2c) $$a \wedge c \preccurlyeq b \rightarrow a \preccurlyeq b \vee \bar{c}.$$
$$a \preccurlyeq b \vee c \rightarrow a \wedge \bar{c} \preccurlyeq b.$$

(2d) For formulae $c(p^\tau)$ in which x^τ does not occur,
$$a \preccurlyeq \textstyle\bigwedge_{x^\tau} c(x^\tau) \rightarrow a \preccurlyeq c(p^\tau),$$
$$\textstyle\bigvee_{x^\tau} c(x^\tau) \preccurlyeq b \rightarrow c(p^\tau) \preccurlyeq b;$$
and, if p^τ does not occur in a nor b,
$$a \preccurlyeq c(p^\tau) \rightarrow a \preccurlyeq \textstyle\bigwedge_{x^\tau} c(x^\tau).$$
$$c(p^\tau) \preccurlyeq b \rightarrow \textstyle\bigvee_{x^\tau} c(x^\tau) \preccurlyeq b.$$

We call (2a)–(2d) the "rules" of the calculus. The propositions to the left of $\rightarrow$ are called the "premisses," the proposition to the right of $\rightarrow$ is called the "conclusion."

The calculus is described by the complete definition of the concept of formula and theorem. We call it shortly the "deductive calculus."

Comments to the deductive calculus. (A) If we restrict ourselves to ordinal numbers $< \omega$, then our calculus is equivalent to the "classical" calculus of the *Principia mathematica* if the axiom of reducibility is omitted there. A formula is classically deducible exactly if $1 \preccurlyeq a$ holds in our calculus, and $a \preccurlyeq b$ holds exactly if the formula $a \to b$ is classically deducible. We shall not undertake the proof of equivalence here, because it results easily from the equivalence of Gentzen's sequent calculus with the classical calculus. If one extends the sequent calculus to the capacity of expression of our calculus, then the sequent $a_1, \cdots, a_m \to b_1, \cdots, b_n$ holds exactly if $a_1 \wedge \cdots \wedge a_m \preccurlyeq b_1 \vee \cdots \vee b_n$ holds.

(B) For each formula pair a, b holds: $1 \preccurlyeq 0 \rightarrow a \preccurlyeq b$. Thus, if $1 \preccurlyeq 0$ were to hold, then in the logical interpretation each proposition would imply each other proposition, i.e. the calculus would be contradictory. The proof of consistency has thus to show that $1 \preccurlyeq 0$ does not hold.

(C) The axiom (1b) vs. (1c) corresponds to the classical axiom of comprehension vs. extensionality. The axiom (1d) corresponds to the axiom of infinity and postulates the existence of an irreflexive, transitive binary relation in the individual domain whose domain is the whole domain. For our proof of consistency, it is of no concern whether this or another equivalent form of the axiom of infinity is postulated.

(D) The concept of formula is defined constructively. The set of formulae is the smallest set of signs that satisfies the conditions (F1)–(F3).

Therefore the following "formula induction" holds: if a claim holds

1. for each prime formula,
2. for $a \wedge b$, $a \vee b$, $\bar{a}$, $\bigwedge_{x^\tau} c(x^\tau)$, $\bigvee_{x^\tau} c(x^\tau)$ if for a, b, and $c(p^\tau)$,

then it holds for each formula.

The concept of theorem is defined constructively as well. The set of theorems is the smallest set of formula pairs that satisfies conditions (1) and (2).

The following theorem induction holds: if a claim holds

1. for each axiom,
2. for each conclusion of a rule whose premisses are theorems if for the premisses,

then it holds for each theorem.

(E) The following "duality principle" holds for the rules of the calculus: if one swaps in each formula pair the left formula with the right one and, in doing so, simultaneously $\wedge$ with $\vee$ and $\bigwedge$ with $\bigvee$, then each rule transforms again into a rule.

At this, negations may remain unchanged. But if one wants to extend the duality also to the theorems, then in addition a must always be swapped with $\bar{a}$ and 0 with 1.

(F) It suffices to restrict oneself to "proper" formulae, i.e. to formulae that contain neither 0 nor 1. If c is proper, then we write $\preccurlyeq c$ instead of $1 \preccurlyeq c$ and $c \preccurlyeq$ instead of $c \preccurlyeq 0$. We write $\preccurlyeq$ instead of $1 \preccurlyeq 0$. If in addition we leave aside the theorems $0 \preccurlyeq c$ and $c \preccurlyeq 1$, then we obtain a calculus in which only proper formulae occur. We call this calculus the "proper deductive calculus".

6. An inductive calculus. We use the same "signs" as for the deductive calculus and add $+$ and $>$.

We take over the definition of "types" and "free" vs. "bound variables" from the deductive calculus.

For the "inductive calculus" to be constructed we define the concept of constants simultaneously with the concept of formula. As "constants of type 0" we take:

[C1] 1.

[C2] With a also $a+1$.

(For communicating the constants of type 0 we use a^0, b^0, $\cdots$.) As "constant of type τ" we take for $\tau = \nu\,(\tau_1 \cdots \tau_n)$ and each formula of νth order $c(p_1^{\tau_1}, \cdots, p_n^{\tau_n})$ in which $x_1^{\tau_1}, \cdots, x_n^{\tau_n}$ do not occur and which is without variables and constants of types of order ν: $(x_1^{\tau_1} \cdots x_n^{\tau_n})^\nu\, c(x_1^{\tau_1}, \cdots, x_n^{\tau_n})$. (For communicating these constants we use a^τ, b^τ, $\cdots$.)

As "formulae of νth order" ($\nu > 0$) we take:

[F1] $(p^0 > q^0)$.

$(p^\tau p_1^{\tau_1} \cdots p_n^{\tau_n})$ for $\tau = \mu(\tau_1\ \cdots\ \tau_n)$ and $\mu \leq \nu$.

With $c(p^\tau)$ also $c(a^\tau)$ for each constant a^τ. These formulae are called the "prime formulae." The formulae $a^0 > b^0$ are called "numerical formulae."

[F2] With a, b, $c(p^\tau)$ also $a \wedge b$, $a \vee b$, $\bar{a}$, $\bigwedge_{x^\tau} c(x^\tau)$, $\bigvee_{x^\tau} c(x^\tau)$ if x^τ does not occur in $c(p^\tau)$ and if for the order μ of τ holds $\mu < \nu$.

The logical interpretation of the formulae is to be carried out as in the deductive calculus. The constants of 0th type are to be interpreted as the natural numbers. $p^0 > q^0$ is to be read as "p^0 greater than q^0." The constants $a^\tau = (x_1^{\tau_1} \cdots x_n^{\tau_n})^\nu\, a(x_1^{\tau_1}, \cdots, x_n^{\tau_n})$ are to be interpreted as "the relation of νth order between $x_1^{\tau_1}, \cdots, x_n^{\tau_n}$ defined by $a(x_1^{\tau_1}, \cdots, x_n^{\tau_n})$."

We define the concept of theorem as in the deductive calculus for proper formulae:

[1] For numerical formulae $c = (a^0 > b^0)$ that are correct vs. false on the basis of the interpretation in terms of content of a^0 and b^0 as natural numbers: $\preccurlyeq c$ vs. $c \preccurlyeq$.
These theorems are called the "numerical theorems."

[2] Structure rule. If $a_1 \wedge \cdots \wedge a_m \preccurlyeq b_1 \vee \cdots \vee b_n$ holds, then by the following changes to the left or right formula arises again a theorem: association, i.e. the grouping of terms by brackets may be changed; transposition, i.e. the sequential arrangement of the terms may be changed; contraction, i.e. of several equal terms one may be omitted.

[3a]
$$a \preccurlyeq b \rightarrow a \preccurlyeq b \vee c.$$
$$a \preccurlyeq b \rightarrow a \wedge c \preccurlyeq b.$$

[3b]
$$a \preccurlyeq b_1 \vee c\,,\, a \preccurlyeq b_2 \vee c \rightarrow a \preccurlyeq (b_1 \wedge b_2) \vee c.$$
$$a_1 \wedge c \preccurlyeq b\,,\, a_2 \wedge c \preccurlyeq b \rightarrow (a_1 \vee a_2) \wedge c \preccurlyeq b.$$

[3c]
$$a \wedge c \preccurlyeq b \rightarrow a \preccurlyeq b \vee \bar{c}.$$
$$a \preccurlyeq b \vee c \rightarrow a \wedge \bar{c} \preccurlyeq b.$$

[3d] For formulae $c(p^\tau)$ in which x^τ does not occur, if p^τ does not occur in a nor b:
$$a \preccurlyeq b \vee c(p^\tau) \rightarrow a \preccurlyeq b \vee \textstyle\bigwedge_{x^\tau} c(x^\tau),$$
$$a \wedge c(p^\tau) \preccurlyeq b \rightarrow a \wedge \textstyle\bigvee_{x^\tau} c(x^\tau) \preccurlyeq b.$$

For constants a^τ:
$$a \wedge c(a^\tau) \preccurlyeq b \rightarrow a \wedge \textstyle\bigwedge_{x^\tau} c(x^\tau) \preccurlyeq b,$$
$$a \preccurlyeq b \vee c(a^\tau) \rightarrow a \preccurlyeq b \vee \textstyle\bigvee_{x^\tau} c(x^\tau).$$

In these rules [3], a and b may also be omitted if $a \wedge c$ and $b \vee c$ are replaced by c.

[4] Induction rule: a formula pair $C(p^\tau)$ is a theorem if for each constant a^τ of type τ the formula pair $C(a^\tau)$ is a theorem.

[5] Rule of constants. In order to formulate this rule we need the concept of "elimination of constants." Let a^τ be a constant of νth order
$$a^\tau = (x_1^{\tau_1} \cdots x_n^{\tau_n})^\nu\, a(x_1^{\tau_1}, \cdots, x_n^{\tau_n}).$$
By "elimination" of a^τ, we understand the mapping of the set of formulae into itself that will be defined in the following way. Let c/a^τ be the image of c.

(i) For each prime formula beginning with a^τ:
$$a^\tau c_1 \cdots c_n / a^\tau = a(c_1, \cdots, c_n).$$
For each other prime formula c: $c/a^\tau = c$.

(ii)
$$(c_1 \wedge c_2)/a^\tau = c_1/a^\tau \wedge c_2/a^\tau.$$
$$(c_1 \vee c_2)/a^\tau = c_1/a^\tau \vee c_2/a^\tau.$$
$$\bar{c}/a^\tau = \overline{c/a^\tau}.$$

For $c(p^\sigma)/a^\tau = c'(p^\sigma)$:
$$(\textstyle\bigwedge_{x^\sigma} c(x^\sigma))/a^\tau = \textstyle\bigwedge_{x^\sigma} c'(x^\sigma),$$
$$(\textstyle\bigvee_{x^\sigma} c(x^\sigma))/a^\tau = \textstyle\bigvee_{x^\sigma} c'(x^\sigma).$$

The image c/a^τ is obviously defined by this for each formula c. If C designates the formula pair c_1, c_2, then let C/a^τ be the formula pair $c_1/a^\tau, c_2/a^\tau$. Now we can formulate the rule of constants: "if C/a^τ is a theorem, then so is C."

We call [2]–[5] the “rules” of the calculus.

The calculus is described by the complete definition of the concept of formula and theorem. We call this calculus shortly the “inductive calculus.”

Comments to the inductive calculus. [A] In the inductive calculus holds according to [3a]

$$\preccurlyeq \rightarrow a \preccurlyeq b$$

for each formula pair a, b. The inductive calculus is obviously consistent in the sense that $\preccurlyeq$ is not a theorem. In fact, there is no rule that could have $\preccurlyeq$ as conclusion. For each rule—except the structure rules—the conclusion contains at least one proper formula. The structure rules trivially cannot have $\preccurlyeq$ as conclusion, as long as the premiss is different from $\preccurlyeq$.

[B] Instead of the axioms of the deductive calculus appear alone the numerical theorems of the inductive calculus. At defining these theorems, use is made of the interpretation in terms of content.

[C] The induction rule yields a conclusion out of infinitely many premisses. But the infinite set of premisses is constructively defined, as the set of the constants a^τ is defined constructively. Therefore the induction rule is constructively admissible.

[D] The concept of formula and theorem is again defined constructively as in the deductive calculus.

Therefore “formula induction of 1st kind” also holds: if a claim holds
1. for each prime formula,
2. for $a \wedge b$, $a \vee b$, $\bar{a}$, $\bigwedge_{x^\tau} c(x^\tau)$, $\bigvee_{x^\tau} c(x^\tau)$ if for a, b, $c(a^\tau)$,
then it holds for each formula.

“Theorem induction”: if a claim holds
1. for each numerical theorem,
2. for the conclusion of a rule whose premisses are theorems if for these premisses,
then it holds for each theorem.

It is essential for the inductive calculus that also the following “formula induction of 2nd kind” is valid: if a claim holds
1. for numerical formulae,
2.1 for $a \wedge b$, $a \vee b$, $\bar{a}$ if for a, b,
2.2 for c if for c/a^τ,
2.3 for $c(p^\tau)$, $\bigwedge_{x^\tau} c(x^\tau)$, $\bigvee_{x^\tau} c(x^\tau)$ if for each $c(a^\tau)$,
then it holds for each formula.

Proof. A claim that fulfils 1., 2., holds at first for each formula $(p^0 > q^0)$ according to 1. and 2.3, thus for each formula of 1st order without variables and constants of 1st order. If the claim holds for each formula of νth order without variables and constants of νth order, then it holds according to 2.2 for each formula $a^\tau p_1^{\tau_1} \cdots p_n^{\tau_n}$ in which a^τ is a constant of order ν, thus according to 2.3 for each prime formula of νth order, i.e. for the prime formulae of $\nu + 1$st order without variables and constants of $\nu + 1$st order. According to 2.1 and 2.3, the validity for each formula of $\nu + 1$st order without variables and constants of $\nu + 1$st order follows from this. Thereby the formula induction of 2nd kind is proved.

[E] The same duality principle holds for the rules of the inductive calculus as in the deductive calculus.

7. The consistency of the deductive calculus. We prove that the proper deductive calculus is a part of the inductive calculus. First each proper formula of the deductive calculus is obviously also a formula of the inductive calculus. We extend therefore the proper deductive calculus if we replace its definition of formulae by the definition of formulae of the inductive calculus, but keep its definition of the concept of theorem.

We have then to prove in addition that each theorem of the proper deductive calculus is also a theorem of the inductive calculus. This claim on all deductive theorems is to be proved by a theorem induction. Thus we have to prove that

(I) the axioms of the proper deductive calculus are inductive theorems,

(II) the conclusion of a deductive rule is an inductive theorem if the premisses are inductive theorems.

We shall prove the claims (I) and (II) by the formula and theorem inductions valid for the inductive calculus.

(1a) Axiom $c \preccurlyeq c$. We prove by formula induction of 2nd kind that for each formula c holds $c \preccurlyeq c$.

1. $c \preccurlyeq c$ holds for numerical formulae c, for $\preccurlyeq c$ or $c \preccurlyeq$ holds, from which in each case arises $c \preccurlyeq c$ according to [3a].

2.1 Let $c_1 \preccurlyeq c_1$ and $c_2 \preccurlyeq c_2$ hold. Then follows, because of $c_1 \preccurlyeq c_1 \to c_1 \wedge c_2 \preccurlyeq c_1$, $c_2 \preccurlyeq c_2 \to c_1 \wedge c_2 \preccurlyeq c_2$, and $c_1 \wedge c_2 \preccurlyeq c_1$, $c_1 \wedge c_2 \preccurlyeq c_2 \to c_1 \wedge c_2 \preccurlyeq c_1 \wedge c_2$, also $c_1 \wedge c_2 \preccurlyeq c_1 \wedge c_2$. As well follows $c_1 \vee c_2 \preccurlyeq c_1 \vee c_2$. Because of $c \preccurlyeq c \to c \wedge \bar{c} \preccurlyeq$ and $c \wedge \bar{c} \preccurlyeq \to \bar{c} \preccurlyeq \bar{c}$ follows also $\bar{c} \preccurlyeq \bar{c}$.

2.2 Let $c/a^\tau \preccurlyeq c/a^\tau$ hold. Then $c \preccurlyeq c$ holds also according to the rule of constants.

2.3 Let $c(a^\tau) \preccurlyeq c(a^\tau)$ hold for each a^τ. Then follows first $c(p^\tau) \preccurlyeq c(p^\tau)$ according to the induction rule. Further follows, because of $c(a^\tau) \preccurlyeq c(a^\tau) \to \bigwedge_{x^\tau} c(x^\tau) \preccurlyeq c(a^\tau)$, $\bigwedge_{x^\tau} c(x^\tau) \preccurlyeq c(a^\tau)$ for each a^τ; thus $\bigwedge_{x^\tau} c(x^\tau) \preccurlyeq c(p^\tau)$ according to the induction rule, and from this $\bigwedge_{x^\tau} c(x^\tau) \preccurlyeq \bigwedge_{x^\tau} c(x^\tau)$. As well follows $\bigvee_{x^\tau} c(x^\tau) \preccurlyeq \bigvee_{x^\tau} c(x^\tau)$.

(1b) Axiom of comprehension: for $\tau = \nu\,(\tau_1 \cdots \tau_n)$ and formulae of νth order $c = c(p_1^{\tau_1}, \cdots, p_n^{\tau_n})$,

$$\preccurlyeq \bigvee_{x^\tau} \bigwedge_{x_1^{\tau_1}} \cdots \bigwedge_{x_n^{\tau_n}} (x^\tau x_1^{\tau_1} \cdots x_n^{\tau_n} \leftrightarrow c(x_1^{\tau_1}, \cdots, x_n^{\tau_n})).$$

According to observation (1a) holds $c(p_1^{\tau_1}, \cdots, p_n^{\tau_n}) \preccurlyeq c(p_1^{\tau_1}, \cdots, p_n^{\tau_n})$. From this follows $\preccurlyeq c(p_1^{\tau_1}, \cdots, p_n^{\tau_n}) \to c(p_1^{\tau_1}, \cdots, p_n^{\tau_n})$, $\preccurlyeq c(p_1^{\tau_1}, \cdots, p_n^{\tau_n}) \leftrightarrow c(p_1^{\tau_1}, \cdots, p_n^{\tau_n})$, thus $\preccurlyeq \bigwedge_{x_1^{\tau_1}} \cdots \bigwedge_{x_n^{\tau_n}} (c(x_1^{\tau_1}, \cdots, x_n^{\tau_n}) \leftrightarrow c(x_1^{\tau_1}, \cdots, x_n^{\tau_n}))$. According to the rule of constants follows for $c^\tau = (y_1^{\tau_1} \cdots y_n^{\tau_n})^\nu\, c(y_1^{\tau_1}, \cdots, y_n^{\tau_n})$, if $c(p_1^{\tau_1}, \cdots, p_n^{\tau_n})$ is without variables and constants of νth order:

$$\preccurlyeq \bigwedge_{x_1^{\tau_1}} \cdots \bigwedge_{x_n^{\tau_n}} (c^\tau x_1^{\tau_1} \cdots x_n^{\tau_n} \leftrightarrow c(x_1^{\tau_1}, \cdots, x_n^{\tau_n})).$$

From this

$$\preccurlyeq \bigvee_{x^\tau} \bigwedge_{x_1^{\tau_1}} \cdots \bigwedge_{x_n^{\tau_n}} (x^\tau x_1^{\tau_1} \cdots x_n^{\tau_n} \leftrightarrow c(x_1^{\tau_1}, \cdots, x_n^{\tau_n})).$$

If c contains a constant a^σ of order ν and the axiom of comprehension holds for c/a^σ, then according to [5] also for c. If $c = c(p^\sigma)$ contains a free variable p^σ of order ν and the axiom of comprehension holds for all $c(a^\sigma)$, then according to [4] also for c. From this follows the axiom of comprehension for each formula of νth order.

(1c) Axiom of extensionality:

$$\bigwedge_{x_1^{\tau_1}} \cdots \bigwedge_{x_n^{\tau_n}} (p^\tau x_1^{\tau_1} \cdots x_n^{\tau_n} \leftrightarrow q^\tau x_1^{\tau_1} \cdots x_n^{\tau_n}) \preccurlyeq c(p^\tau) \to c(q^\tau).$$

We write for the left formula for abbreviating $p^\tau \equiv q^\tau$, and shall for each $c(p^\tau)$ prove $p^\tau \equiv q^\tau \preccurlyeq c(p^\tau) \leftrightarrow c(q^\tau)$. We use for this the following induction, that results immediately from the formula induction of 2nd kind: if a claim holds

1. for each prime formula $p^\tau a_1^{\tau_1} \cdots a_n^{\tau_n}$ and for each formula that does not contain p^τ,

2.1 for $a \wedge b$, $a \vee b$, $\bar{a}$ if for a, b,

2.2 for c if for c/b^σ,

2.3 for $c(q^\sigma)$, $\bigwedge_{x^\sigma} c(x^\sigma)$, $\bigvee_{x^\sigma} c(x^\sigma)$ if for each $c(b^\sigma)$ and $b^\sigma \neq p^\tau$,

then it holds for each formula c.

1. According to (1a) holds

$$p^\tau a_1^{\tau_1} \cdots a_n^{\tau_n} \leftrightarrow q^\tau a_1^{\tau_1} \cdots a_n^{\tau_n} \preccurlyeq p^\tau a_1^{\tau_1} \cdots a_n^{\tau_n} \leftrightarrow q^\tau a_1^{\tau_1} \cdots a_n^{\tau_n},$$

from which follows $p^\tau \equiv q^\tau \preccurlyeq p^\tau a_1^{\tau_1} \cdots a_n^{\tau_n} \leftrightarrow q^\tau a_1^{\tau_1} \cdots a_n^{\tau_n}$.

For 2.1–2.3 it suffices to prove the following:

$$\begin{aligned} &a \preccurlyeq c_1 \leftrightarrow c_2\,,\, a \preccurlyeq d_1 \leftrightarrow d_2 \to a \preccurlyeq (c_1 \wedge d_1) \leftrightarrow (c_2 \wedge d_2). \\ \text{(a)}\qquad &a \preccurlyeq c_1 \leftrightarrow c_2\,,\, a \preccurlyeq d_1 \leftrightarrow d_2 \to a \preccurlyeq (c_1 \vee d_1) \leftrightarrow (c_2 \vee d_2). \\ &a \preccurlyeq c_1 \leftrightarrow c_2 \to a \preccurlyeq \bar{c}_1 \leftrightarrow \bar{c}_2. \end{aligned}$$

(b) $a \preccurlyeq c_1/b^\sigma \leftrightarrow c_2/b^\sigma \to a \preccurlyeq c_1 \leftrightarrow c_2$ if b^σ does not occur in a.

(c) If $a \preccurlyeq c_1(b^\sigma) \leftrightarrow c_2(b^\sigma)$ for each b^σ and q^σ does not occur in a, then

$$a \preccurlyeq c_1(q^\sigma) \leftrightarrow c_2(q^\sigma),$$

$$a \preccurlyeq \bigwedge_{x^\sigma} c_1(x^\sigma) \leftrightarrow \bigwedge_{x^\sigma} c_2(x^\sigma), \qquad a \preccurlyeq \bigvee_{x^\sigma} c_1(x^\sigma) \leftrightarrow \bigvee_{x^\sigma} c_2(x^\sigma).$$

We need for these two auxiliary rules.

For $a \preccurlyeq c \wedge d \to a \preccurlyeq c$, we prove the stronger AUXILIARY RULE 1:

$$a \preccurlyeq b \vee (c \wedge d) \vee \cdots \vee (c \wedge d) \to a \preccurlyeq b \vee c.$$

This is a claim on all inductive theorems. It claims that for each theorem holds: if a theorem C has the form $a \preccurlyeq b \vee (c \wedge d) \vee \cdots \vee (c \wedge d)$, then holds $a \preccurlyeq b \vee c$. For the proof we may therefore apply a theorem induction.

[1] For numerical theorems nothing is to be proved. We consider the inductive rules and assume the validity of the claim for the premisses.

[2] Structure rules. Let C be the conclusion of a structure rule. If one applies the induction hypothesis to the premisses, then one obtains a theorem from which may at once be inferred $a \preccurlyeq b \vee c$ by a structure rule.

[3a] If C is conclusion of an inductive rule [3a], then the premiss has the form $a_1 \preccurlyeq b_1 \vee (c \wedge d) \vee \cdots \vee (c \wedge d)$, and from $a_1 \preccurlyeq b_1 \vee c$ (possibly $a_1 \preccurlyeq b_1$) may be inferred $a \preccurlyeq b \vee c$.

[3b] If C is conclusion of a rule [3b], then we have as premisses $a \preccurlyeq b \vee c \vee (c \wedge d) \vee \cdots$, $a \preccurlyeq b \vee d \vee (c \wedge d) \vee \cdots$, or yet premisses of the form $a_1 \preccurlyeq b_1 \vee (c \wedge d) \vee \cdots$, so that from $a_1 \preccurlyeq b_1 \vee c$ may at once again also $a \preccurlyeq b \vee c$ be inferred by a rule [3b]. In the first case follows $a \preccurlyeq b \vee c \vee c$, thus $a \preccurlyeq b \vee c$.

[3c] If C is conclusion of a rule [3c], then the premiss has the form $a_1 \preccurlyeq b_1 \vee (c \wedge d) \vee \cdots \vee (c \wedge d)$, and from $a_1 \preccurlyeq b_1 \vee c$ we may infer $a \preccurlyeq b \vee c$.

[3d] The same as for [3c] holds verbatim.

[4] If $C = C(p^\tau)$ and C is conclusion out of the premisses $C(a^\tau)$ for each a^τ, then the induction hypothesis yields—if we designate the pair $a, b \vee c$ by $D = D(p^\tau)$—at once that $D(a^\tau)$ is a theorem for each a^τ. Thus D is also a theorem.

[5] If C is conclusion of a rule of constants, then we have the premiss C/a^τ, from which follows according to the induction hypothesis that D/a^τ is also a theorem. Thus D is also a theorem.

Thereby auxiliary rule 1 is proved. As one sees, all of the single steps are trivial. We shall therefore not treat them anymore in the following similar proofs.

For $a \preccurlyeq b \vee \bar{c} \rightarrow a \wedge c \preccurlyeq b$, we prove by theorem induction the stronger AUXILIARY RULE 2:

$$a \preccurlyeq b \vee \bar{c} \vee \cdots \vee \bar{c} \rightarrow a \wedge c \preccurlyeq b.$$

For numerical theorems nothing is to be proved. The treatment of rules [2]–[5] is trivial in all cases. Let only [3c] be emphasised:

$$a \wedge c \preccurlyeq b \vee \bar{c} \vee \cdots \vee \bar{c} \rightarrow a \preccurlyeq b \vee \bar{c} \vee \bar{c} \vee \cdots \vee \bar{c}.$$

The induction hypothesis yields $a \wedge c \wedge c \preccurlyeq b$, thus $a \wedge c \preccurlyeq b$ also follows. Thereby auxiliary rule 2 is proved.

Under addition of these auxiliary rules to the rules of the inductive calculus, we may now undertake the formula induction for the axiom of extensionality, and indeed with exactly the same inferences as for the axiom $c \preccurlyeq c$. From $a \preccurlyeq c_1 \leftrightarrow c_2$ we may now in fact first infer $a \preccurlyeq c_1 \rightarrow c_2$ and then $a \wedge c_1 \preccurlyeq c_2$. Everything remaining is then to be concluded as under (1a).

(1d) Axiom of infinity: for $\tau = 1(00)$,

$$\preccurlyeq \bigvee_{x^\tau} (\bigwedge_{x^0} \overline{x^\tau x^0 x^0} \wedge \bigwedge_{x^0} \bigvee_{y^0} x^\tau x^0 y^0 \wedge \bigwedge_{x^0} \bigwedge_{y^0} \bigwedge_{z^0} ((x^\tau x^0 y^0 \wedge x^\tau y^0 z^0) \rightarrow x^\tau x^0 z^0)).$$

We prove for $a^\tau = (u^0 v^0)^1\, v^0 > u^0$

$$\preccurlyeq \bigwedge_{x^0} \overline{a^\tau x^0 x^0}, \qquad \preccurlyeq \bigwedge_{x^0} \bigvee_{y^0} a^\tau x^0 y^0,$$
$$\preccurlyeq \bigwedge_{x^0} \bigwedge_{y^0} \bigwedge_{z^0} ((a^\tau x^0 y^0 \wedge a^\tau y^0 z^0) \rightarrow a^\tau x^0 z^0);$$

from which the axiom follows at once. Because of the rule of constants it suffices to prove

$$\preccurlyeq \bigwedge_{x^0} \overline{x^0 > x^0}, \qquad \preccurlyeq \bigwedge_{x^0} \bigvee_{y^0} y^0 > x^0,$$
$$\preccurlyeq \bigwedge_{x^0} \bigwedge_{y^0} \bigwedge_{z^0} ((y^0 > x^0 \wedge z^0 > y^0) \rightarrow z^0 > x^0).$$

For each a^0 holds $a^0 > a^0 \preccurlyeq$; from which $\preccurlyeq \overline{a^0 > a^0}$, and according to the induction rule $\preccurlyeq \overline{p^0 > p^0}$, thus $\preccurlyeq \bigwedge_{x^0} \overline{x^0 > x^0}$ follows. Moreover $\preccurlyeq a^0 + 1 > a^0$ holds for each a^0; from which $\preccurlyeq \bigvee_{y^0} y^0 > a^0$ and according to the induction rule $\preccurlyeq \bigvee_{y^0} y^0 > p^0$. Thus $\preccurlyeq \bigwedge_{x^0} \bigvee_{y^0} y^0 > x^0$ follows. For each a^0, b^0, c^0 holds finally $\preccurlyeq c^0 > a^0$ or $c^0 > b^0 \preccurlyeq$ or $b^0 > a^0 \preccurlyeq$; from which in each case follows $b^0 > a^0 \wedge c^0 > b^0 \preccurlyeq c^0 > a^0$ according to [3a]. Thus holds also $q^0 > p^0 \wedge r^0 > q^0 \preccurlyeq r^0 > p^0$ according to the induction rule, from which $\preccurlyeq (q^0 > p^0 \wedge r^0 > q^0) \rightarrow r^0 > p^0$ and $\preccurlyeq \bigwedge_{x^0} \bigwedge_{y^0} \bigwedge_{z^0} ((y^0 > x^0 \wedge z^0 > y^0) \rightarrow z^0 > x^0)$ follows.

(2) Now it still remains to prove the deductive rules for the inductive calculus.

(2a) $a \preccurlyeq c \,\mathbf{,}\, c \preccurlyeq b \rightarrow a \preccurlyeq b$.

We prove the stronger AUXILIARY RULE 3:

$$a_1 \preccurlyeq b_1 \vee c \vee \cdots \vee c \,\mathbf{,}\, a_2 \wedge c \wedge \cdots \wedge c \preccurlyeq b_2 \rightarrow a_1 \wedge a_2 \preccurlyeq b_1 \vee b_2.$$

For this we use the formula induction of 2nd kind for c.

1. Let c be a numerical formula. We prove auxiliary rule 3 by theorem induction for each theorem $a_1 \preccurlyeq b_1 \vee c \vee \cdots \vee c$.

1.1 Let $a_1 \preccurlyeq b_1 \vee c \vee \cdots \vee c$ be a numerical theorem. We prove auxiliary rule 3 by theorem induction for each theorem $a_2 \wedge c \wedge \cdots \wedge c \preccurlyeq b_2$.

1.1.1 Let $a_2 \wedge c \wedge \cdots \wedge c \preccurlyeq b_2$ be a numerical theorem. Auxiliary rule 3 is valid, because $\preccurlyeq c$ and $c \preccurlyeq$ do not hold simultaneously.

1.1.2 Let $a_2 \wedge c \wedge \cdots \wedge c \preccurlyeq b_2$ be a conclusion of an inductive rule, and let auxiliary rule 3 be valid for the premisses. The treatment of rules [2]–[5] is trivial in each case because c is a numerical formula.

1.2 Let $a_1 \preccurlyeq b_1 \vee c \vee \cdots \vee c$ be conclusion of an inductive rule, and let auxiliary rule 3 be valid for the premisses. The treatment of rules [2]–[5] is trivial as under 1.1.2.

2.1 Let auxiliary rule 3 be valid for $c = c_1$ and $c = c_2$. We prove it from this for $c = c_1 \wedge c_2$, $c = c_1 \vee c_2$, and $c = \bar{c}_1$. According to auxiliary rule 1 holds

$$a_1 \preccurlyeq b_1 \vee (c_1 \wedge c_2) \vee \cdots \vee (c_1 \wedge c_2) \rightarrow a_1 \preccurlyeq b_1 \vee c_1.$$
$$a_1 \preccurlyeq b_1 \vee (c_1 \wedge c_2) \vee \cdots \vee (c_1 \wedge c_2) \rightarrow a_1 \preccurlyeq b_1 \vee c_2.$$

Moreover holds $a_2 \wedge (c_1 \wedge c_2) \wedge \cdots \wedge (c_1 \wedge c_2) \preccurlyeq b_2 \rightarrow a_2 \wedge c_1 \wedge c_2 \preccurlyeq b_2$. But according to hypothesis holds $a_1 \preccurlyeq b_1 \vee c_1 \,\mathbf{,}\, a_2 \wedge c_1 \wedge c_2 \preccurlyeq b_2 \rightarrow a_1 \wedge a_2 \wedge c_2 \preccurlyeq b_1 \vee b_2$ and $a_1 \preccurlyeq b_1 \vee c_2 \,\mathbf{,}\, a_1 \wedge a_2 \wedge c_2 \preccurlyeq b_1 \vee b_2 \rightarrow a_1 \wedge a_1 \wedge a_2 \preccurlyeq b_1 \vee b_1 \vee b_2$. By contraction arises $a_1 \wedge a_2 \preccurlyeq b_1 \vee b_2$. Dually to auxiliary rule 1 holds $a \wedge (c_1 \vee c_2) \wedge \cdots \wedge (c_1 \vee c_2) \preccurlyeq b \rightarrow a \wedge c_1 \preccurlyeq b$, and by its aid follows as just also the validity of auxiliary rule 3 for $c = c_1 \vee c_2$. Finally holds according to auxiliary rule 2, $a_1 \preccurlyeq b_1 \vee \bar{c}_1 \vee \cdots \vee \bar{c}_1 \rightarrow a_1 \wedge c_1 \preccurlyeq b_1$; and dually to that holds $a_2 \wedge \bar{c}_1 \wedge \cdots \wedge \bar{c}_1 \preccurlyeq b_2 \rightarrow a_2 \preccurlyeq b_2 \vee c_1$. But according to hypothesis holds $a_2 \preccurlyeq b_2 \vee c_1 \,\mathbf{,}\, a_1 \wedge c_1 \preccurlyeq b_1 \rightarrow a_1 \wedge a_2 \preccurlyeq b_1 \vee b_2$.

2.2 Let auxiliary rule 3 be valid for $c = d/a^\tau$. We prove it from this for $c = d$.

For this we use AUXILIARY RULE 4: if C is a theorem, then also C/a^τ.

The proof by theorem induction is trivial for each step.

We designate the theorems appearing in auxiliary rule 3 by C_1, C_2, and C_3, so that it states: if C_1 and C_2 are theorems, then also C_3. According to auxiliary rule 4 follows first that C_1/a^τ and C_2/a^τ are theorems. According to hypothesis follows from that that C_3/a^τ is a theorem. According to the rule of constants, then also C_3 is a theorem.

2.3 Let auxiliary rule 3 be valid for each $c = d(a^\tau)$, where a^τ runs through all constants of type τ.

We prove it from this for $c = d(p^\tau)$ exactly correspondingly to 2.2 by using AUXILIARY RULE 5: if $C(p^\tau)$ is a theorem, then also $C(a^\tau)$ for each a^τ.

The proof is again trivial for each step.

It remains yet to show the validity of auxiliary rule 3 for $c = \bigwedge_{x^\tau} d(x^\tau)$ and $c = \bigvee_{x^\tau} d(x^\tau)$.

First, we prove again by a trivial theorem induction AUXILIARY RULE 6:

$$a_1 \preccurlyeq b_1 \vee \bigwedge_{x^\tau} d(x^\tau) \vee \cdots \vee \bigwedge_{x^\tau} d(x^\tau) \rightarrow a_1 \preccurlyeq b_1 \vee d(p^\tau).$$

If we choose for p^τ a variable that does not occur in a_1, $b_1 \vee \bigwedge_{x^\tau} d(x^\tau)$, then auxiliary rule 5 yields $a_1 \preccurlyeq b_1 \vee d(a^\tau)$ for each a^τ. We have to show now—under hypothesis of auxiliary rule 3 for $c = d(a^\tau)$ and the validity of $a_1 \preccurlyeq b_1 \vee d(a^\tau)$ for each a^τ:

$$a_2 \wedge \bigwedge_{x^\tau} d(x^\tau) \wedge \cdots \wedge \bigwedge_{x^\tau} d(x^\tau) \preccurlyeq b_2 \rightarrow a_1 \wedge a_2 \preccurlyeq b_1 \vee b_2.$$

We prove this claim by theorem induction.

2.3.1 Let $a_2 \wedge \bigwedge_{x^\tau} d(x^\tau) \wedge \cdots \preccurlyeq b_2$ be a numerical theorem. The claim is then trivially valid.

2.3.2 Let $a_2 \wedge \bigwedge_{x^\tau} d(x^\tau) \wedge \cdots \preccurlyeq b_2$ be conclusion of an inductive rule, and let the claim be valid for the premisses. The treatment of each rule is trivial, except the one case of rule [3d]:

$$a_2 \wedge d(a^\tau) \wedge \bigwedge_{x^\tau} d(x^\tau) \wedge \cdots \preccurlyeq b_2 \rightarrow a_2 \wedge \bigwedge_{x^\tau} d(x^\tau) \wedge \cdots \preccurlyeq b_2.$$

As the claim is valid for the premisses, $a_1 \wedge a_2 \wedge d(a^\tau) \preccurlyeq b_1 \vee b_2$ follows. Further holds according to hypothesis $a_1 \preccurlyeq b_1 \vee d(a^\tau)$ and

$$a_1 \preccurlyeq b_1 \vee d(a^\tau)\,,\, a_1 \wedge a_2 \wedge d(a^\tau) \preccurlyeq b_1 \vee b_2 \rightarrow a_1 \wedge a_1 \wedge a_2 \preccurlyeq b_1 \vee b_1 \vee b_2,$$

from which $a_1 \wedge a_2 \preccurlyeq b_1 \vee b_2$ follows.

Thereby auxiliary rule 3 is proved for $c = \bigwedge_{x^\tau} d(x^\tau)$.

The proof for $c = \bigvee_{x^\tau} d(x^\tau)$ proceeds dually to this; and auxiliary rule 3 is proved in general.

The further rules of the deductive calculus now make no difficulties anymore. The deductive rules (2b) and (2c) are contained in the inductive rules [2], [3a]–[3c]. Of the deductive rules (2d) two are contained in the inductive rules [3d], the other two in auxiliary rule 6 and the dual auxiliary rule.

Thereby all deductive axioms are recognised as inductive theorems, and all deductive rules as also valid in the inductive calculus.

Thus each proper deductive theorem is also an inductive theorem; in particular $\preccurlyeq$ is not a theorem in the proper deductive calculus, because $\preccurlyeq$ is not a theorem in the inductive calculus. Thereby the consistency of the deductive calculus is proved.

8. The independence of the axiom of reducibility. With the method of §7 one may also prove the consistency of other similar calculuses. The relation of identity may e.g. be added to the deductive calculus, and the axiom of extensionality replaced by the axioms of identity:

$(c_1) \preccurlyeq p^\tau = p^\tau$.

$(c_2)\ p^\tau = q^\tau \preccurlyeq c(p^\tau) \to c(q^\tau)$.

For the proof of consistency one has then to modify the inductive calculus in the following way. To the prime formulae one adds $p^\tau = q^\tau$, to the numerical formulae $a^\tau = b^\tau$. These numerical formulae are interpreted as "a^τ and b^τ are signs of the same form." The axiom (c_1) is then an inductive theorem, because $\preccurlyeq a^\tau = a^\tau$ is a numerical theorem for each constant a.

(c_2) follows as well, as for two constants a^τ, b^τ not of the same the form always holds $a^\tau = b^\tau \preccurlyeq$; but for constants a^τ, b^τ of the same form $c(a^\tau) \preccurlyeq c(b^\tau)$, thus $\preccurlyeq c(a^\tau) \to c(b^\tau)$.

According to [3a] follows in each case $a^\tau = b^\tau \preccurlyeq c(a^\tau) \to c(b^\tau)$.

The consistency of the deductive calculus is even conserved if one adds axioms that postulate in terms of content the equipotence of the set of individuals with the set of relations of type τ:

(e) For $\sigma = \nu + 1\,(\tau\ 0)$, if τ of order ν,

$$\preccurlyeq \bigvee_{x^\sigma}(\bigwedge_{x^\tau} \bigvee_{x^0} x^\sigma x^\tau x^0 \wedge \bigwedge_{z^0} \bigwedge_{x^\tau} \bigwedge_{y^\tau}((x^\sigma x^\tau z^0 \wedge x^\sigma y^\tau z^0) \to x^\tau = y^\tau)).$$

For the proof of consistency we modify the inductive calculus once more. To the prime formulae $p^\tau \vdash p^0$ is being added, to the numerical formulae $a^\tau \vdash a^0$. In order to interpret these numerical formulae, we carry out an enumeration of the constants of type τ. Such an enumeration is possible, as the set of all formulae is countable. We interpret $a^\tau \vdash a^0$ then as "a^τ has in the enumeration of the constants of type τ the number a^0." On the basis of this interpretation there is to each a^τ an a^0 with $a^\tau \vdash a^0$. Thus for each a^τ holds $\preccurlyeq \bigvee_{x^0} a^\tau \vdash x^0$; from which follows $\preccurlyeq \bigvee_{x^0} p^\tau \vdash x^0$ and $\preccurlyeq \bigwedge_{x^\tau} \bigvee_{x^0} x^\tau \vdash x^0$. For arbitrary constants a^τ, b^τ, and c^0 holds moreover $\preccurlyeq a^\tau = b^\tau$ or $a^\tau \vdash c^0 \preccurlyeq$ or $b^\tau \vdash c^0 \preccurlyeq$. In each case results according to [3a] $a^\tau \vdash c^0 \wedge b^\tau \vdash c^0 \preccurlyeq a^\tau = b^\tau$, from which follows

$$p^\tau \vdash r^0 \wedge q^\tau \vdash r^0 \preccurlyeq p^\tau = q^\tau, \qquad \preccurlyeq (p^\tau \vdash r^0 \wedge q^\tau \vdash r^0) \to p^\tau = q^\tau,$$
$$\preccurlyeq \bigwedge_{z^0} \bigwedge_{x^\tau} \bigwedge_{y^\tau}((x^\tau \vdash z^0 \wedge y^\tau \vdash z^0) \to x^\tau = y^\tau).$$

According to the rule of constants holds therefore for $a^\sigma = (u^\tau v^0)^{\nu+1}\, u^\tau \vdash v^0$,

$$\preccurlyeq \bigwedge_{x^\tau} \bigvee_{x^0} a^\sigma x^\tau x^0, \qquad \bigwedge_{z^0} \bigwedge_{x^\tau} \bigwedge_{y^\tau}((a^\sigma x^\tau z^0 \wedge a^\sigma y^\tau z^0) \to x^\tau = y^\tau).$$

If we bind these two formulae by $\wedge$, then a formula $d(a^\sigma)$ arises; and $\preccurlyeq d(a^\sigma)$ holds, from which $\preccurlyeq \bigvee_{x^\sigma} d(x^\sigma)$ follows. This is axiom (e).

We show in the end that in the modified deductive calculus thereby proven to be consistent, the axiom of reducibility is refutable. A simple case of this axiom is: for $\rho = \nu + 2\,(0)$, if $\tau = \nu\,(0)$, $\preccurlyeq \bigwedge_{x^\rho} \bigvee_{x^\tau} \bigwedge_{x^0} x^\rho x^0 \leftrightarrow x^\tau x^0$. First holds for $\sigma = \nu + 1\,(\tau\ 0)$, $d(p^\sigma) \preccurlyeq (p^\sigma p^\tau r^0 \wedge p^\sigma q^\tau r^0) \to p^\tau = q^\tau$ and $p^\tau = q^\tau \preccurlyeq \overline{q^\tau r^0} \to \overline{p^\tau r^0}$; from which follows $d(p^\sigma) \wedge p^\sigma p^\tau r^0 \wedge p^\sigma q^\tau r^0 \wedge \overline{q^\tau r^0} \preccurlyeq \overline{p^\tau r^0}$ and $d(p^\sigma) \wedge p^\sigma p^\tau r^0 \wedge \bigvee_{x^\tau}(p^\sigma x^\tau r^0 \wedge \overline{x^\tau r^0}) \preccurlyeq \overline{p^\tau r^0}$. Because of $p^\sigma p^\tau r^0 \wedge \overline{p^\tau r^0} \preccurlyeq \bigvee_{x^\tau}(p^\sigma x^\tau r^0 \wedge \overline{x^\tau r^0})$ results elementarily

$$d(p^\sigma) \wedge p^\sigma p^\tau r^0 \wedge (p^\tau r^0 \leftrightarrow \bigvee_{x^\tau}(p^\sigma x^\tau r^0 \wedge \overline{x^\tau r^0})) \preccurlyeq\ ,$$
$$d(p^\sigma) \wedge p^\sigma p^\tau r^0 \wedge (p^\rho r^0 \leftrightarrow \bigvee_{x^\tau}(p^\sigma x^\tau r^0 \wedge \overline{x^\tau r^0})) \wedge (p^\rho r^0 \leftrightarrow p^\tau r^0) \preccurlyeq\ .$$

Further follows now

$$d(p^\sigma) \wedge \bigvee_{x^0} p^\sigma p^\tau x^0 \wedge \bigwedge_{x^0}(p^\rho x^0 \leftrightarrow \bigvee_{x^\tau}(p^\sigma x^\tau x^0 \wedge \overline{x^\tau x^0})) \wedge \bigwedge_{x^0}(p^\rho x^0 \leftrightarrow p^\tau x^0) \preccurlyeq\ ;$$

thus because of $d(p^\sigma) \preccurlyeq \bigwedge_{x^\tau} \bigvee_{x^0} p^\sigma x^\tau x^0$,

$$d(p^\sigma) \wedge \bigwedge_{x^0}(p^\rho x^0 \leftrightarrow \bigvee_{x^\tau}(p^\sigma x^\tau x^0 \wedge \overline{x^\tau x^0})) \wedge \bigvee_{x^\tau} \bigwedge_{x^0}(p^\rho x^0 \leftrightarrow x^\tau x^0) \preccurlyeq\ ;$$

because of $\preccurlyeq \bigvee_{x^\rho} \bigwedge_{x^0}(x^\rho x^0 \leftrightarrow \bigvee_{x^\tau}(p^\sigma x^\tau x^0 \wedge \overline{x^\tau x^0}))$,

$$d(p^\sigma) \wedge \bigwedge_{x^\rho} \bigvee_{x^\tau} \bigwedge_{x^0}(x^\rho x^0 \leftrightarrow x^\tau x^0) \preccurlyeq\ ;$$

and because of $\preccurlyeq \bigvee_{x^\sigma} d(x^\sigma)$,

$$\bigwedge_{x^\rho} \bigvee_{x^\tau} \bigwedge_{x^0}(x^\rho x^0 \leftrightarrow x^\tau x^0) \preccurlyeq\ ,$$

q.e.d.

UNIVERSITY BONN