Metric unconditionality and Fourier analysis

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Résumé Nous étudions plusieurs propriétés fonctionnelles d'inconditionnalité isométrique et presqu'isométrique en les exprimant à l'aide de multiplicateurs. Parmi ceux-ci, la notion la plus générale est celle de "propriété d'approximation inconditionnelle métrique". Nous la caractérisons parmi les espaces de Banach de cotype fini par une propriété simple d'"inconditionnalité par blocs". En nous ramenant à des multiplicateurs de Fourier, nous étudions cette propriété dans les sous-espaces des espaces de Banach de fonctions sur le cercle qui sont engendrés par une suite de caractères e^{int}. Nous étudions aussi les suites basiques inconditionnelles isométriques et presqu'isométriques de caractères, en particulier les ensembles de Sidon de constante asymptotiquement 1. Nous obtenons dans chaque cas des propriétés combinatoires sur la suite. La propriété suivante des normes L^p est cruciale pour notre étude : si p est un entier pair, $\int |f|^p = \int |f^{p/2}|^2 = \sum |\widehat{f^{p/2}}(n)|^2$ est une expression polynomiale en les coefficients de Fourier de f et $\bar{f}$. Nous proposons d'ailleurs une estimation précise de la constante de Sidon des ensembles à la Hadamard.

Zusammenfassung Verschiedene funktionalanalytische isometrische und fast isometrische Unbedingtheitseigenschaften werden mittels Multiplikatoren untersucht. Am allgemeinsten ist die metrische unbedingte Approximationseigenschaft gefasst. Wir charakterisieren diese für Banachräume mit endlichem Kotyp durch eine einfache "blockweise" Unbedingtheit. Daraufhin betrachten wir genauer den Fall von Funktionenräumen auf dem Einheitskreis, die durch eine Folge von Frequenzen e^{int} aufgespannt werden. Wir untersuchen isometrisch und fast isometrisch unbedingte Basisfolgen von Frequenzen, unter anderem Sidonmengen mit einer Konstante asymptotisch zu 1. Für jeden Fall erhalten wir kombinatorische Eigenschaften der Folge. Die folgende Eigenschaft der L^p Normen ist entscheidend für diese Arbeit: Ist p eine gerade Zahl, so ist $\int |f|^p = \int |f^{p/2}|^2 = \sum |\widehat{f^{p/2}}(n)|^2$ ein polynomialer Ausdruck der Fourierkoeffizienten von f und $\bar{f}$. Des weiteren erhalten wir eine genaue Abschätzung der Sidonkonstante von Hadamardfolgen.

Abstract We study several functional properties of isometric and almost isometric unconditionality and state them as a property of families of multipliers. The most general such notion is that of "metric unconditional approximation property". We characterise this "(*umap*)" by a simple property of "block unconditionality" for spaces with nontrivial cotype. We focus on subspaces of Banach spaces of functions on the circle spanned by a sequence of characters e^{int}. There (*umap*) may be stated in terms of Fourier multipliers. We express (*umap*) as a simple combinatorial property of this sequence. We obtain a corresponding result for isometric and almost isometric basic sequences of characters. Our study uses the following crucial property of the L^p norm for even p: $\int |f|^p = \int |f^{p/2}|^2 = \sum |\widehat{f^{p/2}}(n)|^2$ is a polynomial expression in the Fourier coefficients of f and $\bar{f}$. As a byproduct, we get a sharp estimate of the Sidon constant of sets *à la* Hadamard.

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1 A general introduction in French

1.1 Position du problème

Ce travail se situe au croisement de l'analyse fonctionnelle et de l'analyse harmonique. Nous allons donner des éléments de réponse à la question générale suivante.
Question 1.1 Quelle est la validité de la représentation

$$f \sim \sum \varrho_q \,\mathrm{e}^{\mathrm{i}\vartheta_q}\,\mathrm{e}_q \tag{1}$$

de la fonction f comme série de fréquences e_q d'intensité ϱ_q et de phase ϑ_q ?
Les réponses seront donnés en termes de l'espace de fonctions $X \ni f$ et du spectre $E \supseteq \{q : \varrho_q > 0\}$.
Considérons par exemple les deux questions classiques suivantes dans le cadre des espaces de Banach homogènes de fonctions sur le tore $\mathbb{T}$, des fréquences de Fourier $\mathrm{e}_q(t) = \mathrm{e}^{\mathrm{i}qt}$ et des coefficients de Fourier

$$\varrho_q \,\mathrm{e}^{\mathrm{i}\vartheta_q} = \int \mathrm{e}_{-q}\, f = \widehat{f}(q).$$

Question 1.1.1 Est-ce que pour les fonctions $f \in X$ à spectre dans E

$$\Big\| f - \sum_{|q|\le n} \varrho_q \,\mathrm{e}^{\mathrm{i}\vartheta_q}\,\mathrm{e}_q \Big\|_X \xrightarrow[n\to\infty]{} 0\,?$$

Cela revient à demander : est-ce que la suite $\{\mathrm{e}_q\}_{q\in E}$ rangée par valeur absolue $|q|$ croissante est une base de X_E ? En d'autres termes, la suite des multiplicateurs idempotents relatifs $T_n : X_E \to X_E$ définie par

$$T_n\,\mathrm{e}_q = \begin{cases} \mathrm{e}_q & \text{si } |q| \le n \\ 0 & \text{sinon} \end{cases}$$

est-elle uniformément bornée sur n ? Soit $E = \mathbb{Z}$. Un élément de réponse classique est le suivant.

$$\|T_n\|_{\mathrm{L}^2(\mathbb{T})\to\mathrm{L}^2(\mathbb{T})} = 1\ ,\ \ \|T_n\|_{\mathrm{L}^1(\mathbb{T})\to\mathrm{L}^1(\mathbb{T})} = \|T_n\|_{\mathscr{C}(\mathbb{T})\to\mathscr{C}(\mathbb{T})} \asymp \log n.$$

On sait de plus que les T_n sont aussi uniformément bornés sur $\mathrm{L}^p(\mathbb{T})$, $1 < p < \infty$.
Question 1.1.2 Est-ce que la somme de la série $\sum \varrho_q \,\mathrm{e}^{\mathrm{i}\vartheta_q}\,\mathrm{e}_q$ dépend de l'ordre dans lequel on somme les fréquences ? Cette question est équivalente à la suivante : la nature de $\sum \varrho_q \,\mathrm{e}^{\mathrm{i}\vartheta_q}\,\mathrm{e}_q$ dépend-elle des phases ϑ_q ? En termes fonctionnels, $\{\mathrm{e}_q\}_{q\in E}$ forme-t-elle une suite basique inconditionnelle dans X ? Cette question s'énonce aussi en termes de multiplicateurs relatifs : la famille des $T_\epsilon : X_E \to X_E$ avec

$$T_\epsilon\,\mathrm{e}_q = \epsilon_q\,\mathrm{e}_q \ \text{ et } \epsilon_q = \pm 1$$

est-elle uniformément bornée sur les choix de signes ϵ ? Un élément de réponse classique est le suivant. Soit $E = \mathbb{Z}$. Alors

$$\|T_\epsilon\|_{\mathrm{L}^2(\mathbb{T})\to\mathrm{L}^2(\mathbb{T})} = 1;$$

si $p \neq 2$, il existe un choix de signes ϵ tel que T_ϵ n'est pas borné sur $\mathrm{L}^p(\mathbb{T})$.
Question 1.1.3 Peut-on améliorer ce phénomène en restreignant le spectre E ? Cette question mène à l'étude des sous-ensembles lacunaires de $\mathbb{Z}$, et a été traitée en détail par Walter Rudin.
Nous choisissons la notion de multiplicateur relatif comme dictionnaire entre l'analyse harmonique et l'analyse fonctionnelle. Nous développons une technique pour le calcul de la norme de familles $\{T_\epsilon\}$ de multiplicateurs relatifs. Celle-ci nous permet de traiter les questions suivantes.
Question 1.1.4 Est-ce que la norme de $f \in X_E$ dépend seulement de l'intensité ϱ_q de ses fréquences e_q, et non pas de leur phase ϑ_q ? Cela revient à demander si $\{\mathrm{e}_q\}_{q\in E}$ est une suite basique 1-inconditionnelle complexe dans X.
Question 1.1.5 Est-ce que l'on a pour tout choix de signes "réel" $\pm$

$$\Big\|\sum_{q\in E} \pm a_q\,\mathrm{e}_q\Big\|_X = \Big\|\sum_{q\in E} a_q\,\mathrm{e}_q\Big\|_X\,?$$

En d'autres mots, est-ce que $\{\mathrm{e}_q\}_{q\in E}$ est une suite basique 1-inconditionnelle réelle dans X ?
La réponse est décevante dans le cas des espaces $\mathrm{L}^p(\mathbb{T})$, p non entier pair : seules les fonctions dont le spectre a au plus deux éléments vérifient ces deux propriétés. Pour mieux cerner le phénomène, nous proposons d'introduire la question presqu'isométrique suivante.
Question 1.1.6 Est-ce que la norme de $f \in X_E$ dépend arbitrairement peu de la phase ϑ_q de ses fréquences e_q ? De manière précise, dans quel cas existe-t-il, pour chaque $\varepsilon > 0$, un sous-ensemble $F \subseteq E$ fini tel que

$$\Big\| \sum_{q\in E\setminus F} \varrho_q \,\mathrm{e}^{\mathrm{i}\vartheta_q}\, \mathrm{e}_q \Big\|_X \le (1+\varepsilon) \Big\| \sum_{q\in E\setminus F} \varrho_q \,\mathrm{e}_q \Big\|_X \text{ ?}$$

Dans le cas $X = \mathscr{C}(\mathbb{T})$, cela signifiera que E est un ensemble de constante de Sidon "asymptotiquement 1". De même, peut-on choisir pour chaque $\varepsilon > 0$ un ensemble fini F tel que pour tout choix de signe "réel" $\pm$

$$\Big\| \sum_{q\in E\setminus F} \pm a_q \,\mathrm{e}_q \Big\|_X \le (1+\varepsilon) \Big\| \sum_{q\in E\setminus F} a_q \,\mathrm{e}_q \Big\|_X \text{ ?}$$

Toutes ces questions s'agrègent autour d'un fait bien connu : sommer la série de Fourier de f est une très mauvaise manière d'approcher la fonction f dès que l'erreur considérée n'est pas quadratique. On sait qu'il est alors utile de rechercher des méthodes de sommation plus lisses, c'est-à-dire d'autres suites approximantes plus régulières. Il s'agit là de suites d'opérateurs de rang fini sur X_E qui approchent ponctuellement l'identité de X_E. Nous pourrons toujours supposer que ces opérateurs sont des multiplicateurs. Une première question est la suivante.
Question 1.1.7 Existe-t-il une suite approximante $\{T_n\}$ de multiplicateurs idempotents ? Cela revient à demander : existe-t-il une décomposition de X_E en sous-espaces X_{E_k} de dimension finie avec

$$X_E = \bigoplus X_{E_k} \quad \text{et} \quad A_k : X_E \to X_{E_k} \ , \ \mathrm{e}_q \mapsto \begin{cases} \mathrm{e}_q & \text{si } q \in E_k \\ 0 & \text{sinon} \end{cases} \tag{2}$$

telle que la suite des $T_n = A_1 + \cdots + A_n$ est uniformément bornée sur n ? Soit $E = \mathbb{Z}$. Alors la réponse est identique à la réponse de la question 1.1.1.
Mais nous pouvons produire dans ce cadre plus général des décompositions inconditionnelles de X_E en réponse à la question suivante.
Question 1.1.8 Pour quels espaces X et spectres E existe-t-il une décomposition comme ci-dessus telle que la famille des multiplicateurs

$$\sum_{k=1}^{n} \epsilon_k A_k \quad \text{avec } n \ge 1 \text{ et } \epsilon_k = \pm 1 \tag{3}$$

est uniformément bornée ? Littlewood et Paley ont montré que la partition de $\mathbb{Z}$ en $\mathbb{Z} = \bigcup E_k$ avec $E_0 = \{0\}$ et $E_k = \{j : 2^{k-1} \le |j| < 2^k\}$ donne une décomposition inconditionnelle des espaces $\mathrm{L}^p(\mathbb{T})$ avec $1 < p < \infty$. D'après la réponse à la question 1.1.7, ce n'est pas le cas *a fortiori* des espaces $\mathrm{L}^1(\mathbb{T})$ et $\mathscr{C}(\mathbb{T})$. Une étude fine de telles partitions a été entreprise par Kathryn Hare et Ivo Klemes.
Notre technique permet de traiter la question suivante.
Question 1.1.9 Pour quels espaces X et spectres E existe-t-il une décomposition du type (2) telle que

$$\Big\| \sum \epsilon_k A_k f \Big\|_X = \|f\|_X \text{ pour tout choix de signes } \epsilon_k \text{ ?}$$

La réponse dépendra de la nature du choix de signes, qui peut être réel ou complexe.
Il est instructif de noter que l'espace de Hardy $H^1(\mathbb{T})$ n'admet pas de décomposition du type (2). $H^1(\mathbb{T})$ admet néanmoins des suites approximantes de multiplicateurs et il existe même des suites approximantes de multiplicateurs inconditionnelles au sens où la famille (3) est uniformément bornée. Cela motive la question suivante, qui est la plus générale dans notre contexte.
Question 1.1.10 Quels sont les espaces X et spectres E tels que pour chaque $\varepsilon > 0$ il existe une suite approximante $\{T_n\}$ sur X_E telle que

$$\sup_{\text{signes } \epsilon_n} \Big\| \sum \epsilon_n (T_n - T_{n-1}) \Big\|_X \le 1 + \varepsilon$$

En termes fonctionnels, X_E a-t-il la propriété d'approximation inconditionnelle métrique ? Il faudra distinguer le cas des signes complexes et réels.

1.2 Propriété d'approximation inconditionnelle métrique

Comme nos questions distinguent les choix de signe réel et complexe, nous proposons pour la fluidité de l'exposé de fixer un choix de signes $\mathbb{S}$ qui sera $\mathbb{S} = \mathbb{T} = \{\epsilon \in \mathbb{C} : |\epsilon| = 1\}$ dans le cas complexe et $\mathbb{S} = \mathbb{D} = \{-1, 1\}$ dans le cas réel.
Seule la question 1.1.10 n'impose pas au préalable de forme particulière à la suite de multiplicateurs qui est censée réaliser la propriété considérée. Afin d'établir un lien entre la (*umap*) et la structure du spectre E, nous faisons le détour par une étude générale de cette propriété dans le cadre des espaces de Banach séparables.

1.2.1 Amorce et queue d'un espace de Banach

Peter G. Casazza et Nigel J. Kalton ont découvert le critère suivant :

Proposition 1.2.1 *Soit X un espace de Banach séparable. X a la (umap) si et seulement s'il existe une suite approximante $\{T_k\}$ telle que*

$$\sup_{\epsilon\in\mathbb{S}} \|T_k + \epsilon(\mathrm{Id} - T_k)\|_{\mathscr{L}(X)} \xrightarrow[k\to\infty]{} 1.$$

Ceci exprime que la constante d'inconditionnalité entre l'amorce T_kX et la queue $(\mathrm{Id} - T_k)X$ de l'espace X s'améliore asymptotiquement jusqu'à l'optimum pour $k \to \infty$.
La (*umap*) s'exprime de manière plus élémentaire encore si l'on choisit d'autres notions adaptées d'amorce et de queue. Nous proposons en particulier la définition suivante.

Définition 1.2.2 *Soit τ une topologie d'espace vectoriel topologique sur X. X a la propriété $(u(\tau))$ de τ-inconditionnalité si pour chaque $x \in X$ et toute suite bornée $\{y_j\}$ τ-nulle l'oscillation*

$$\operatorname*{osc}_{\epsilon\in\mathbb{S}} \|x + \epsilon y_j\|_X = \sup_{\delta,\epsilon\in\mathbb{S}} \big(\|x + \epsilon y_j\| - \|x + \delta y_j\|\big)$$

forme elle-même une suite nulle.

Nous avons alors le théorème suivant.

Théorème 1.2.3 *Soit X un espace de Banach séparable de cotype fini avec la propriété $(u(\tau))$. Si X admet une suite approximante $\{T_k\}$ inconditionnelle et commutative telle que $T_kx \xrightarrow{\tau} x$ uniformément sur la boule unité B_X, alors des combinaisons convexes successives $\{U_j\}$ de $\{T_k\}$ réalisent la (umap).*

Esquisse de preuve. On construit ces combinaisons convexes successives par le biais de décompositions skipped blocking. En effet, la propriété $(u(\tau))$ a l'effet suivant sur $\{T_k\}$. Pour chaque $\varepsilon > 0$, il existe une sous-suite $\{S_k = T_{n_k}\}$ telle que toute suite de blocs $S_{b_k} - S_{a_k}$ obtenue en sautant les blocs $S_{a_{k+1}} - S_{b_k}$ se somme de manière $(1 + \varepsilon)$-inconditionnelle.
Soit $n \geq 1$. Pour chaque j, $1 \leq j \leq n$, la suite de blocs obtenue en sautant $S_{kn+j} - S_{kn+j-1}$ pour $k \geq 0$ est $(1 + \varepsilon)$-inconditionnelle. Il s'agit alors d'estimer la moyenne sur j de ces suites de blocs. On obtient une suite approximante et l'hypothèse de cotype fini permet de contrôler l'apport des blocs sautés.
Alors X a la (*umap*) parce que n et ε sont arbitraires. ■

1.2.2 Amorce et queue en termes de spectre de Fourier

Lorsqu'on considère l'espace invariant par translation X_E, une amorce et une queue naturelle sont les espaces X_F et $X_{E\setminus G}$ pour F et G des sous-ensembles finis de E. Nous avons concrètement le lemme suivant.

Lemme 1.2.4 *X_E a $(u(\tau_f))$, où τ_f est la topologie*

$$f_n \xrightarrow{\tau_f} 0 \iff \forall k\ \widehat{f_n}(k) \to 0$$

de convergence simple des coefficients de Fourier, si et seulement si E est bloc-inconditionnel dans X au sens suivant : quels que soient $\varepsilon > 0$ et $F \subseteq E$ fini, il existe $G \subseteq E$ fini tel que pour $f \in B_{X_F}$ et $g \in B_{X_{E\setminus G}}$

$$\operatorname*{osc}_{\epsilon\in\mathbb{S}} \|f + \epsilon g\|_X = \sup_{\delta,\epsilon\in\mathbb{S}} \big(\|f + \epsilon g\| - \|f + \delta g\|\big) \leq \varepsilon.$$

Le théorème 1.2.3 s'énonce donc ainsi dans ce contexte particulier.

Théorème 1.2.5 *Soit $E \subseteq \mathbb{Z}$ et X un espace de Banach homogène de fonctions sur le tore $\mathbb{T}$. Si X_E a la (umap), alors E est bloc-inconditionnel dans X. Inversement, si E est bloc-inconditionnel dans X et de plus X_E a la propriété d'approximation inconditionnelle et un cotype fini, alors X_E a la (umap). En particulier, on a*
(i) Soit $1 < p < \infty$. $\mathrm{L}^p_E(\mathbb{T})$ a la (umap) si et seulement si E est bloc-inconditionnel dans $\mathrm{L}^p(\mathbb{T})$.
(ii) $\mathrm{L}^1_E(\mathbb{T})$ a la (umap) si et seulement si $\mathrm{L}^1_E(\mathbb{T})$ a la propriété d'approximation inconditionnelle et E est bloc-inconditionnel dans $\mathrm{L}^1(\mathbb{T})$.
(iii) Si E est bloc-inconditionnel dans $\mathscr{C}(\mathbb{T})$ et E est un ensemble de Sidon, alors $\mathscr{C}_E(\mathbb{T})$ a la (umap).

Donnons une application de ce théorème.

Proposition 1.2.6 *Soit $E = \{n_k\} \subseteq \mathbb{Z}$. Si n_{k+1}/n_k est un entier impair pour tout k, alors $\mathscr{C}_E(\mathbb{T})$ a la (umap) réelle.*

Preuve. Comme E est nécessairement un ensemble de Sidon, il suffit de vérifier que E est bloc-inconditionnel. Soient $\varepsilon > 0$ et $F \subseteq E \cap [-n, n]$. Soit l tel que $|n_l| \geq \pi n/\varepsilon$ et $G = \{n_1, \dots, n_{l-1}\}$. Soit $f \in B_{\mathscr{C}_F}$ et $g \in B_{\mathscr{C}_{E \setminus G}}$. Alors $g(t + \pi/n_l) = -g(t)$ par hypothèse et

$$|f(t + \pi/n_l) - f(t)| \leq \pi/|n_l| \cdot \|f'\|_\infty \leq \pi n/|n_l| \leq \varepsilon$$

par l'inégalité de Bernstein. Alors, pour un certain $u \in \mathbb{T}$

$$\begin{aligned} \|f - g\|_\infty &= |f(u) + g(u + \pi/n_l)| \\ &\leq |f(u + \pi/n_l) + g(u + \pi/n_l)| + \varepsilon \\ &\leq \|f + g\|_\infty + \varepsilon. \end{aligned}$$

Donc E est bloc-inconditionnel au sens réel. ■

En particulier, soit la suite géométrique $G = \{3^k\}$. Alors $\mathscr{C}_G(\mathbb{T})$ et $\mathscr{C}_{G \cup -G}(\mathbb{T})$ ont la (*umap*) réelle.
Question 1.2.7 Qu'en est-il de la (*umap*) complexe et qu'en est-il de la suite géométrique $G = \{2^k\}$?

1.3 Norme de multiplicateurs et conditions combinatoires

Nous proposons ici une méthode uniforme pour répondre aux questions 1.1.4, 1.1.5, 1.1.6, 1.1.9 et 1.1.10. En effet, les questions 1.1.4, 1.1.5 et 1.1.6 reviennent à évaluer l'oscillation de la norme

$$\Theta(\epsilon, a) = \|\epsilon_0 a_0 \, \mathrm{e}_{r_0} + \dots + \epsilon_m a_m \, \mathrm{e}_{r_m}\|_X.$$

La question 1.1.9 revient à évaluer l'oscillation de la norme

$$\begin{aligned} \Psi(\epsilon, a) &= \Theta((\overbrace{1, \dots, 1}^{j}, \overbrace{\epsilon, \dots, \epsilon}^{m-j}), a) \\ &= \|a_0 \, \mathrm{e}_{r_0} + \dots + a_j \, \mathrm{e}_{r_j} + \epsilon a_{j+1} \, \mathrm{e}_{r_{j+1}} + \dots + \epsilon a_m \, \mathrm{e}_{r_m}\|_X \end{aligned}$$

Par le théorème 1.2.5, la question 1.1.10 revient à étudier cette même expression dans le cas particulier où on fait un saut de grandeur arbitraire entre r_j et r_{j+1}.
Dans le cas des espaces $X = \mathrm{L}^p(\mathbb{T})$, p entier pair, ces normes sont des polynômes en ϵ, ϵ^{-1}, a et $\bar{a}$. Dans le cas des espaces $X = \mathrm{L}^p(\mathbb{T})$, p non entier pair, elles s'expriment comme des séries. Il n'y a pas moyen d'exprimer ces normes comme fonction $\mathscr{C}^\infty$ pour $X = \mathscr{C}(\mathbb{T})$.
Soit $X = \mathrm{L}^p(\mathbb{T})$. Développons $\Theta(\epsilon, a)$. Posons $q_i = r_i - r_0$. On peut supposer $\epsilon_0 = 1$ et $a_0 = 1$. Nous utilisons la notation suivante :

$$\binom{x}{\alpha} = \frac{x(x-1)\cdots(x-n+1)}{\alpha_1!\alpha_2!\dots} \quad \text{pour } \alpha \in \mathbb{N}^m \text{ tel que } \sum \alpha_i = n$$

Alors, si $|a_1|, \dots, |a_m| < 1/m$ lorsque p n'est pas un entier pair et sans restriction sinon,

$$\begin{aligned}
\Theta(\epsilon, a) &= \int \Big| \sum_{n\geq 0} \binom{p/2}{n} \Big(\sum_{i=1}^{m} \epsilon_i a_i \,\mathrm{e}_{q_i} \Big)^n \Big|^2 \\
&= \int \Big| \sum_{n\geq 0} \binom{p/2}{n} \sum_{\substack{\alpha:\alpha_1,\dots,\alpha_m\geq 0 \\ \alpha_1+\dots+\alpha_m=n}} \binom{n}{\alpha} \epsilon^\alpha a^\alpha \,\mathrm{e}_{\sum \alpha_i q_i} \Big|^2 \\
&= \int \Big| \sum_{\alpha\in\mathbb{N}^m} \binom{p/2}{\alpha} \epsilon^\alpha a^\alpha \,\mathrm{e}_{\sum \alpha_i q_i} \Big|^2 \\
&= \sum_{R\in\mathscr{R}} \Big| \sum_{\alpha\in R} \binom{p/2}{\alpha} \epsilon^\alpha a^\alpha \Big|^2 \\
&= \sum_{\alpha\in\mathbb{N}^m} \binom{p/2}{\alpha}^2 |a|^{2\alpha} + \sum_{\substack{\alpha\neq\beta\in\mathbb{N}^m \\ \alpha\sim\beta}} \binom{p/2}{\alpha} \binom{p/2}{\beta} \epsilon^{\alpha-\beta} a^\alpha \bar{a}^\beta \qquad (4)
\end{aligned}$$

où $\mathscr{R}$ est la partition de $\mathbb{N}^m$ induite par la relation d'équivalence

$$\alpha \sim \beta \Leftrightarrow \sum \alpha_i q_i = \sum \beta_i q_i.$$

Nous pouvons répondre immédiatement aux questions 1.1.4 et 1.1.5 pour $X = \mathrm{L}^p(\mathbb{T})$.

1.3.1 Question 1.1.4 : suites basiques 1-inconditionnelles complexes

Soient $r_0, \dots r_m$ sont choisis dans E, alors (4) doit être constante pour $a \in \{|z| < 1/m\}^m$ et $\epsilon \in \mathbb{T}^m$. Cela veut dire que pour tous $\alpha \neq \beta \in \mathbb{N}^m$,

$$\sum \alpha_i q_i \neq \sum \beta_i q_i \quad \text{ou} \quad \binom{p/2}{\alpha}\binom{p/2}{\beta} = 0.$$

■ Si p n'est pas un entier pair, alors $\binom{p/2}{\alpha}\binom{p/2}{\beta} \neq 0$ pour tous $\alpha, \beta \in \mathbb{N}^m$ et on a les relations arithmétiques suivantes sur $q_1, q_2, 0$:

$$\begin{aligned}
\overbrace{q_1 + \dots + q_1}^{|q_2|} &= \overbrace{q_2 + \dots + q_2}^{|q_1|} && \text{si } q_1 q_2 > 0; \\
\overbrace{q_1 + \dots + q_1}^{|q_2|} + \overbrace{q_2 + \dots + q_2}^{|q_1|} &= 0 && \text{sinon.}
\end{aligned}$$

Il suffit donc de prendre respectivement

$$(\alpha, \beta) = \big((|q_2|, 0, \dots), (|q_1|, 0, \dots) \big)$$

et

$$(\alpha, \beta) = \big((|q_2|, |q_1|, 0, \dots), (0, \dots) \big)$$

pour conclure que $\{r_0, r_1, r_2\}$ n'est pas une suite basique 1-inconditionnelle complexe dans $\mathrm{L}^p(\mathbb{T})$ si p n'est pas un entier pair.

■ Si p est un entier pair, $\binom{p/2}{\alpha}\binom{p/2}{\beta} = 0$ si et seulement si

$$\sum \alpha_i > p/2 \quad \text{ou} \quad \sum \beta_i > p/2.$$

On obtient que E est une suite basique 1-inconditionnelle dans $\mathrm{L}^p(\mathbb{T})$ si et seulement si E est "p-indépendant", c'est-à-dire que $\sum \alpha_i (r_i - r_0) \neq \sum \beta_i (r_i - r_0)$ pour tous $r_0, \dots, r_m \in E$ et $\alpha \neq \beta \in \mathbb{N}^m$ tels que $\sum \alpha_i, \sum \beta_i \leq p/2$. Cette condition est équivalente à : tout entier $n \in \mathbb{Z}$ s'écrit de manière au plus unique comme somme de $p/2$ éléments de E.

1.3.2 Question 1.1.5 : suites basiques 1-inconditionnelles réelles

Les suites basiques 1-inconditionnelles réelles et complexes coïncident et la réponse à la question 1.1.5 est identique à la réponse à la question 1.1.4. En effet, dès qu'une relation arithmétique $\sum(\alpha_i - \beta_i)q_i$ pèse sur E, on peut supposer que $\alpha_i - \beta_i$ est impair pour au moins un i en simplifiant la relation par le plus grand diviseur commun des $\alpha_i - \beta_i$. Mais alors (4) n'est pas une fonction constante pour ϵ_i réel. Cette propriété est propre au tore $\mathbb{T}$. En effet, par exemple la suite des fonctions de Rademacher est 1-inconditionnelle réelle dans $\mathscr{C}(\mathbb{D}^\infty)$, alors que sa constante d'inconditionnalité complexe est $\pi/2$.

1.3.3 Question 1.1.6 : suites basiques inconditionnelles métriques

On peut même tirer des conséquences utiles du calcul de (4) dans le cas presqu'isométrique. Il faut pour cela prendre la précaution suivante qui permet un passage à la limite. Soit $0 < \varrho < 1/m$. Alors

$$\bigl\{\Theta\colon \mathbb{S}^m \times \{|z| \le \varrho\}^m \to \mathbb{R}^+ : q_1, \dots, q_m \in \mathbb{Z}^m\bigr\}$$

est un sous-ensemble relativement compact de $\mathscr{C}^\infty(\mathbb{S}^m \times \{|z| \le \varrho\}^m)$. Il en découle que si E est une suite basique inconditionnelle métrique, alors certains coefficients de (4) deviennent arbitrairement petits lorsque $q_1, \dots, q_m$ sont choisis grands.

Donnons deux conséquences de ce raisonnement.

Proposition 1.3.1 *Soit $E \subseteq \mathbb{Z}$.*

(i) Soit p un entier pair. Si E est une suite basique inconditionnelle métrique réelle, alors E est en fait une suite basique 1-inconditionnelle complexe à un ensemble fini près.

(ii) Si E est un ensemble de Sidon de constante asymptotiquement 1, alors

$$\langle\zeta, E\rangle = \sup_{G \subseteq E \text{ fini}} \inf\bigl\{|\zeta_1 p_1 + \dots + \zeta_m p_m| : p_1, \dots, p_m \in E \setminus G \text{ distincts}\bigr\} > 0$$

pour tout $m \ge 1$ et $\zeta \in \mathbb{Z}^{*m}$.

On peut exprimer cette dernière propriété en disant que la relation arithmétique ζ ne persiste pas sur E.

1.3.4 Question 1.1.10 : propriété d'approximation inconditionnelle métrique

On peut appliquer la technique du paragraphe précédent en observant que si X_E a la (*umap*), alors

$$\operatorname*{osc}_{\epsilon \in \mathbb{S}} \Psi(\epsilon, a) \xrightarrow[r_{j+1}, \dots, r_m \in E \to \infty]{} 0.$$

Définition 1.3.2 *E a la propriété $(\mathscr{J}_n)$ de bloc-indépendance si pour tout $F \subseteq E$ fini il existe $G \subseteq E$ fini tel que si un $k \in \mathbb{Z}$ admet deux représentations comme somme de n éléments de $F \cup (E \setminus G)$*

$$p_1 + \dots + p_n = k = p'_1 + \dots + p'_n,$$

alors

$$\bigl|\{j : p_j \in F\}\bigr| \quad \text{et} \quad \bigl|\{j : p'_j \in F\}\bigr|$$

sont égaux (choix de signes complexe $\mathbb{S} = \mathbb{T}$) ou de même parité (choix de signes réel $\mathbb{S} = \mathbb{D}$).

Théorème 1.3.3 *Soit $E \subseteq \mathbb{Z}$.*

(i) Si $X = \mathrm{L}^p(\mathbb{T})$, p entier pair, alors $\mathrm{L}^p_E(\mathbb{T})$ a la (umap) si et seulement si E satisfait $(\mathscr{J}_{p/2})$.

(ii) Si $X = \mathrm{L}^p(\mathbb{T})$, p non entier pair, ou $X = \mathscr{C}(\mathbb{T})$, alors X_E a la (umap) seulement si E satisfait

$$\langle\zeta, E\rangle = \sup_{G \subseteq E \text{ fini}} \inf\bigl\{|\zeta_1 p_1 + \dots + \zeta_m p_m| : p_1, \dots, p_m \in E \setminus G \text{ distincts}\bigr\} > 0$$

pour tout $m \ge 1$ et $\zeta \in \mathbb{Z}^{*m}$ tel que $\sum \zeta_i$ est non nul (cas complexe) ou impair (cas réel).

On obtient la hiérarchie suivante.

$\mathscr{C}_E(\mathbb{T})$ a (*umap*) $\Rightarrow$ $\mathrm{L}^p_E(\mathbb{T})$ a (*umap*), p non entier pair $\Rightarrow \dots \Rightarrow$ $\mathrm{L}^{2n+2}_E(\mathbb{T})$ a (*umap*) $\Rightarrow$ $\mathrm{L}^{2n}_E(\mathbb{T})$ a (*umap*) $\Rightarrow \dots \Rightarrow$ $\mathrm{L}^2_E(\mathbb{T})$ a (*umap*).

Nous pouvons répondre à la question 1.2.7. Soit $G = \{j^k\}$ avec $j \in \mathbb{Z} \setminus \{-1, 0, 1\}$ et considérons $\zeta = (j, -1)$. Alors $\langle\zeta, G\rangle = 0$. Donc $\mathscr{C}_G(\mathbb{T})$ n'a pas la (*umap*) complexe. $\mathscr{C}_G(\mathbb{T})$ n'a pas la (*umap*) réelle si j est pair.

1.3.5 Deux exemples

À l'aide de nos conditions arithmétiques, nous sommes à même de prouver la proposition suivante.

Proposition 1.3.4 *Soit $\sigma > 1$ et E la suite des parties entières de σ^k. Alors les assertions suivantes sont équivalentes.*
(i) σ est un nombre transcendant.
(ii) $\mathrm{L}^p_E(\mathbb{T})$ a la (umap) complexe pour tout p entier pair.
(iii) E est une suite basique inconditionnelle métrique dans chaque $\mathrm{L}^p(\mathbb{T})$, p entier pair.
(iv) Pour chaque m donné, la constante de Sidon des sous-ensembles à m éléments de queues de E est asymptotiquement 1.

Nous obtenons aussi la proposition suivante.

Proposition 1.3.5 *Soit E la suite des bicarrés. $\mathrm{L}^p_E(\mathbb{T})$ a la (umap) réelle seulement si $p = 2$ ou $p = 4$.*

Preuve. E ne satisfait pas la propriété de bloc-indépendance $(\mathscr{I}_3)$ réelle. En effet, Ramanujan a découvert l'égalité suivante pour tout n :

$$(4n^5 - 5n)^4 + (6n^4 - 3)^4 + (4n^4 + 1)^4 = (4n^5 + n)^4 + (2n^4 - 1)^4 + 3^4. \qquad \blacksquare$$

1.4 Impact de la croissance du spectre

Nous démontrons de manière directe le résultat positif suivant.

Théorème 1.4.1 *Soit $E = \{n_k\} \subseteq \mathbb{Z}$ tel que $n_{k+1}/n_k \to \infty$. Alors la suite des projections associée à E réalise la (umap) complexe dans $\mathscr{C}_E(\mathbb{T})$ et E est un ensemble de Sidon de constante asymptotiquement 1. Dans l'hypothèse où les rapports n_{k+1}/n_k sont tous entiers, la réciproque vaut.*

Corollaire 1.4.2 *Alors E est une suite basique inconditionnelle métrique dans tout espace de Banach homogène X de fonctions sur $\mathbb{T}$. De plus, X_E a la (umap) complexe.*

Esquisse de preuve. Nous prouvons concrètement que si $n_{k+1}/n_k \to \infty$, alors quel que soit $\varepsilon > 0$ il existe $l \geq 1$ tel que pour toute fonction $f = \sum a_k \, \mathrm{e}_{n_k}$

$$\|f\|_\infty \geq (1-\varepsilon)\Big(\Big\|\sum_{k \leq l} a_k \, \mathrm{e}_{n_k}\Big\|_\infty + \sum_{k > l} |a_k|\Big). \tag{5}$$

Cela revient à dire que la suite $\{\pi_k\}$ de projections associée à la base E réalise la $1/(1-\varepsilon)$-(*uap*). Pour obtenir l'inégalité (5), on utilise une récurrence basée sur l'idée suivante.
Soit $u \in \mathbb{T}$ tel que $\|\pi_k f\|_\infty = |\pi_k f(u)|$. Il existe alors $v \in \mathbb{T}$ tel que

$$|u - v| \leq \pi/|n_{k+1}| \quad \text{et} \quad |\pi_k f(u) + a_{k+1} \, \mathrm{e}_{n_{k+1}}(v)| = \|\pi_k f\|_\infty + |a_{k+1}|.$$

De plus, dans ce cas,

$$|\pi_k f(u) - \pi_k f(v)| \leq |u - v| \, \|\pi_k f'\|_\infty \leq \pi |n_k/n_{k+1}| \, \|\pi_k f\|_\infty.$$

En résumé, $a_{k+1} \, \mathrm{e}_{n_{k+1}}$ a le même argument que $\pi_k f$ très près du maximum de $|\pi_k f|$, et $\pi_k f$ varie peu. Mais alors

$$\begin{aligned} \|\pi_k f(t) + a_{k+1} \, \mathrm{e}_{n_{k+1}}\|_\infty &\geq |\pi_k f(v) + a_{k+1} \, \mathrm{e}_{n_{k+1}}(v)| \\ &\geq \|\pi_k f\|_\infty + |a_{k+1}| - \pi |n_k/n_{k+1}| \|\pi_k f\|_\infty \\ &= (1 - \pi |n_k/n_{k+1}|)\|\pi_k f\|_\infty + |a_{k+1}|. \end{aligned}$$

On obtient (5) en réitérant cet argument. $\blacksquare$
Notre technique donne d'ailleurs l'estimation suivante de la constante de Sidon des ensembles de Hadamard.

Corollaire 1.4.3 *Soit $E = \{n_k\} \subseteq \mathbb{Z}$ et $q > \sqrt{\pi^2/2 + 1}$. Si $|n_{k+1}| \geq q|n_k|$, alors la constante de Sidon de E est inférieure ou égale à $1 + \pi^2/(2q^2 - 2 - \pi^2)$.*

Nous prouvons que cette estimation est optimale au sens où l'ensemble $E = \{0, 1, q\}$, $q \geq 2$, a pour constante d'inconditionnalité réelle dans $\mathscr{C}(\mathbb{T})$

$$\big(\cos(\pi/(2q))\big)^{-1} \geq 1 + \pi^2/8 \, q^{-2}.$$

2 Introduction

We study isometric and almost isometric counterparts to the following two properties of a separable Banach space Y:

(ubs) Y is the closed span of an unconditional basic sequence;

(uap) Y admits an unconditional finite dimensional expansion of the identity.

We focus on the case of translation invariant spaces of functions on the torus group $\mathbb{T}$, which will provide us with a bunch of natural examples. Namely, let E be a subset of $\mathbb{Z}$ and X be one of the spaces $\mathrm{L}^p(\mathbb{T})$ $(1 \le p < \infty)$ or $\mathscr{C}(\mathbb{T})$. If $\{\mathrm{e}^{\mathrm{i}nt}\}_{n\in E}$ is an unconditional basic sequence ((*ubs*) for short) in X, then E is known to satisfy strong conditions of lacunarity: E must be in Rudin's class $\Lambda(q)$, $q = p \vee 2$, and a Sidon set respectively. We raise the following question: what kind of lacunarity is needed to get the following stronger property:

(umbs) E is a metric unconditional basic sequence in X: for any $\varepsilon > 0$, one may lower its unconditionality constant to $1+\varepsilon$ by removing a finite set from it.

In the case of $\mathscr{C}(\mathbb{T})$, E is a (*umbs*) exactly when E is a Sidon set with constant asymptotically 1.
In the same way, call $\{T_k\}$ an approximating sequence (a.s. for short) for Y if the T_k's are finite rank operators that tend strongly to the identity on Y; if such a sequence exists, then Y has the bounded approximation property. Denote by $\Delta T_k = T_k - T_{k-1}$ the difference sequence of T_k. Following Rosenthal (see [18, §1]), we then say that Y has the unconditional approximation property ((*uap*) for short) if it admits an a.s. $\{T_k\}$ such that for some C

$$\Big\| \sum_{k=1}^{n} \epsilon_k \Delta T_k \Big\|_{\mathscr{L}(Y)} \le C \qquad \text{for all } n \text{ and scalar } \epsilon_k \text{ with } |\epsilon_k| = 1. \tag{6}$$

By the uniform boundedness principle, (6) means exactly that $\sum \Delta T_k y$ converges unconditionally for all $y \in Y$. We now ask the following question: which conditions on E do yield the corresponding almost isometric (metric for short) property, first introduced by Casazza and Kalton [8, §3]?

(umap) The span $Y = X_E$ of E in X has the metric unconditional approximation property: for any $\varepsilon > 0$, one may lower the constant C in (6) to $1+\varepsilon$ by choosing an adequate a.s. $\{T_k\}$.

Several kinds of metric, *i. e.* almost isometric properties have been investigated in the last decade (see [26]). There is a common feature to these notions since Kalton's [35]: they can be reconstructed from a corresponding interaction between some break and some tail of the space. We prove that (*umap*) is characterised by almost 1-unconditionality between a specific break and tail, that we coin "block unconditionality".

Property (*umap*) has been studied by Li [43] for $X = \mathscr{C}(\mathbb{T})$. He obtains remarkably large examples of such sets E, in particular Hilbert sets. Thus, the second property seems to be much weaker than the first (although we do not know whether $\mathscr{C}_E(\mathbb{T})$ has (*umap*) for all (*umbs*) E in $\mathscr{C}(\mathbb{T})$: for sets of the latter kind, the natural sequence of projections realises (*uap*) in $\mathscr{C}_E(\mathbb{T})$, but we do not know whether it achieves (*umap*)).

In fact, both problems lead to strong arithmetical conditions on E that are somewhat complementary to the property of quasi-independence (see [59, §3]). In order to obtain them, we apply Forelli's [19, Prop. 2] and Plotkin's [60, Th. 1.4] techniques in the study of isometric operators on L^p: see Theorem 3.4.2 and Lemma 8.1.4. This may be done at once for the projections associated to basic sequences of characters. In the case of general metric unconditional approximating sequences, however, we need a more thorough knowledge of their connection with the structure of E: this is the duty of Theorem 7.2.3. As in Forelli's and Plotkin's results, we obtain that the spaces $X = \mathrm{L}^p(\mathbb{T})$ with p an even integer play a special rôle. For instance, they are the only spaces which admit 1-unconditional basic sequences $E \subseteq \mathbb{Z}$ with more than two elements: see Proposition 3.2.1.

There is another fruitful point of view: we may consider elements of E as random variables on the probability space $(\mathbb{T}, dm)$. They have uniform distribution and if they were independent, then our questions would have trivial answers. In fact, they are strongly dependent: for any $k, l \in \mathbb{Z}$, Rosenblatt's [64] strong mixing coefficient

$$\sup\bigl\{|m[A \cap B] - m[A]m[B]| : A \in \sigma(\mathrm{e}^{\mathrm{i}kt}) \text{ and } B \in \sigma(\mathrm{e}^{\mathrm{i}lt})\bigr\}$$

has its maximum value, 1/4. But lacunarity of E enhances their independence in several weaker senses (see [2]). Properties (*umap*) and (*umbs*) can be seen as an expression of almost independence of elements of E in the "additive sense", *i. e.* when appearing in sums. We show their relationship to the notions of pseudo-independence (see [54, §4.2]) and almost i.i.d. sequences (see [1]).

The gist of our results is the following: almost isometric properties for spaces X_E in "little" Fourier analysis may be read as a smallness property of E. They rely in an essential way on the arithmetical structure of E and distinguish between real and complex properties. In the case of $\mathrm{L}^{2n}(\mathbb{T})$, n integer, these arithmetical conditions are in finite number and turn out to be sufficient, because the norm of trigonometric polynomials is a polynomial expression in these spaces. Furthermore, the number of conditions increases with n in that case. In the remaining cases of $\mathrm{L}^p(\mathbb{T})$, p not an even integer, and $\mathscr{C}(\mathbb{T})$, these arithmetical conditions are infinitely many and become much more coercive. In particular, if our properties are satisfied in $\mathscr{C}(\mathbb{T})$, then they are satisfied in all spaces $\mathrm{L}^p(\mathbb{T})$, $1 \le p < \infty$.

We now turn to a detailed discussion of our results: in Section 3, we first characterise the sets E and values p such that E is a 1-unconditional basic sequence in $\mathrm{L}^p(\mathbb{T})$ (Prop. 3.2.1). Then we show how to treat similarly the almost isometric case and obtain a range of arithmetical conditions $(\mathscr{I}_n)$ on E (Th. 3.4.2). These conditions turn out to be identical whether one considers real or complex unconditionality: this is surprising and in sharp contrast to what happens when $\mathbb{T}$ is replaced by the Cantor group. They also do not distinguish amongst $\mathrm{L}^p(\mathbb{T})$ spaces with p not an even integer and $\mathscr{C}(\mathbb{T})$, but single out $\mathrm{L}^p(\mathbb{T})$ with p an even integer: this property does not "interpolate". This is similar to the phenomena of equimeasurability (see [40, introduction]) and $\mathscr{C}^\infty$-smoothness of norms (see [10, Chapter V]). These facts may also be appreciated from the point of view of natural renormings of the Hilbert space $\mathrm{L}^2_E(\mathbb{T})$.

In Section 4, of purely arithmetical nature, we give many examples of 1-unconditional and metric unconditional basic sequences through an investigation of property $(\mathscr{I}_n)$. As expected with lacunary series, number theoretic conditions show up (see especially Prop. 4.3.1).

In Section 5, we first return to the general case of a separable Banach space Y and show how to connect the metric unconditional approximation property with a simple property of "block unconditionality". Then a skipped blocking technique invented by Bourgain and Rosenthal [5] gives a canonical way to construct an a.s. that realises (*umap*) (Th. 5.3.1).

In Section 6, we introduce the p-additive approximation property ℓ_p-(*ap*) and its metric counterpart, ℓ_p-(*map*). It may be described as simply as (*umap*). Then we connect ℓ_p-(*map*) with the work of Godefroy, Kalton, Li and Werner [36, 22] on subspaces of L^p which are almost isometric to ℓ_p.

Section 7 focusses on (*uap*) and (*umap*) in the case of translation invariant subspaces X_E. The property of block unconditionality may then be expressed in terms of "break" and "tail" of E: see Theorem 7.2.3.

In Section 8, we proceed as in Section 3 to obtain a range of arithmetical conditions $(\mathscr{J}_n)$ for (*umap*) and metric unconditional (*fdd*) (Th. 8.2.1 and Prop. 8.2.4). These conditions are similar to $(\mathscr{I}_n)$, but are decidedly weaker: see Proposition 9.1.2(i). This time, real and complex unconditionality differ; again spaces $\mathrm{L}^p(\mathbb{T})$ with even p are singled out.

In Section 9, we continue the arithmetical investigation begun in Section 4 with property $(\mathscr{J}_n)$ and obtain many examples for the 1-unconditional and the metric unconditional approximation property.

However, the main result of Section 10, Theorem 10.3.1, shows how a rapid (and optimal) growth condition on E allows avoiding number theory in any case considered. We therefore get a new class of examples for (*umbs*), in particular Sidon sets of constant asymptotically 1, and (*umap*). We also prove that $\mathscr{C}_{\{3^k\}}(\mathbb{T})$ has real (*umap*) and that this is due to the oddness of 3 (Prop. 10.1.1). A sharp estimate of the Sidon constant of Hadamard sets is obtained as a byproduct (Cor. 10.4.1).

Section 11 uses combinatorial tools to give some rough information about the size of sets E that satisfy our arithmetical conditions. In particular, we answer a question of Li [43]: for $X = \mathscr{C}(\mathbb{T})$ and for $X = \mathrm{L}^p(\mathbb{T})$, $p \neq 2, 4$, the maximal density d^* of E is zero if X_E has (*umap*) (Prop. 11.2). For $X = \mathrm{L}^4(\mathbb{T})$, our technique falls short of the expected result: we just know that if $\mathrm{L}^4_{E\cup\{a\}}(\mathbb{T})$ has (*umap*) for every $a \in \mathbb{Z}$, then $d^*(E) = 0$.

Section 12 is an attempt to describe the relationship between these notions and probabilistic independence. Specifically the Rademacher and Steinhaus sequences show the way to a connection between metric unconditionality and the almost i.i.d. sequences of [1]. We note further that the arithmetical property $(\mathscr{I}_\infty)$ of Section 3 is equivalent to Murai's [54, §4.2] property of pseudo-independence.

In Section 13, we collect our results on metric unconditional basic sequences of characters and (*umap*) in translation invariant spaces. We conclude with open questions.

Notation and definitions Sections 3, 7, 8 and 10 will take place in the following framework. $(\mathbb{T}, dm)$ denotes the compact abelian group $\{z \in \mathbb{C} : |z| = 1\}$ endowed with its Haar measure dm; $m[A]$ is the measure of a subset $A \subseteq \mathbb{T}$. Let $\mathbb{D} = \{-1, 1\}$. $\mathbb{S}$ will denote either the complex ($\mathbb{S} = \mathbb{T}$) or real ($\mathbb{S} = \mathbb{D}$) choice of signs. For a real function f on $\mathbb{S}$, the oscillation of f is

$$\operatorname*{osc}_{\epsilon\in\mathbb{S}} f(\epsilon) = \sup_{\epsilon\in\mathbb{S}} f(\epsilon) - \inf_{\epsilon\in\mathbb{S}} f(\epsilon).$$

We shall study homogeneous Banach spaces X of functions on $\mathbb{T}$ [38, Chapter I.2], and especially the peculiar behaviour of the following ones: $\mathrm{L}^p(\mathbb{T})$ $(1 \leq p < \infty)$, the space of p-integrable functions with the norm $\|f\|_p = (\int |f|^p dm)^{1/p}$, and $\mathscr{C}(\mathbb{T})$, the space of continuous functions with the norm $\|f\|_\infty = \max\{|f(t)| : t \in \mathbb{T}\}$. $\mathscr{M}(\mathbb{T})$ is the dual of $\mathscr{C}(\mathbb{T})$ realised as Radon measures on $\mathbb{T}$.
The dual group $\{\mathrm{e}_n : z \mapsto z^n : n \in \mathbb{Z}\}$ of $\mathbb{T}$ is identified with $\mathbb{Z}$. We write $|B|$ for the cardinal of a set B. For a not necessarily increasing sequence $E = \{n_k\}_{k\geq 1} \subseteq \mathbb{Z}$, let $\mathscr{P}_E(\mathbb{T})$ be the space of trigonometric polynomials spanned by [the characters in] E. Let X_E be the translation invariant subspace of those elements in X whose Fourier transform vanishes off E: for all $f \in X_E$ and $n \notin E$, $\widehat{f}(n) = \int f(t)\,\mathrm{e}_{-n}(t) dm(t) = 0$. X_E is also the closure of $\mathscr{P}_E(\mathbb{T})$ in homogeneous X [38, Th. 2.12]. Denote by $\pi_k : X_E \to X_E$ the orthogonal projection onto $X_{\{n_1,\ldots,n_k\}}$. It is given by

$$\pi_k(f) = \widehat{f}(n_1)\,\mathrm{e}_{n_1} + \cdots + \widehat{f}(n_k)\,\mathrm{e}_{n_k}.$$

Then the π_k commute. They form an a.s. for X_E if and only if E is a basic sequence. For a finite or cofinite $F \subseteq E$, π_F is similarly the orthogonal projection of X_E onto X_F.
Sections 5 and 6 consider the general case of a separable Banach space X. B_X is the unit ball of X and Id denotes the identity operator on X. For a given sequence $\{U_k\}$, its difference sequence is $\Delta U_k = U_k - U_{k-1}$ (where $U_0 = 0$).
The functional notions of (*ubs*), (*umbs*) are defined in 3.1.1. The functional notions of a.s., (*uap*) and (*umap*) are defined in 5.1.1. Properties ℓ_p-(*ap*) and ℓ_p-(*map*) are defined in 6.1.1. The functional property $(\mathscr{U})$ of block unconditionality is defined in 7.2.1. The sets of arithmetical relations Z^m and Z^m_n are defined before 3.2.1. The arithmetical properties $(\mathscr{I}_n)$ of almost independence and $(\mathscr{J}_n)$ of block independence are defined in 3.4.1 and 8.1.2 respectively. The pairing $\langle \zeta, E \rangle$ is defined before 4.1.1.

3 Metric unconditional basic sequences of characters (*umbs*)

3.1 Definitions. Isomorphic case

We start with the definition of metric unconditional basic sequences ((*umbs*) for short). $\mathbb{S} = \mathbb{T} = \{\epsilon \in \mathbb{C} : |\epsilon| = 1\}$ (*vs.* $\mathbb{S} = \mathbb{D} = \{-1, 1\}$) is the complex (*vs.* real) choice of signs.

Definition 3.1.1 *Let $E \subseteq \mathbb{Z}$ and X be a homogeneous Banach space on $\mathbb{T}$.*
(i) [37] E is an unconditional basic sequence (ubs) in X if there is a constant C such that

$$\Big\|\sum_{q\in G}\epsilon_q a_q\,\mathrm{e}_q\Big\|_X \le C\Big\|\sum_{q\in G}a_q\,\mathrm{e}_q\Big\|_X \tag{7}$$

for all finite subsets $G \subseteq E$, coefficients $a_q \in \mathbb{C}$ and signs $\epsilon_q \in \mathbb{T}$ (vs. $\epsilon_q \in \mathbb{D}$). The infimum of such C is the complex (vs. real) unconditionality constant of E in X. If $C = 1$ works, then E is a complex (vs. real) 1-(ubs) in X.
(ii) E is a complex (vs. real) metric unconditional basic sequence (umbs) in X if for each $\varepsilon > 0$ there is a finite set F such that the complex (vs. real) unconditionality constant of $E \setminus F$ is less than $1+\varepsilon$.

Note that $\mathbb{Z}$ itself is an (*ubs*) in $\mathrm{L}^p(\mathbb{T})$ if and only if $p = 2$ by Khinchin's inequality. The same holds in the framework of the Cantor group $\mathbb{D}^\infty$ and its dual group of Walsh functions: their common feature with the e_n is that their modulus is everywhere equal to 1 (see [39]).
The following facts are folklore.

Proposition 3.1.2 *Let Y be a Banach space.*
(i) If $\big\|\sum \epsilon_k y_k\big\|_Y \le C\big\|\sum y_k\big\|_Y$ for all $\epsilon_k \in \mathbb{T}$ (vs. $\epsilon_k \in \mathbb{D}$), then this holds automatically for all complex (vs. real) ϵ_k with $|\epsilon_k| \le 1$.
(ii) Real and complex unconditionality are isomorphically $\pi/2$-equivalent.

Proof. (i) follows by convexity. (ii) Let us use the fact that the complex unconditionality constant of the Rademacher sequence is $\pi/2$ [69]:

$$\begin{aligned}\sup_{\delta_k\in\mathbb{T}}\Big\|\sum \delta_k y_k\Big\|_Y &= \sup_{y^*\in Y^*}\sup_{\delta_k\in\mathbb{T}}\sup_{\epsilon_k=\pm1}\Big|\sum \delta_k\langle y^*, y_k\rangle\epsilon_k\Big| \\ &\le \pi/2\sup_{y^*\in Y^*}\sup_{\epsilon_k=\pm1}\Big|\sum\langle y^*, y_k\rangle\epsilon_k\Big| = \pi/2\sup_{\epsilon_k=\pm1}\Big\|\sum \epsilon_k y_k\Big\|_Y.\end{aligned}$$

Taking the Rademacher sequence in $\mathscr{C}(\mathbb{D}^\infty)$, we see that $\pi/2$ is optimal. ∎

In fact, if (7) holds, then E is a basis of its span in X, which is X_E [38, Th. 2.12]. We have the following relationship between the unconditionality constants of E in $\mathscr{C}(\mathbb{T})$ and in a homogeneous Banach space X on $\mathbb{T}$.

Proposition 3.1.3 *Let $E \subseteq \mathbb{Z}$ and X be a homogeneous Banach space on $\mathbb{T}$.*
(i) The complex (vs. real) unconditionality constant of E in X is at most the complex (vs. real) unconditionality constant of E in $\mathscr{C}(\mathbb{T})$.
(ii) If E is a (ubs) (vs. 1-(ubs), (umbs)) in $\mathscr{C}(\mathbb{T})$, then E is a (ubs) (vs. 1-(ubs), (umbs)) in X.

This follows from the well-known (see *e.g.* [27])

Lemma 3.1.4 *Let $E \subseteq \mathbb{Z}$ and X be a homogeneous Banach space on $\mathbb{T}$. Let T be a multiplier on $\mathscr{C}_E(\mathbb{T})$. Then T is also a multiplier on X_E and*

$$\|T\|_{\mathscr{L}(X_E)} \le \|T\|_{\mathscr{L}(\mathscr{C}_E)}.$$

Proof. The linear functional $f \mapsto Tf(0)$ on $\mathscr{C}_E(\mathbb{T})$ extends to a measure $\mu \in \mathscr{M}(\mathbb{T})$ such that $\|\mu\|_{\mathscr{M}} = \|T\|_{\mathscr{L}(\mathscr{C}_E)}$. Let $\check\mu(t) = \mu(-t)$. Then $Tf = \check\mu * f$ for $f \in \mathscr{P}_E(\mathbb{T})$ and

$$\|T\|_{\mathscr{L}(X_E)} \le \|\check\mu\|_{\mathscr{M}} = \|T\|_{\mathscr{L}(\mathscr{C}_E)}.$$ ∎

Question 3.1.5 There is no interpolation theorem for such relative multipliers. The forthcoming Theorem 3.4.2 shows that there can be no metric interpolation. Is it possible that one cannot interpolate multipliers at all between $\mathrm{L}^p_E(\mathbb{T})$ and $\mathrm{L}^q_E(\mathbb{T})$?

Note that conversely, [20] furnishes the example of an $E \subseteq \mathbb{Z}$ such that the π_k are uniformly bounded on $\mathrm{L}^1_E(\mathbb{T})$ but not on $\mathscr{C}_E(\mathbb{T})$.
It is known that E is an (*ubs*) in $\mathscr{C}(\mathbb{T})$ (*vs.* in $\mathrm{L}^p(\mathbb{T})$) if and only if it is a Sidon (*vs.* $\Lambda(2 \vee p)$) set. To see this, let us recall the relevant definitions.

Definition 3.1.6 *Let $E \subseteq \mathbb{Z}$.*
(i) *[32] E is a Sidon set if there is a constant C such that*

$$\sum_{q\in G} |a_q| \le C \Big\| \sum_{q\in G} a_q \, \mathrm{e}_q \Big\|_\infty \quad \textit{for all finite } G \subseteq E \textit{ and } a_q \in \mathbb{C}.$$

The infimum of such C is E's Sidon constant.
(ii) *[66, Def. 1.5] Let $p > 1$. E is a $\Lambda(p)$ set if there is a constant C such that $\|f\|_p \le C\|f\|_1$ for $f \in \mathscr{P}_E(\mathbb{T})$.*

In fact, the Sidon constant of E is the complex unconditionality constant of E in $\mathscr{C}(\mathbb{T})$. Thus E is a complex (*umbs*) in $\mathscr{C}(\mathbb{T})$ if and only if tails of E have their Sidon constant arbitrarily close to 1. We may also say: E's Sidon constant is asymptotically 1.
Furthermore, E is a $\Lambda(2 \vee p)$ set if and only if $\mathrm{L}^p_E(\mathbb{T}) = \mathrm{L}^2_E(\mathbb{T})$. Therefore $\Lambda(2 \vee p)$ sets are (*ubs*) in $\mathrm{L}^p(\mathbb{T})$. Conversely, if E is an (*ubs*) in $\mathrm{L}^p(\mathbb{T})$, then by Khinchin's inequality

$$\Big\| \sum_{q\in G} a_q \, \mathrm{e}_q \Big\|_p^p \approx \text{average} \Big\| \sum_{q\in G} \pm a_q \, \mathrm{e}_q \Big\|_p^p \approx \Big(\sum_{q\in G} |a_q|^2 \Big)^{p/2} = \Big\| \sum_{q\in G} a_q \, \mathrm{e}_q \Big\|_2^p$$

for all finite $G \subseteq E$ (see [66, proof of Th. 3.1]). This shows also that the $\Lambda(p)$ set constant and the unconditionality constant in $\mathrm{L}^p(\mathbb{T})$ are connected *via* the constants in Khinchin's inequality; whereas Sidon sets have their unconditionality constant in $\mathrm{L}^p(\mathbb{T})$ uniformly bounded, the $\Lambda(p)$ set constant of infinite sets grows at least like $\sqrt{p}$ [66, Th. 3.4].

3.2 Isometric case: 1-unconditional basic sequences of characters

The corresponding isometric question: when is E a complex 1-(*ubs*)? admits a rather easy answer. To this end, introduce the following notation for arithmetical relations: let $\mathrm{A}_n = \big\{\alpha = \{\alpha_p\}_{p\ge 1} : \alpha_p \in \mathbb{N} \;\&\; \alpha_1 + \alpha_2 + \cdots = n\big\}$. If $\alpha \in \mathrm{A}_n$, all but a finite number of the α_p vanish and the multinomial number

$$\binom{n}{\alpha} = \frac{n!}{\alpha_1!\alpha_2!\ldots}$$

is well defined. Let $\mathrm{A}_n^m = \{\alpha \in \mathrm{A}_n : \alpha_p = 0 \text{ for } p > m\}$. Note that A_n^m is finite. We call E n-independent if every integer admits at most one representation as the sum of n elements of E, up to a permutation. In terms of arithmetical relations, this yields

$$\sum \alpha_i p_i = \sum \beta_i p_i \Rightarrow \alpha = \beta \text{ for } \alpha, \beta \in \mathrm{A}_n^m \text{ and distinct } p_1, \ldots, p_m \in E.$$

This notion is studied in [12] where it is called birelation. In Rudin's [66, §1.6(*b*)] notation, the number $r_n(E;k)$ of representations of $k \in \mathbb{Z}$ as a sum of n elements of E is at most $n!$ for all k if E is n-independent (the converse if false). This may also be expressed in the framework of arithmetical relations

$$\mathrm{Z}^m = \{\zeta \in \mathbb{Z}^{*m} : \zeta_1 + \cdots + \zeta_m = 0\} \quad \& \quad \mathrm{Z}_n^m = \{\zeta \in \mathrm{Z}^m : |\zeta_1| + \cdots + |\zeta_m| \le 2n\}.$$

Note that Z_n^m is finite, and void if $m > 2n$. Then E is n-independent if and only if

$$\sum \zeta_i p_i \ne 0 \quad \text{for all } \zeta \in \mathrm{Z}_n^m \text{ and distinct } p_1, \ldots, p_m \in E.$$

We shall prefer to treat arithmetical relations in terms of Z_n^m rather than A_n^m.

Proposition 3.2.1 *Let $E \subseteq \mathbb{Z}$.*
(i) *E is a complex 1-*(ubs)* in $\mathrm{L}^p(\mathbb{T})$, p not an even integer, or in $\mathscr{C}(\mathbb{T})$, if and only if E has at most two elements.*
(ii) *If p is an even integer, then E is a complex 1-*(ubs)* in $\mathrm{L}^p(\mathbb{T})$ if and only if E is $p/2$-independent. There is a constant $C_p > 1$ depending only on p, such that either E is a complex 1-*(ubs)* in $\mathrm{L}^p(\mathbb{T})$ or the complex unconditionality constant of E in $\mathrm{L}^p(\mathbb{T})$ is at least C_p.*

Proof. (i) By Proposition 3.1.3(ii), if E is not a complex 1-(ubs) in some $\mathrm{L}^p(\mathbb{T})$, then neither in $\mathscr{C}(\mathbb{T})$. Let p be not an even integer. We may suppose $0 \in E$; let $\{0, k, l\} \subseteq E$. If we had $\|1 + \mu a \,\mathrm{e}_k + \nu b \,\mathrm{e}_l\|_p = \|1 + a \,\mathrm{e}_k + b \,\mathrm{e}_l\|_p$ for all $\mu, \nu \in \mathbb{T}$, then

$$\begin{aligned}\int |1 + a \,\mathrm{e}_k + b \,\mathrm{e}_l|^p dm &= \int |1 + \mu a \,\mathrm{e}_k + \nu b \,\mathrm{e}_l|^p dm(\mu) dm(\nu) dm \\ &= \int |1 + \mu a + \nu b|^p dm(\mu) dm(\nu).\end{aligned}$$

Denoting by $\theta_i \colon (\epsilon_1, \epsilon_2) \mapsto \epsilon_i$ the projections of $\mathbb{T}^2$ onto $\mathbb{T}$, this would mean that $\|1 + a \,\mathrm{e}_k + b \,\mathrm{e}_l\|_p = \|1 + a\theta_1 + b\theta_2\|_{\mathrm{L}^p(\mathbb{T}^2)}$ for all $a, b \in \mathbb{C}$. By [67, Th. I], $(\mathrm{e}_k, \mathrm{e}_l)$ and (θ_1, θ_2) would have the same distribution. This is false, since θ_1 and θ_2 are independent random variables while e_k and e_l are not.
(ii) Let $q_1, \dots, q_m \in E$ be distinct and $\epsilon_1, \dots, \epsilon_m \in \mathbb{T}$. By the multinomial formula for the power $p/2$ and Bessel–Parseval's formula, we get

$$\begin{aligned}\Big\|\sum_{i=1}^m \epsilon_i a_i \,\mathrm{e}_{q_i}\Big\|_p^p &= \int \Big|\sum_{\alpha \in \mathrm{A}^m_{p/2}} \binom{p/2}{\alpha} \prod_{i=1}^m (\epsilon_i a_i)^{\alpha_i} \,\mathrm{e}_{\sum \alpha_i q_i}\Big|^2 dm \\ &= \sum_{A \in \mathscr{R}_q} \Big|\sum_{\alpha \in A} \binom{p/2}{\alpha} \prod_{i=1}^m (\epsilon_i a_i)^{\alpha_i}\Big|^2 \\ &= \sum_{\alpha \in \mathrm{A}^m_{p/2}} \binom{p/2}{\alpha}^2 \prod_{i=1}^m |a_i|^{2\alpha_i} + \sum_{\substack{\alpha \neq \beta \in \mathrm{A}^m_{p/2} \\ \alpha \sim \beta}} \binom{p/2}{\alpha} \binom{p/2}{\beta} \prod_{i=1}^m \epsilon_i^{\alpha_i - \beta_i} a_i^{\alpha_i} \overline{a_i}^{\beta_i}, \end{aligned} \tag{8}$$

where $\mathscr{R}_q$ is the partition of $\mathrm{A}^m_{p/2}$ induced by the equivalence relation $\alpha \sim \beta \Leftrightarrow \sum \alpha_i q_i = \sum \beta_i q_i$. If E is $p/2$-independent, the second sum in (8) is void and E is a 1-(ubs).
Furthermore, suppose E is not $p/2$-independent and let $q_1, \dots, q_m \in E$ be a minimal set of distinct elements of E such that there are $\alpha, \beta \in \mathrm{A}^m_{p/2}$ with $\alpha \sim \beta$. Then $m \leq p$. Take $a_i = 1$ in the former computation: then the clearly nonzero oscillation of (8) for $\epsilon_1, \dots, \epsilon_m \in \mathbb{T}$ does only depend on $\mathscr{R}_q$ and thus is finitely valued. This yields C_p. ∎

Example 3.2.2 Let us treat explicitly the case $p = 4$. If E is not 2-independent, then one of the two following arithmetic relations occurs on E:

$$2q_1 = q_2 + q_3 \quad \text{or} \quad q_1 + q_2 = q_3 + q_4.$$

In the first case, we may assume $q_2 < q_1 < q_3$ and thus

$$2q_2 < q_1 + q_2 < 2q_1 = q_2 + q_3 < q_1 + q_3 < 2q_3.$$

Let $\varrho > 0$. Then

$$\int |\,\mathrm{e}_{q_1} + \varrho \,\mathrm{e}_{q_2} + \epsilon\varrho \,\mathrm{e}_{q_3}|^4 dm = 1 + 6\varrho^4 + 4\varrho^2(2 + \Re\epsilon).$$

Taking $\epsilon = -1$ and $\epsilon = 1$, $\varrho = 6^{-1/4}$, we see that E's real unconditionality constant is at least the fourth root of $2\sqrt{6} - 3$. In fact, E's real and complex unconditionality constants coincide with this value.
In the second case, we may assume $q_1 < q_3 < q_4 < q_2$ and thus

$$2q_1 < q_1 + q_3 < q_1 + q_4, 2q_3 < q_1 + q_2 = q_3 + q_4 < q_2 + q_3, 2q_4 < q_2 + q_4 < 2q_2.$$

We may further assume $q_1 + q_4 \neq 2q_3$ and $q_2 + q_3 \neq 2q_4$: otherwise the first case occurs. Then

$$\int |\,\mathrm{e}_{q_1} + \mathrm{e}_{q_2} + \mathrm{e}_{q_3} + \epsilon \,\mathrm{e}_{q_4}|^4 dm = 28 + 8\Re\epsilon.$$

Thus E's real unconditionality constant must be at least $(9/5)^{1/4}$. In fact, E's real and complex unconditionality constants coincide with this value.

From these two cases we conclude that $C_2 = (9/5)^{1/4} \approx 1.16$ is the optimal choice for the constant in Proposition 3.2.1(ii).

Remark 3.2.3 We shall compute explicitly the Sidon constant of sets with three elements and show that it is equal to the real unconditionality constant in that case. This provides an alternative proof and a generalisation of Prop. 3.2.1 (i) for $\mathscr{C}(\mathbb{T})$.

Remark 3.2.4 In fact the conclusion in (ii) holds also if we assume that E is just a real 1-(ubs). If we have some arithmetical relation $\alpha \sim \beta$, we may assume that $\alpha_i - \beta_i$ is odd for one i at least. Indeed, we may simplify all $\alpha_i - \beta_i$ by their greatest common divisor and this yields another arithmetical relation $\sum(\alpha'_i - \beta'_i)q_i = 0$. But then the oscillation of (8) is again clearly nonzero for $\epsilon_1, \ldots, \epsilon_m \in \mathbb{D}$.

Remark 3.2.5 We shall see in Remark 3.3.3 that (i) also holds in the real setting. This is a property of $\mathbb{T}$ and fails for the Cantor group $\mathbb{D}^\infty$: the Rademacher sequence forms a real 1-(ubs) in $\mathscr{C}(\mathbb{D}^\infty)$ but is clearly not complex 1-unconditional in any space $\mathrm{L}^p(\mathbb{D}^\infty)$, $p \neq 2$: see Section 12 and [69].

Question 3.2.6 There are nevertheless subspaces of $\mathrm{L}^p(\mathbb{T})$, p not an even integer, and $\mathscr{C}(\mathbb{T})$ with 1-unconditional bases, like sequences of functions with disjoint support. What about spaces $\mathrm{L}^p_E(\mathbb{T})$ and $\mathscr{C}_E(\mathbb{T})$, in particular when E is finite ? Are there 1-unconditional bases that do not consist of characters ?

Remark 3.2.7 For each even integer $p \geq 4$, there are $p/2$-independent sets that are not $\Lambda(p + \varepsilon)$ for any $\varepsilon > 0$: such maximal $\Lambda(p)$ sets are constructed in [66].

3.3 Almost isometric case. A computation

As 1-(ubs) are thus a quite exceptional phenomenon and distinguish so harshly between even integers and all other reals, one may wonder what kind of behaviour its almost isometric counterpart will bring about. In the proof of Proposition 3.2.1(i), we used the fact that the e_n, seen as random variables, are dependent: the L^p norm for even integer p is just somewhat blind to this because it keeps the interaction of the random variables down to a finite number of arithmetical relations. The contrast with the other L^p norms becomes clear when we try to compute explicitly an expression of type $\left\|\sum \epsilon_q a_q \, \mathrm{e}_q\right\|_p$ for any $p \in [1, \infty[$. This sort of seemingly brutal computation has been applied successfully in [19, Prop. 2] and [60, Th. 1.4] to study isometric operators on L^p, p not an even integer.
We now undertake this tedious computation as preparatory work for Theorem 3.4.2, Lemma 8.1.4 and Proposition 8.2.4. Let us fix some more notation: for $x \in \mathbb{R}$ and $\alpha \in \mathrm{A}_n$, put

$$\binom{x}{\alpha} = \binom{x}{n}\binom{n}{\alpha}.$$

This generalised multinomial number is nonzero if and only if $x \geq n$ or $x \notin \mathbb{N}$.

Computational lemma 3.3.1 *Let $\mathbb{S} = \mathbb{T}$ or $\mathbb{S} = \mathbb{D}$ in the complex and real case respectively. Let $1 \leq p < \infty$ and $m \geq 1$. Put*

$$\varphi_q(\epsilon, z, t) = \left|1 + \sum_{i=1}^{m} \epsilon_i z_i \, \mathrm{e}_{q_i}(t)\right|^p \quad , \quad \Phi_q(\epsilon, z) = \int \varphi_q(\epsilon, z, t)\, dm(t)$$

for $q = (q_1, \ldots, q_m) \in \mathbb{Z}^m$, $\epsilon = (\epsilon_1, \ldots, \epsilon_m) \in \mathbb{S}^m$ and $z = (z_1, \ldots, z_m) \in D^m$, where D is the disc $\{|w| \leq \varrho\} \subseteq \mathbb{C}$ for some $0 < \varrho < 1/m$. Define the equivalence relation $\alpha \sim \beta \Leftrightarrow \sum \alpha_i q_i = \sum \beta_i q_i$. Then

$$\Phi_q(\epsilon, z) = \sum_{\alpha \in \mathbb{N}^m} \binom{p/2}{\alpha}^2 \prod |z_i|^{2\alpha_i} + \sum_{\substack{\alpha \neq \beta \in \mathbb{N}^m \\ \alpha \sim \beta}} \binom{p/2}{\alpha}\binom{p/2}{\beta} \prod z_i^{\alpha_i} \overline{z_i}^{\beta_i} \epsilon_i^{\alpha_i - \beta_i}. \tag{9}$$

Furthermore, $\{\Phi_q : q \in \mathbb{Z}^m\}$ is a relatively compact subset of $\mathscr{C}^\infty(\mathbb{S}^m \times D^m)$.

Proof. The function Φ_q is infinitely differentiable on the compact set $\mathbb{S}^m \times D^m$. Furthermore the family $\{\Phi_q : q_1, \dots, q_m \in \mathbb{Z}\}$ is bounded in $\mathscr{C}^\infty(\mathbb{S}^m \times D^m)$ and henceforth relatively compact by Montel's theorem. Let us compute φ_q. By the expansion of the function $(1+w)^{p/2}$, analytic on the unit disc, and the multinomial formula, we have

$$\begin{aligned}\varphi_q(\epsilon, z) &= \left|\sum_{a\ge 0}\binom{p/2}{a}\left(\sum_{i=1}^m \epsilon_i z_i \,\mathrm{e}_{q_i}\right)^a\right|^2\\ &= \left|\sum_{a\ge 0}\binom{p/2}{a}\sum_{\alpha\in \mathrm{A}_a^m}\binom{a}{\alpha}\prod(\epsilon_i z_i)^{\alpha_i}\,\mathrm{e}_{\sum\alpha_i q_i}\right|^2\\ &= \left|\sum_{\alpha\in\mathbb{N}^m}\binom{p/2}{\alpha}\prod(\epsilon_i z_i)^{\alpha_i}\,\mathrm{e}_{\sum\alpha_i q_i}\right|^2.\end{aligned}$$

Let $\mathscr{R}_q$ be the partition of $\mathbb{N}^m$ induced by $\sim$. Then, by Bessel–Parseval's formula

$$\Phi_q(\epsilon, z) = \sum_{A\in\mathscr{R}_q}\left|\sum_{\alpha\in A}\binom{p/2}{\alpha}\prod(\epsilon_i z_i)^{\alpha_i}\right|^2$$

and this gives (9) by expanding the modulus. ∎

Remark 3.3.2 If $m \ge 2$, this expansion has a finite number of terms if and only if p is an even integer: then and only then $\binom{p/2}{\alpha} = 0$ for $\sum \alpha_i > p/2$, whereas $\mathscr{R}_q$ contains clearly some class with two elements and thus an infinity thereof. For example, we have the following arithmetical relation on q_1, q_2 or $q_1, q_2, 0$ respectively:

$$\overbrace{q_1 + \cdots + q_1}^{|q_2|} = \overbrace{q_2 + \cdots + q_2}^{|q_1|} \qquad \text{if } \operatorname{sgn} q_1 = \operatorname{sgn} q_2;$$
$$\overbrace{q_1 + \cdots + q_1}^{|q_2|} + \overbrace{q_2 + \cdots + q_2}^{|q_1|} = 0 \qquad \text{if not.}$$

Remark 3.3.3 This shows that Proposition 3.2.1(i) holds also in the real setting: we may suppose that $0 \in E$; take $m = 2$ and choose $q_1, q_2 \in E$. One of the two relations in Remark 3.3.2 yields an arithmetical relation on E with at least one odd coefficient, as done in Remark 3.2.4. But then (9) contains terms nonconstant in $\epsilon_1 \in \mathbb{D}$ or in $\epsilon_2 \in \mathbb{D}$ and thus E cannot be a real 1-unconditional basic sequence in $\mathrm{L}^p(\mathbb{T})$.

We return to our computation.

Computational lemma 3.3.4 *Let $r = (r_0, \dots, r_m) \in E^{m+1}$ and put $q_i = r_i - r_0$ $(1 \le i \le m)$. Define*

$$\Theta_r(\epsilon, z) = \int\left|\mathrm{e}_{r_0} + \sum_{i=1}^m \epsilon_i z_i\,\mathrm{e}_{r_i}\right|^p = \Phi_q(\epsilon, z) \tag{10}$$

Let $\zeta_0, \dots, \zeta_m \in \mathbb{Z}^*$ and

$$(\gamma_i, \delta_i) = (-\zeta_i \vee 0, \zeta_i \vee 0) \qquad (1 \le i \le m). \tag{11}$$

If the arithmetical relation

$$\zeta_0 r_0 + \cdots + \zeta_m r_m = 0 \qquad \textit{while} \qquad \zeta_0 + \cdots + \zeta_m = 0 \tag{12}$$

holds, then the coefficient of $\prod z_i^{\gamma_i}\overline{z_i}^{\delta_i}\epsilon_i^{\gamma_i-\delta_i}$ in (9) is $\binom{p/2}{\gamma}\binom{p/2}{\delta}$ and thus independent of r. If $\sum|\zeta_i| \le p$ or p is not an even integer, this coefficient is nonzero.

Proof. We have $\delta_i - \gamma_i = \zeta_i$, $\sum\gamma_i - \sum\delta_i = \zeta_0$ and $\sum\gamma_i + \sum\delta_i = |\zeta_1| + \cdots + |\zeta_m|$, so that $\sum\gamma_i \vee \sum\delta_i = \frac{1}{2}\sum|\zeta_i|$. Moreover $\sum(\delta_i - \gamma_i)q_i = \sum\zeta_i r_i = 0$, so that $\gamma \sim \delta$. ∎

3.4 Almost independent sets of integers. Main theorem

The Computational lemmas suggest the following definition.

Definition 3.4.1 *Let $E \subseteq \mathbb{Z}$.*
(i) E enjoys the property $(\mathscr{I}_n)$ of almost n-independence provided there is a finite subset $G \subseteq E$ such that $E \setminus G$ is n-independent, i. e. $\zeta_1 r_1 + \cdots + \zeta_m r_m \neq 0$ for all $\zeta \in \mathbb{Z}_n^m$ and $r_1, \ldots, r_m \in E \setminus G$.
(ii) E enjoys exactly $(\mathscr{I}_n)$ if furthermore it fails $(\mathscr{I}_{n+1})$.
(iii) E enjoys $(\mathscr{I}_\infty)$ if it enjoys $(\mathscr{I}_n)$ for all n, i. e. for any $\zeta \in \mathbb{Z}^m$ there is a finite set G such that $\zeta_1 r_1 + \cdots + \zeta_m r_m \neq 0$ for $r_1, \ldots, r_m \in E \setminus G$.

Note that property $(\mathscr{I}_1)$ is void and that $(\mathscr{I}_{n+1}) \Rightarrow (\mathscr{I}_n)$. This property is also stable under unions with a finite set. The preceding computations yield

Theorem 3.4.2 *Let $E = \{n_k\} \subseteq \mathbb{Z}$ and $1 \leq p < \infty$.*
(i) Suppose p is an even integer. Then E is a real, and at the same times complex, (umbs) in $\mathrm{L}^p(\mathbb{T})$ if and only if E enjoys $(\mathscr{I}_{p/2})$. If $(\mathscr{I}_{p/2})$ holds, there is in fact a finite $G \subseteq E$ such that $E \setminus G$ is a 1-(ubs) in $\mathrm{L}^p(\mathbb{T})$.
(ii) If p is not an even integer and E is a real or complex (umbs) in $\mathrm{L}^p(\mathbb{T})$, then E enjoys $(\mathscr{I}_\infty)$.

Proof. Sufficiency in (i) follows directly from Proposition 3.2.1: if $E \setminus G$ is $p/2$-independent, then $E \setminus G$ is a real and complex 1-(*ubs*).
Let us prove the necessity of the arithmetical property. We keep the notation of Computational lemmas 3.3.1 and 3.3.4. Assume E fails $(\mathscr{I}_n)$ and let $\zeta_0, \ldots, \zeta_m \in \mathbb{Z}^*$ with $\sum \zeta_i = 0$ and $\sum |\zeta_i| \leq 2n$ such that for each $l \geq 1$ there are distinct $r_0^l, \ldots, r_m^l \in E \setminus \{n_1, \ldots, n_l\}$ with $\zeta_0 r_0^l + \cdots + \zeta_m r_m^l = 0$. One may furthermore assume that at least one of the ζ_i is not even.
Assume E is a (*umbs*) in $\mathrm{L}^p(\mathbb{T})$. Then the oscillation of Θ_r in (10) satisfies

$$\underset{\epsilon \in \mathbb{S}^m}{\operatorname{osc}} \Theta_{r^l}(\epsilon, z) \xrightarrow[l \to \infty]{} 0 \tag{13}$$

for each $z \in D^m$. We may assume that the sequence of functions Θ_{r^l} converges in $\mathscr{C}^\infty(\mathbb{S}^m \times D^m)$ to a function Θ. Then by (13), $\Theta(\epsilon, z)$ is constant in ϵ for each $z \in D^m$: in particular, its coefficient of $\prod z_i^{\gamma_i} \overline{z_i}^{\delta_i} \epsilon_i^{\gamma_i - \delta_i}$ is zero. (Note that at least one of the $\gamma_i - \delta_i$ is not even). This is impossible by Computational lemma 3.3.4 if p is either not an even integer or if $p \geq 2n$. ■

Corollary 3.4.3 *Let $E \subseteq \mathbb{Z}$. If E is a (umbs) in $\mathscr{C}(\mathbb{T})$, that is E's Sidon constant is asymptotically 1, then E enjoys $(\mathscr{I}_\infty)$. The converse does not hold.*

Proof. Necessity follows from Theorem 3.4.2 and Proposition 3.1.3(ii). There is a counterexample to the converse in [66, Th. 4.11]: Rudin constructs a set E that enjoys $(\mathscr{I}_\infty)$ while E is not even a Sidon set. ■

For p an even integer, Sections 4 and 11 will provide various examples of (*umbs*) in $\mathrm{L}^p(\mathbb{T})$. Proposition 10.2.1 gives a general growth condition on E under which it is an (*umbs*).
As we do not know any partial converse to Theorem 3.4.2(ii) and Corollary 3.4.3, the sole known examples of (*umbs*) in $\mathrm{L}^p(\mathbb{T})$, p not an even integer, and $\mathscr{C}(\mathbb{T})$ are those given by Theorem 10.3.1. This theorem will therefore provide us with Sidon sets of constant asymptotically 1. Note, however, that Li [43, Th. 4] already constructed implicitly such a Sidon set by using Kronecker's theorem.

4 Examples of metric unconditional basic sequences

After a general study of the arithmetical property of almost independence $(\mathscr{I}_n)$, we shall investigate three classes of subsets of $\mathbb{Z}$: integer geometric sequences, more generally integer parts of real geometric sequences, and polynomial sequences.

4.1 General considerations

The quantity

$$\begin{aligned}\langle\zeta,E\rangle &= \sup_{G\subseteq E \text{ finite}} \inf\bigl\{|\zeta_1p_1+\cdots+\zeta_mp_m| : p_1,\ldots,p_m\in E\setminus G \text{ distinct}\bigr\}\\ &= \lim_{l\to\infty} \inf\bigl\{|\zeta_1p_1+\cdots+\zeta_mp_m| : p_1,\ldots,p_m\in \{n_l,n_{l+1},\ldots\} \text{ distinct}\bigr\},\end{aligned}$$

where $\{n_k\}=E$, plays a key rôle. We have

Proposition 4.1.1 *Let $E=\{n_k\}\subseteq\mathbb{Z}$.*
(i) E enjoys $(\mathscr{I}_n)$ if and only if $\langle\zeta,E\rangle\neq 0$ for all $\zeta\in\mathrm{Z}_n^m$. If $\langle\zeta,E\rangle<\infty$ for some $\zeta_1,\ldots,\zeta_m\in\mathbb{Z}^*$, then E fails $(\mathscr{I}_{|\zeta_1|+\cdots+|\zeta_m|})$. Thus E enjoys $(\mathscr{I}_\infty)$ if and only if $\langle\zeta,E\rangle=\infty$ for all $\zeta_1,\ldots,\zeta_m\in\mathbb{Z}^*$.
(ii) Suppose E is an increasing sequence. If E enjoys $(\mathscr{I}_2)$, then the pace $n_{k+1}-n_k$ of E tends to infinity.
(iii) Suppose $jF+s, kF+t\subseteq E$ for an infinite F, $j\neq k\in\mathbb{Z}^*$ and $s,t\in\mathbb{Z}$. Then E fails $(\mathscr{I}_{|j|+|k|})$.
(iv) Let $E'=\{n_k+m_k\}$ with $\{m_k\}$ bounded. Then $\langle\zeta,E\rangle=\infty$ if and only if $\langle\zeta,E'\rangle=\infty$. Thus $(\mathscr{I}_\infty)$ is stable under bounded perturbations of E.

Proof. (i) Suppose $\langle\zeta,E\rangle<\infty$. Then there is an $h\in\mathbb{Z}$ such that there are sequences $p_1^l,\ldots,p_m^l\in\{n_k\}_{k\geq l}$ with $\sum\zeta_ip_i^l=h$ and $\{p_1^{l+1},\ldots,p_m^{l+1}\}$ is disjoint from $\{p_1^l,\ldots,p_m^l\}$ for all $l\geq 1$. As $\sum\zeta_ip_i^l-\sum\zeta_ip_i^{l+1}=0$ for $l\geq 1$, E fails $(\mathscr{I}_{|\zeta_1|+\cdots+|\zeta_m|})$.
(ii) Indeed, $\langle(1,-1),E\rangle=\infty$.
(iii) Put $\zeta=(j,-k)$. Then $\langle\zeta,E\rangle<\infty$. ■

4.2 Geometric sequences

Let $G=\{j^k\}_{k\geq 0}$ with $j\in\mathbb{Z}\setminus\{-1,0,1\}$. Then $G,jG\subseteq G$: so G fails $(\mathscr{I}_{|j|+1})$. In order to check $(\mathscr{I}_{|j|})$ for G, let us study more carefully the following Diophantine equation:

$$\sum_{i=1}^m \zeta_i j^{k_i}=0 \quad\text{with}\quad \zeta\in\mathbb{N}^*\times\mathbb{Z}^{*m-1} \;\&\; \sum_{i=1}^m|\zeta_i|\leq 2|j| \;\&\; k_1<\cdots<k_m. \tag{14}$$

Suppose (14) holds. Then necessarily $m\geq 2$ and $\zeta_1+\sum_{i=2}^m\zeta_ij^{k_i-k_1}=0$. Hence $j\mid\zeta_1$ and $\zeta_1\geq|j|$. As $\zeta_1<2|j|$, $\zeta_1=|j|$. Then $\operatorname{sgn} j+\sum_{i=2}^m\zeta_ij^{k_i-k_1-1}=0$. Hence $k_2=k_1+1$ and $j\mid\operatorname{sgn} j+\zeta_2$. As $|\zeta_2|\leq|j|$, $\zeta_2\in\{-\operatorname{sgn} j, j-\operatorname{sgn} j\}$. If $\zeta_2=j-\operatorname{sgn} j$, then $m=3$, $k_3=k_1+2$ and $\zeta_3=-1$. If $\zeta_2=-\operatorname{sgn} j$, then $m=2$: otherwise, $j\mid\zeta_3$ as before and $|\zeta_1|+|\zeta_2|+|\zeta_3|>2|j|$. Thus (14) has exactly two solutions:

$$|j|\cdot j^k+(-\operatorname{sgn} j)\cdot j^{k+1}=0 \;\&\; |j|\cdot j^k+(j-\operatorname{sgn} j)\cdot j^{k+1}+(-1)\cdot j^{k+2}=0. \tag{15}$$

If j is positive, this shows that G enjoys $(\mathscr{I}_j)$: both solutions yield $\sum\zeta_i\neq 0$. If j is negative, G enjoys $(\mathscr{I}_{|j|-1})$, but the second solution of (14) shows that G fails $(\mathscr{I}_{|j|})$.

4.3 Algebraic and transcendental numbers

An interesting feature of property $(\mathscr{I}_\infty)$ is that it distinguishes between algebraic and transcendental numbers. A similar fact has already been noticed by Murai [54, Prop. 26, Cor. 28].

Proposition 4.3.1 *Let $E=\{n_k\}\subseteq\mathbb{Z}$.*
(i) If $n_{k+1}/n_k\to\sigma$ where $\sigma>1$ is transcendental, then $\langle\zeta,E\rangle=\infty$ for any $\zeta_1,\ldots,\zeta_m\in\mathbb{Z}^*$. Thus E enjoys $(\mathscr{I}_\infty)$.
(ii) Write $[x]$ for the integer part of a real x. Let $n_k=[\sigma^k]$ with $\sigma>1$ algebraic. Let $P(x)=\zeta_0+\cdots+\zeta_dx^d$ be the corresponding polynomial of minimal degree. Then $\langle\zeta,E\rangle<\infty$ and E fails $(\mathscr{I}_{|\zeta_0|+\cdots+|\zeta_d|})$.

Note that part (ii) is very restrictive on the speed of convergence of n_{k+1}/n_k to σ: even if we take into account Proposition 4.1.1(iv), it requires that

$$|n_{k+1}/n_k-\sigma|\preccurlyeq\sigma^{-k}.$$

Proof. (i) Suppose on the contrary that we have ζ and sequences $p_1^l < \cdots < p_m^l$ in E that tend to infinity such that $\zeta_1 p_1^l + \cdots + \zeta_m p_m^l = 0$. As the sequences $\{p_i^l/p_m^l\}_l$ $(1 \le i \le m)$ are bounded, we may assume they are converging — and by hypothesis, they converge either to 0, say for $i < j$, or to σ^{-d_i} for $d_i \in \mathbb{N}$ and $i \ge j$. But then $\zeta_j \sigma^{-d_j} + \cdots + \zeta_m \sigma^{-d_m} = 0$ and σ is algebraic.
(ii) Apply Proposition 4.1.1(i) with ζ:

$$|\zeta_0[\sigma^k] + \cdots + \zeta_d[\sigma^{k+d}]| = |\zeta_0([\sigma^k] - \sigma^k) + \cdots + \zeta_d([\sigma^{k+d}] - \sigma^{k+d})| \le \sum |\zeta_i|. \qquad \blacksquare$$

4.4 Polynomial sequences

Let us first give some numerical evidence for the classical case of sets of dth powers. The table below reads as follows: "the set $E = \{k^d\}$ for d the value in the first column fails the property in the second column by the counterexample given in the third column." Indeed, each such counterexample to n-independence yields arbitrarily large counterexamples.

$\{k^d\}$	fails	by counterexample
d=2	$(\mathscr{I}_2)$	$7^2 + 1^2 = 2 \cdot 5^2$ or $18^2 + 1^2 = 15^2 + 10^2$ [11, book II, problem 9]
d=3	$(\mathscr{I}_2)$	$12^3 + 1^3 = 10^3 + 9^3$ [6, due to Frénicle]
d=4	$(\mathscr{I}_2)$	$158^4 + 59^4 = 134^4 + 133^4$
		or $12231^4 + 2903^4 = 10381^4 + 10203^4$ [16]
d=5	$(\mathscr{I}_3)$	$67^5 + 28^5 + 24^5 = 62^5 + 54^5 + 3^5$ (another first in [52])
d=6	$(\mathscr{I}_3)$	$23^6 + 15^6 + 10^6 = 22^6 + 19^6 + 3^6$ [62]
d=7	$(\mathscr{I}_4)$	$149^7 + 123^7 + 14^7 + 10^7 = 146^7 + 129^7 + 90^7 + 15^7$ [13]
d=8	$(\mathscr{I}_5)$	$43^8 + 20^8 + 11^8 + 10^8 + 1^8 = 41^8 + 35^8 + 32^8 + 28^8 + 5^8$: see [14]
d=9	$(\mathscr{I}_6)$	$23^9 + 18^9 + 14^9 + 2 \cdot 13^9 + 1^9 =$
		$22^9 + 21^9 + 15^9 + 10^9 + 9^9 + 5^9$ [41]
d=10	$(\mathscr{I}_7)$	$38^{10} + 33^{10} + 2 \cdot 26^{10} + 15^{10} + 8^{10} + 1^{10} = 36^{10} + 35^{10} + 32^{10} +$
		$29^{10} + 24^{10} + 23^{10} + 22^{10}$ (another first in [52])

Table 4.4.1

Note that a positive answer to Euler's conjecture — for $k \ge 5$ $a^k + b^k = c^k + d^k$ has only trivial solutions in integers — would imply that the set of kth powers has $(\mathscr{I}_2)$. This conjecture has been neither proved nor disproved for any value of $k \ge 5$ (see [71] and [14]).
Now let $E = \{n_k\} \subseteq \mathbb{Z}$ be a set of polynomial growth: $|n_k| \preccurlyeq k^d$ for some $d \ge 1$. Then $|E \cap [-n, n]| \succcurlyeq n^{1/d}$ and by [66, Th. 3.6], E fails the $\Lambda(p)$ property for $p > 2d$ and E fails *a fortiori* $(\mathscr{I}_{d+1})$. In the special case $E = \{P(k)\}$ for a polynomial P of degree d, we can exhibit a huge explicit arithmetical relation. Recall that

$$\Delta^j P(k) = \sum_{i=0}^{j} \binom{j}{i} (-1)^i P(k-i) \ , \ \sum_{i=0}^{j} \binom{j}{i} (-1)^i = 0 \ , \ \sum_{i=0}^{j} \binom{j}{i} = 2^j. \qquad (16)$$

As $\Delta^{d+1} P(k) = 0$, this makes E fail $(\mathscr{I}_{2^d})$, which is coarse.

Conclusion By Theorem 3.4.2, property $(\mathscr{I}_n)$ yields directly (*umbs*) in the spaces $\mathrm{L}^{2p}(\mathbb{T})$, $p \le n$ integer. But we do not know whether $(\mathscr{I}_\infty)$ ensures (*umbs*) in spaces $\mathrm{L}^p(\mathbb{T})$, p not an even integer.

5 Metric unconditional approximation property

As we investigate simultaneously real and complex (*umap*), it is convenient to introduce a subgroup $\mathbb{S}$ of $\mathbb{T}$ corresponding to each case. Thus, if $\mathbb{S} = \mathbb{D} = \{-1, 1\}$, then the following applies to real (*umap*). If $\mathbb{S} = \mathbb{T} = \{\epsilon \in \mathbb{C} : |\epsilon| = 1\}$, it applies to complex (*umap*).
He who is first and foremost interested in the application to harmonic analysis may concentrate on the equivalence $(ii) \Leftrightarrow (iv)$ in Theorem 5.3.1 and then pass on to Section 7.

5.1 Definition

We start with defining the metric unconditional approximation property ((*umap*) for short). Recall that $\Delta T_k = T_k - T_{k-1}$ (where $T_0 = 0$).

Definition 5.1.1 *Let X be a separable Banach space.*

(*i*) *A sequence $\{T_k\}$ of operators on X is an approximating sequence (a.s. for short) if each T_k has finite rank and $\|T_k x - x\| \to 0$ for every $x \in X$. If X admits an a.s., it has the bounded approximation property. An a.s. of commuting projections is a finite-dimensional Schauder decomposition ((fdd) for short).*

(*ii*) *[18] X has the unconditional approximation property (uap) if there are an a.s. $\{T_k\}$ and a constant C such that*

$$\Big\| \sum_{k=1}^{n} \epsilon_k \Delta T_k \Big\| \le C \quad \text{for all } n \text{ and } \epsilon_k \in \mathbb{S}. \tag{17}$$

The (uap) constant is the least such C.

(*iii*) *[8, §3] X has the metric unconditional approximation property (umap) if it has (uap) with constant $1+\varepsilon$ for any $\varepsilon > 0$.*

Property (*ii*) is the approximation property which most appropriately generalises the unconditional basis property. It has first been introduced by Pełczyński and Wojtaszczyk [57]. They showed that it holds if and only if X is a complemented subspace of a space with an unconditional (*fdd*). By [44, Th. 1.g.5], this implies that X is subspace of a space with an unconditional basis. Thus, neither $\mathrm{L}^1([0,1])$ nor $\mathscr{C}([0,1])$ share (*uap*).
Property (*iii*) has been introduced by Casazza and Kalton as an extreme form of metric approximation. It has been studied in [8, §3], [23, §8,9], [22] and [21, §IV].
There is a simple and very useful criterion for (*umap*):

Proposition 5.1.2 ([8, Th. 3.8] and [23, Lemma 8.1]) *Let X be a separable Banach space. X has (umap) if and only if there is an a.s. $\{T_k\}$ such that*

$$\sup_{\epsilon \in \mathbb{S}} \|(\mathrm{Id} - T_k) + \epsilon T_k\| \xrightarrow[k\to\infty]{} 1. \tag{18}$$

If (18) holds, we say that $\{T_k\}$ realises (*umap*). A careful reading of the above mentioned proof also gives the following results for a.s. that satisfy $T_{n+1}T_n = T_n$.

Proposition 5.1.3 *Let X be a separable Banach space.*

(*i*) *Let $\{T_k\}$ be an a.s. for X such that $T_{n+1}T_n = T_n$. A subsequence $\{T'_k\}$ of $\{T_k\}$ realises 1-(uap) in X if and only if for all $k \ge 1$ and $\epsilon \in \mathbb{S}$*

$$\|\mathrm{Id} - (1+\epsilon)T'_k\| = 1.$$

(*ii*) *X has metric unconditional (fdd) if and only if there is an (fdd) $\{T_k\}$ such that (18) holds.*

5.2 Characterisation of (*umap*). Block unconditionality

We want to characterise (*umap*) in an even simpler way than Proposition 5.1.2. Relation (18) and the method of [36, Th. 4.2], suggest considering some unconditionality condition between a certain "break" and a certain "tail" of X. We propose two such notions.

Definition 5.2.1 *Let X be a separable Banach space.*

(*i*) *Let τ be a vector space topology on X. Then X has the property $(u(\tau))$ of τ-unconditionality if for all $u \in X$ and norm bounded sequences $\{v_j\} \subseteq X$ such that $v_j \xrightarrow{\tau} 0$*

$$\operatorname*{osc}_{\epsilon \in \mathbb{S}} \|\epsilon u + v_j\| \to 0. \tag{19}$$

(*ii*) *Let $\{T_k\}$ be a commuting a.s. X has the property $(u(T_k))$ of commuting block unconditionality if for all $\varepsilon > 0$ and $n \ge 1$ we may choose $m \ge n$ such that for all $x \in T_n B_X$ and $y \in (\mathrm{Id} - T_m)B_X$*

$$\operatorname*{osc}_{\epsilon \in \mathbb{S}} \|\epsilon x + y\| \le \varepsilon. \tag{20}$$

Thus, given a commuting a.s. $\{T_k\}$, T_nX is the "break" and $(\mathrm{Id}-T_m)X$ the "tail" of X. We have

Lemma 5.2.2 *Let X be a separable Banach space and $\{T_k\}$ a commuting a.s. for X. The following are equivalent.*
(i) X enjoys $(u(\tau))$ for some vector space topology τ such that $T_nx \xrightarrow{\tau} x$ uniformly for $x \in B_X$;
(ii) X enjoys $(u(T_k))$.

Proof. $(i) \Rightarrow (ii)$. Suppose that (ii) fails: there are $n \geq 1$ and $\varepsilon > 0$ such that for each $m \geq n$, there are $x_m \in T_nB_X$ and $y_m \in (\mathrm{Id}-T_m)B_X$ such that

$$\operatorname*{osc}_{\epsilon\in\mathbb{S}} \|\epsilon x_m + y_m\| > \varepsilon.$$

As T_nB_X is compact, we may suppose by extracting a convergent subsequence that $x_m = x$. Let τ be as in (i): then $y_m \xrightarrow{\tau} 0$ and $(u(\tau))$ must fail.
$(ii) \Rightarrow (i)$. Let us define a vector space topology τ by

$$x_n \xrightarrow{\tau} 0 \quad\Longleftrightarrow\quad \forall k \ \|T_kx_n\| \to 0.$$

Then $T_nx \xrightarrow{\tau} x$ uniformly on B_X. Indeed, $T_k(T_nx - x) = (T_n - \mathrm{Id})T_kx$ and $T_n - \mathrm{Id}$ converges uniformly to 0 on T_kB_X which is norm compact.
Let us check $(u(\tau))$. Let $u \in B_X$ and $\{v_j\} \subseteq B_X$ be such that $v_j \xrightarrow{\tau} 0$. Let $\varepsilon > 0$. There is $n \geq 1$ such that $\|T_nu - u\| \leq \varepsilon$. Choose m such that (20) holds for $x \in T_nB_X$ and $y \in (\mathrm{Id}-T_m)B_X$. Then choose $k \geq 1$ such that $\|T_mv_j\| \leq \varepsilon$ for $j \geq k$. We have, for any $\epsilon \in \mathbb{S}$,

$$\begin{aligned}\|\epsilon u + v_j\| &\leq \|\epsilon T_nu + (\mathrm{Id}-T_m)v_j\| + \|T_nu - u\| + \|T_mv_j\|\\ &\leq \|T_nu + (\mathrm{Id}-T_m)v_j\| + 3\varepsilon \leq \|u + v_j\| + 5\varepsilon.\end{aligned}$$

Thus we have (19). ■
In order to obtain (*umap*) from block independence, we shall have to construct unconditional skipped blocking decompositions.

Definition 5.2.3 *Let X be a separable Banach space. X admits unconditional skipped blocking decompositions if for each $\varepsilon > 0$, there is an unconditional a.s. $\{S_k\}$ such that for all $0 \leq a_1 < b_1 < a_2 < b_2 < \dots$ and $x_k \in (S_{b_k} - S_{a_k})X$*

$$\sup_{\epsilon_k\in\mathbb{S}} \Big\|\sum \epsilon_k x_k\Big\| \leq (1+\varepsilon)\Big\|\sum x_k\Big\|.$$

5.3 Main theorem: convex combinations of multipliers

We have

Theorem 5.3.1 *Consider the following properties for a separable Banach space X.*
(i) There are an unconditional commuting a.s. $\{T_k\}$ and a vector space topology τ such that X enjoys $(u(\tau))$ and $T_kx \xrightarrow{\tau} x$ uniformly for $x \in B_X$;
(ii) X enjoys $(u(T_k))$ for an unconditional commuting a.s. $\{T_k\}$;
(iii) X admits unconditional skipped blocking decompositions;
(iv) X has (umap).
Then $(iv) \Rightarrow (i) \Leftrightarrow (ii) \Rightarrow (iii)$. If X has finite cotype, then $(iii) \Rightarrow (iv)$.

Proof. $(i) \Leftrightarrow (ii)$ holds by Lemma 5.2.2.
$(iv) \Rightarrow (ii)$. By Godefroy–Kalton's [21, Th. IV.1], there is in fact an a.s. $\{T_k\}$ that satisfies (18) such that $T_kT_l = T_{\min(k,l)}$ if $k \neq l$.
Let C be a uniform bound for $\|T_k\|$. Let $\varepsilon > 0$ and $n \geq 1$. There is $m \geq n+2$ such that

$$\sup_{\epsilon\in\mathbb{S}} \|\epsilon T_{m-1} + (\mathrm{Id}-T_{m-1})\| \leq 1 + \varepsilon/2C.$$

Let $x \in T_n B_X$ and $y \in (\mathrm{Id} - T_m)B_X$. As $x - T_{m-1}x = 0$ and $T_{m-1}y = 0$,

$$\epsilon x + y = \epsilon T_{m-1}(x+y) + (\mathrm{Id} - T_{m-1})(x+y),$$

and, for all $\epsilon \in \mathbb{S}$,

$$\|\epsilon x + y\| \leq (1 + \varepsilon/2C)\|x+y\| \leq \|x+y\| + \varepsilon.$$

$(ii) \Rightarrow (iii)$. By a perturbation [70, proof of Lemma III.9.2], we may suppose that $T_k T_l = T_{\min(k,l)}$ if $k \neq l$. Let $\varepsilon > 0$ and choose a sequence of $\eta_j > 0$ such that $1 + \varepsilon_j = \prod_{i \leq j}(1+\eta_i) < 1 + \varepsilon$ for all j. By (ii), there is a subsequence $\{S_j = T_{k_j}\}$ such that $k_0 = 0$ and thus $S_0 = 0$, and

$$\sup_{\epsilon \in \mathbb{S}} \|x + \epsilon y\| \leq (1+\eta_j)\|x+y\| \tag{21}$$

for $x \in (\mathrm{Id} - S_j)X$ and $y \in S_{j-1}X$. Let us show that it is an unconditional skipped blocking decomposition: we shall prove by induction that

$$(H_j) \qquad \begin{cases} \sup_{\epsilon_i \in \mathbb{S}} \Big\| x + \sum_{i=1}^n \epsilon_i x_i \Big\| \leq (1+\varepsilon_j) \Big\| x + \sum_{i=1}^n x_i \Big\| \text{ for } x \in (\mathrm{Id} - S_j)X \\ \text{and } x_i \in (S_{b_i} - S_{a_i})X \ (0 \leq a_1 < b_1 < \cdots < a_n < b_n \leq j-1). \end{cases}$$

- (H_1) trivially holds.
- Assume (H_i) holds for $i < j$. Let x and x_i as in (H_j). Let $\epsilon_i \in \mathbb{S}$. Then

$$\Big\| x + \sum_{i=1}^n \epsilon_i x_i \Big\| \leq (1+\eta_j) \Big\| x + \overline{\epsilon_n} \sum_{i=1}^n \epsilon_i x_i \Big\| = (1+\eta_j) \Big\| x + x_n + \sum_{i=1}^{n-1} \overline{\epsilon_n} \epsilon_i x_i \Big\|$$

by (21). Note that $x + x_n \in (\mathrm{Id} - S_{a_n})X$: an application of (H_{a_n}) yields (H_j).

$(iii) \Rightarrow (iv)$. Let $\varepsilon > 0$, $n > 1$. There is an unconditional skipped blocking decomposition $\{S_k\}$. Let C_u be the *(uap)* constant of $\{S_k\}$. Let

$$V_{i,j} = S_{in+j-1} - S_{(i-1)n+j} \qquad \text{for } 1 \leq j \leq n \text{ and } i \geq 0.$$

The jth skipped blocks are

$$U_j = \mathrm{Id} - \sum_i V_{i,j} = \sum_i \Delta S_{in+j};$$

then $\sum_{j=1}^n U_j = \mathrm{Id}$. Let

$$R_i = \frac{1}{n-1} \sum_{j=1}^n V_{i,j};$$

then R_i has finite rank and

$$R_0 + R_1 + \cdots = (n\mathrm{Id} - \mathrm{Id})/(n-1) = \mathrm{Id}.$$

Thus $W_j = \sum_{i \leq j} R_i$ defines an a.s. We may bound its *(uap)* constant. First, since $\{S_k\}$ is a skipped blocking decomposition,

$$\begin{aligned} \forall x \in B_X \quad \sup_{\epsilon_i \in \mathbb{S}} \Big\| \sum \epsilon_i R_i x \Big\| &\leq \frac{1}{n-1} \sum_{j=1}^n \sup_{\epsilon_i \in \mathbb{S}} \Big\| \sum_i \epsilon_i V_{i,j} x \Big\| \\ &\leq \frac{1+\varepsilon}{n-1} \sum_{j=1}^n \|x - U_j x\| \\ &\leq \frac{1+\varepsilon}{n-1} \Big(n + \sum_{j=1}^n \|U_j x\| \Big). \end{aligned}$$

Let us bound $\sum_1^n \|U_j x\|$. Let $q < \infty$ be the cotype of X and C_c its cotype constant. Then by Hölder's inequality we have for all $x \in B_X$

$$\begin{aligned}\sum \|U_j x\| &\le n^{1-1/q}\Big(\sum \|U_j x\|^q\Big)^{1/q} \\ &\le n^{1-1/q} C_c \cdot \operatorname*{average}_{\pm} \Big\|\sum \pm U_j x\Big\| \le n^{1-1/q} C_c C_u. \end{aligned} \tag{22}$$

Thus the (*uap*) constant of $\{W_j\}$ is at most $(1+\varepsilon)(n + C_c C_u n^{1-1/q})/(n-1)$. As ε is arbitrarily little and n arbitrarily large, X has (*umap*). ∎

Remark 5.3.2 How does Theorem 5.3.1 look in the special cases where τ is the weak or the weak* topology ? They correspond to the classical cases where the a.s. is shrinking *vs.* boundedly complete.

We may remove the cotype assumption in Theorem 5.3.1 (*iii*) ⇒ (*iv*) if the space has the properties of commuting ℓ_1-(*ap*) or ℓ_q-(*fdd*) for $q < \infty$, which will be introduced in Section 6:

Theorem 5.3.3 *Consider the following properties for a separable Banach space X.*
(i) There are a commuting ℓ_1-a.s. or an ℓ_q-(fdd) $\{T_k\}$, $q < \infty$, and a vector space topology τ such that X enjoys $(u(\tau))$ and $T_k x \xrightarrow{\tau} x$ uniformly for $x \in B_X$;
(ii) X enjoys $(u(T_k))$ for a commmuting ℓ_1-a.s. or an ℓ_q-(fdd) $\{T_k\}$, $q < \infty$;
(iii) X admits unconditional skipped blocking decompositions and one may in fact take an ℓ_1-a.s. or an ℓ_q-(fdd) $\{T_k\}$, $q < \infty$, in its definition 5.2.3;
(iv) X has (umap).
Then (i) ⇔ (ii) ⇒ (iii) ⇒ (iv).

Proof. Part (*i*) ⇔ (*ii*) ⇒ (*iii*) goes as before. To prove (*iii*) ⇒ (*iv*), note that in the proof of Theorem 5.3.1 (*iii*) ⇒ (*iv*), one may replace the estimate in (22) by

$$\forall x \in B_X \quad \sum \|U_j x\| \le n^{1-1/q}\Big(\sum \|U_j x\|^q\Big)^{1/q} \le n^{1-1/q} C_\ell,$$

where C_ℓ is the ℓ_1-(*ap*) or the ℓ_q-(*fdd*) constant. ∎

6 The p-additive approximation property ℓ_p-(*ap*)

6.1 Definition

Definition 6.1.1 *Let X be a separable Banach space.*
(i) X has the p-additive approximation property ℓ_p-(ap) if there are an a.s. $\{T_k\}$ and a constant C such that

$$C^{-1}\|x\| \le \Big(\sum \|\Delta T_k x\|^p\Big)^{1/p} \le C\|x\| \tag{23}$$

for all $x \in X$. The ℓ_p-(ap) constant is the least such C.
(ii) X has the metric p-additive approximation property ℓ_p-(map) if it has ℓ_p-(ap) with constant $1+\varepsilon$ for any $\varepsilon > 0$.

Note that ℓ_p-(*ap*) implies (*uap*) and ℓ_p-(*map*) implies (*umap*). Note also that in (23), the left inequality is trivial with $C = 1$ if $p = 1$; the right inequality is always achieved for some C if $p = \infty$.
Property (*ii*) is implicit in Kalton–Werner's [36] investigation of subspaces of L^p that are almost isometric to subspaces of ℓ_p: see Section 6.4.
The proof of Proposition 5.1.2 can be adapted to yield

Proposition 6.1.2 *Let X be a separable Banach space.*
(i) If there is an a.s. $\{T_k\}$ such that

$$\Big(\|x - T_k x\|^p + \|T_k x\|^p\Big)^{1/p} \xrightarrow[k\to\infty]{} 1 \tag{24}$$

uniformly on the unit sphere, then X has ℓ_p-(map). The converse holds if $p = 1$.

(ii) X has a metric ℓ_p-(fdd) if and only if there is an (fdd) $\{T_k\}$ such that (24) holds.

We shall say that $\{T_k\}$ realises ℓ_p-*(map)* if it satisfies (24).

Proof. Let $\{T_k\}$ be an a.s. that satisfies (24) and $\varepsilon > 0$. By a perturbation [30, Lemma 2.4], we may suppose that $T_{k+1}T_k = T_k$. Choose a sequence of $\eta_j > 0$ such that $1 + \varepsilon_k = \prod_{j \le k}(1 + \eta_j) \le 1 + \varepsilon$ for each k. We may assume by taking a subsequence of the T_k's that for all k and $x \in X$,

$$(1+\eta_k)^{-1}\|x\| \le \Big(\|x - T_k x\|^p + \|T_k x\|^p\Big)^{1/p} \le (1+\eta_k)\|x\|. \tag{25}$$

We then prove by induction the hypothesis (H_k)

$$\forall x \in X \quad (1+\varepsilon_k)^{-1}\|x\| \le \Big(\|x - T_k x\|^p + \sum_{j=1}^{k} \|\Delta T_j x\|^p\Big)^{1/p} \le (1+\varepsilon_k)\|x\|.$$

▪ (H_1) is true.
▪ Suppose (H_{k-1}) is true. Let $x \in X$. Note that

$$x - T_k x = (\mathrm{Id} - T_k)(x - T_{k-1}x) \quad , \quad \Delta T_k x = T_k(x - T_{k-1}x).$$

By (25), we get

$$\big(\|x - T_k x\|^p + \|\Delta T_k x\|^p\big)^{1/p} \le (1+\eta_k)\|x - T_{k-1}x\|.$$

Hence, by (H_{k-1}),

$$\Big(\|x - T_k x\|^p + \sum_{j=1}^{k} \|\Delta T_j x\|^p\Big)^{1/p} \le$$

$$\le \quad (1+\eta_k)\Big(\|x - T_{k-1}x\|^p + \sum_{j=1}^{k-1} \|\Delta T_j x\|^p\Big)^{1/p} \le (1+\varepsilon_k)\|x\|.$$

▪ We obtain the lower bound in the same way. Thus the induction is complete.
Hence $\{T_k\}$ realises ℓ_p-*(ap)* with constant $1+\varepsilon$. As ε is arbitrary, X has ℓ_p-*(map)*.
If X has ℓ_1-*(map)*, then for each $\varepsilon > 0$, there is a sequence $\{S_k\}$ such that

$$\|x\| \le \|x - S_k x\| + \|S_k x\| \le \sum \|\Delta S_k x\| \le (1+\varepsilon)\|x\|$$

for all $x \in X$. By a diagonal argument, this gives an a.s. $\{T_k\}$ satisfying (24).
(iii) If X has a metric ℓ_p-*(fdd)*, then for each $\varepsilon > 0$ there is a *(fdd)* $\{T_k\}$ such that (23) holds with $C = 1+\varepsilon$. Then, for all $k \ge 1$,

$$(1-\varepsilon)\|T_k x\| \le \Big(\sum_{j=1}^{k} \|\Delta T_j x\|^p\Big)^{1/p} \le (1+\varepsilon)\|T_k x\|$$

$$(1-\varepsilon)\|x - T_k x\| \le \Big(\sum_{j=k+1}^{\infty} \|\Delta T_j x\|^p\Big)^{1/p} \le (1+\varepsilon)\|x - T_k x\|.$$

Thus

$$(1-\varepsilon)/(1+\varepsilon)\|x\| \le \Big(\|x - T_k x\|^p + \|T_k x\|^p\Big)^{1/p} \le (1+\varepsilon)/(1-\varepsilon)\|x\|.$$

By a diagonal argument, this gives an *(fdd)* $\{T_k\}$ satisfying (24). ∎

Question 6.1.3 What about the converse in Proposition 6.1.2*(i)* for $p > 1$?

6.2 Some consequences of ℓ_p-(*ap*)

We start with the simple

Proposition 6.2.1 *Let X be a separable Banach space.*
(i) If X has ℓ_p-(ap) with constant C, then X is C-isomorphic to a subspace of an ℓ_p-sum of finite dimensional subspaces of X.
(ii) If furthermore X is a subspace of L^q, then X is $(C+\varepsilon)$-isomorphic to a subspace of $(\bigoplus \ell_q^n)_p$ for any given $\varepsilon > 0$.
(iii) In particular, if a subspace of L^p has ℓ_p-(ap) with constant C, then it is $(C+\varepsilon)$-isomorphic to a subspace of ℓ_p for any given $\varepsilon > 0$. If a subspace of L^p has ℓ_p-(map), then it is almost isometric to subspaces of ℓ_p.

Proof. (i) Indeed, $\Phi\colon X \hookrightarrow (\bigoplus \operatorname{im} \Delta T_i)_p$, $x \mapsto \{\Delta T_i x\}_{i\ge 1}$ is an embedding: for all $x \in X$

$$C^{-1}\|x\|_X \le \|\Phi x\| = \Big(\sum \|\Delta T_i x\|_X^p\Big)^{1/p} \le C\|x\|_X.$$

$(ii\,\&\,iii)$ Recall that, given $\varepsilon > 0$, a finite dimensional subspace of L^q is $(1+\varepsilon)$-isomorphic to a subspace of ℓ_q^n for some $n \ge 1$. ■

We have in particular (see [29, §VIII, Def. 7] for the definition of Hilbert sets)

Corollary 6.2.2 *Let $E \subseteq \mathbb{Z}$ be infinite.*
(i) No $\mathrm{L}^q_E(\mathbb{T})$ $(1 \le q < \infty)$ has ℓ_p-(ap) for $p \ne 2$.
(ii) No $\mathscr{C}_E(\mathbb{T})$ has ℓ_q-(ap) for $q \ne 1$. If E is a Hilbert set, then $\mathscr{C}_E(\mathbb{T})$ fails ℓ_1-(ap).

Proof. This is a consequence of Proposition 6.2.1(i): every infinite E contains a Sidon set and thus a $\Lambda(2 \vee p)$ set. So $\mathrm{L}^p_E(\mathbb{T})$ contains ℓ_2. Also, if E is a Hilbert set, then $\mathscr{C}_E(\mathbb{T})$ contains c_0 by [42, Th. 2]. ■

However, there is a Hilbert set E such that $\mathscr{C}_E(\mathbb{T})$ has complex (*umap*): see [43, Th. 10]. The class of sets E such that $\mathscr{C}_E(\mathbb{T})$ has ℓ_1-(*ap*) contains the Sidon sets and Blei's sup-norm-partitioned sets [7].

6.3 Characterisation of ℓ_p-(*map*)

Recall [36, Def. 4.1]:

Definition 6.3.1 *Let X be a separable Banach space.*
(i) Let τ be a vector space topology on X. X enjoys property $(m_p(\tau))$ if for all $x \in X$ and norm bounded sequences $\{y_j\}$ such that $y_j \xrightarrow{\tau} 0$

$$\big|\|x+y_j\| - \big(\|x\|^p + \|y_j\|^p\big)^{1/p}\big| \to 0.$$

(ii) X enjoys the property $(m_p(T_k))$ for a commuting a.s. $\{T_k\}$ if for all $\varepsilon > 0$ and $n \ge 1$ we may choose $m \ge n$ such that for all $x \in B_X$

$$\big|\|T_n x + (\mathrm{Id} - T_m)x\| - \big(\|T_n x\|^p + \|(\mathrm{Id} - T_m)x\|^p\big)^{1/p}\big| \le \varepsilon.$$

Then [36, Th. 4.2] may be read as follows

Theorem 6.3.2 *Let $1 \le p < \infty$ and consider the following properties for a separable Banach space X.*
(i) There are an unconditional commuting a.s. $\{T_k\}$ and a vector space topology τ such that X enjoys $(m_p(\tau))$ and $T_k x \xrightarrow{\tau} x$ uniformly for $x \in B_X$;
(ii) X enjoys the property $(m_p(T_k))$ for an unconditional commuting a.s. $\{T_k\}$.
(iii) X has ℓ_p-(map).
Then $(i) \Leftrightarrow (ii)$. If X has finite cotype, then $(ii) \Rightarrow (iii)$.

As for Theorem 5.3.1, we may remove the cotype assumption if X has commuting ℓ_1-(*ap*) or ℓ_p-(*fdd*), $p < \infty$:

Theorem 6.3.3 *Let* $1 \le p < \infty$*. Consider the following properties for a separable Banach space* X*.*
(i) There are an ℓ_p*-(fdd) (or just a commuting* ℓ_1*-a.s. in the case* $p = 1$*)* $\{T_k\}$ *and a vector space topology* τ *such that* X *enjoys* $(m_p(\tau))$ *and* $T_k x \xrightarrow{\tau} x$ *uniformly for* $x \in B_X$*;*
(ii) X *enjoys* $(m_p(T_k))$ *for an* ℓ_p*-(fdd) (or just a commuting* ℓ_1*-a.s. in the case* $p = 1$*)* $\{T_k\}$*.*
(iii) X *has* ℓ_p*-(map).*
Then (i) $\Leftrightarrow$ *(ii)* $\Rightarrow$ *(iii).*

6.4 Subspaces of L^p with ℓ_p-(*map*)

Although no translation invariant subspace of $\mathrm{L}^p(\mathbb{T})$ has ℓ_p-(*ap*) for $p \ne 2$, Proposition 6.2.1 (*iii*) is not void. By the work of Godefroy, Kalton, Li and Werner [36, 22], we get examples of subspaces of L^p with ℓ_p-(*map*) and even a characterisation of such spaces.
Let us treat the case $p = 1$. Recall first that a space X has the 1-strong Schur property when, given $\delta \in]0, 2]$ and $\varepsilon > 0$, any normalised δ-separated sequence in X contains a subsequence that is $(2/\delta + \varepsilon)$-equivalent to the unit vector basis of ℓ_1 (see [65]). In particular, a gliding hump argument shows that any subspace of ℓ_1 shares this property. By Proposition 6.2.1(*iii*), a space X with ℓ_1-(*map*) also does. Now recall the main theorem of [22]:

Theorem *Let* X *be a subspace of* L^1 *with the approximation property. Then the following properties are equivalent:*
(i) The unit ball of X *is compact and locally convex in measure;*
(ii) X *has (umap) and the 1-strong Schur property;*
(iii) X *is* $(1+\varepsilon)$*-isomorphic to a* w^**-closed subspace* X_ε *of* ℓ_1 *for any* $\varepsilon > 0$*.*
We may then add to these three the fourth equivalent property
(iv) X *has* ℓ_1*-(map).*
Proof. We just showed that (*ii*) holds when X has ℓ_1-(*map*). Now suppose we have (*iii*) and let $\varepsilon > 0$. Thus there is a quotient Z of c_0 such that Z^* has the approximation property and Z^* is $(1+\varepsilon)$-isomorphic to X.
Let us show that any such Z^* has ℓ_1-(*map*). Z has beforehand the metric approximation property, with say $\{R_n\}$, because Z^* has it as a dual separable space. By [24, Th. 2.2], $\{R_n^*\}$ is a metric a.s. in Z^*. Let Q be the canonical quotient map from c_0 onto Z. Let $\{P_n\}$ be the sequence of projections associated to the natural basis of c_0. Then $\{P_n^*\}$ is also an a.s. in ℓ_1. Thus

$$\|P_n^* Q^* x^* - Q^* R_n^* x^*\|_{\ell_1} \to 0 \qquad \text{for any } x^* \in Z^*.$$

By Lebesgue's dominated convergence theorem (see [34, Th. 1]), $QP_n - R_n Q \to 0$ weakly in the space $\mathscr{K}(c_0, Z)$ of compact operators from c_0 to Z. By Mazur's theorem, there are convex combinations $\{C_n\}$ of $\{P_n\}$ and $\{D_n\}$ of $\{R_n\}$ such that $\|QC_n - D_n Q\|_{\mathscr{L}(c_0,Z)} \to 0$. Thus

$$\|C_n^* Q^* - Q^* D_n^*\|_{\mathscr{L}(Z^*,\ell_1)} \to 0. \tag{26}$$

Furthermore $C_n^* : \ell_1 \to \ell_1$ has the form $C_n^*(x_1, x_2, \dots) = (t_1 x_1, t_2 x_2, \dots)$ with $0 \le t_i \le 1$. Therefore, defining $Q^* a = (a_1, a_2, \dots)$,

$$\begin{aligned} \|C_n^* Q^* a\|_1 + \|Q^* a - C_n^* Q^* a\|_1 &= \\ &= \|(t_1 a_1, t_2 a_2, \dots)\|_1 + \|((1-t_1)a_1, (1-t_2)a_2, \dots)\|_1 \\ &= \sum (|t_i| + |1 - t_i|)|a_i| = \sum |a_i| = \|Q^* a\|_1. \end{aligned} \tag{27}$$

As $\{D_n^*\}$ is still an a.s. for Z^*, $\{D_n^*\}$ realises ℓ_1-(*map*) in Z^* by (27), (26) and Proposition 6.1.2(*i*). Thus X has ℓ_1-(*ap*) with constant $1 + 2\varepsilon$. As ε is arbitrary, X has ℓ_1-(*map*). ∎

For $1 < p < \infty$, we have similarly by [36, Th. 4.2]

Proposition 6.4.1 *Let* $1 < p < \infty$ *and* X *be a subspace of* L^p *with the approximation property. The following are equivalent:*
(i) X *is* $(1+\varepsilon)$*-isomorphic to a subspace* X_ε *of* ℓ_p *for any* $\varepsilon > 0$*.*

(*ii*) *X has* ℓ_p*-(map).*

Proof. (*ii*) $\Rightarrow$ (*i*) is in Proposition 6.2.1. For (*i*) $\Rightarrow$ (*ii*), it suffices to prove that any subspace Z of ℓ_p with the approximation property has ℓ_p-(*map*).
As Z is reflexive, Z admits a commuting shrinking a.s. $\{R_n\}$. Let i be the injection of Z into ℓ_p. Let $\{P_n\}$ be the sequence of projections associated to the natural basis of ℓ_p. It is also an a.s. for $\ell_{p'}$. Thus

$$\|i^* P_n^* x^* - R_n^* i^* x^*\|_{Z^*} \to 0 \qquad \text{for any } x^* \in \ell_{p'}.$$

As before, there are convex combinations $\{C_n\}$ of $\{P_n\}$ and $\{D_n\}$ of $\{R_n\}$ such that $\|C_n i - iD_n\| \to 0$. The convex combinations are finite and may be chosen not to overlap, so that for each $n \geq 1$ there is $m > n$ such that

$$\|C_n x + (\mathrm{Id} - C_m)x\| = \big(\|C_n x\|^p + \|(\mathrm{Id} - C_m)x\|^p\big)^{1/p}$$

for $x \in \ell_p$. Thus Z satisfies the property $(m_p(D_n))$. Following the lines of [18, Lemma 1], we observe that $\{D_n\}$ is a commuting unconditional a.s. since $\{P_n\}$ is. By Theorem 6.3.2, Z has ℓ_p-(*map*). ∎

7 (*uap*) and (*umap*) in translation invariant spaces

Recall that $\mathbb{S}$ is a subgroup of $\mathbb{T}$. If $\mathbb{S} = \mathbb{D} = \{-1, 1\}$, the following applies to real (*umap*). If $\mathbb{S} = \mathbb{T} = \{\epsilon \in \mathbb{C} : |\epsilon| = 1\}$, it applies to complex (*umap*).

7.1 General properties. Isomorphic case

$\mathrm{L}^p(\mathbb{T})$ spaces $(1 < p < \infty)$ are known to have an unconditional basis; furthermore, they have an unconditional (*fdd*) in translation invariant subspaces $\mathrm{L}^p_{I_k}(\mathbb{T})$: this is a corollary of Littlewood–Paley theory [45]. One may choose $I_0 = \{0\}$ and $I_k = \,]-2^k, -2^{k-1}] \cup [2^{k-1}, 2^k[$. Thus any $\mathrm{L}^p_E(\mathbb{T})$ $(1 < p < \infty)$ has an unconditional (*fdd*) in translation invariant subspaces $\mathrm{L}^p_{E\cap I_k}(\mathbb{T})$. The spaces $\mathrm{L}^1(\mathbb{T})$ and $\mathscr{C}(\mathbb{T})$, however, do not even have (*uap*).

Proposition 7.1.1 (see [43, Lemma 5, Cor. 6, Th. 7]) *Let $E \subseteq \mathbb{Z}$ and X be a homogeneous Banach space on $\mathbb{T}$.*
(i) If X_E has (umap) (vs. (uap), ℓ_1-(ap) or ℓ_1-(map)), then some a.s. of multipliers realises it.
(ii) Let $F \subseteq E$. If X_E has (umap) (vs. (uap), ℓ_1-(ap) or ℓ_1-(map)), then so does X_F.
(iii) If $\mathscr{C}_E(\mathbb{T})$ has (umap) (vs. (uap)), then so does X_E.

Note the important property that a.s. of multipliers commute and commute with one another.
Whereas (*uap*) is always satisfied for $\mathrm{L}^p_E(\mathbb{T})$ $(1 < p < \infty)$, we have the following generalisation of [43, remark after Th. 7, Prop. 9] for the spaces $\mathrm{L}^1_E(\mathbb{T})$ and $\mathscr{C}_E(\mathbb{T})$. By the method of [21],

Lemma 7.1.2 *If X has (uap) with a commuting a.s. and $X \not\supseteq c_0$, then X is a dual space.*

Proof. Suppose $\{T_n\}$ is a commuting a.s. such that (17) holds. As $X \not\supseteq c_0$, $Px^{**} = \lim T_n^{**} x^{**}$ is well defined for each $x^{**} \in X^{**}$. As $\{T_n\}$ is an a.s., P is a projection onto X. Let us show that $\ker P$ is w^*-closed. Indeed, if $x^{**} \in \ker P$, then

$$\|T_n^{**} x^{**}\| = \lim_m \|T_m T_n^{**} x^{**}\| = \lim_m \|T_n T_m^{**} x^{**}\| = 0$$

and $T_n^{**} x^{**} = 0$. Thus

$$\ker P = \bigcap_n \ker T_n^{**}.$$

Let $M = (\ker P)_\perp$. Then $M^* = X$. ∎

Corollary 7.1.3 *Let $E \subseteq \mathbb{Z}$.*
(i) If $\mathrm{L}^1_E(\mathbb{T})$ has (uap), then E is a Riesz set.
(ii) If $\mathscr{C}_E(\mathbb{T})$ has (uap) and $\mathscr{C}_E(\mathbb{T}) \not\supseteq c_0$, then E is a Rosenthal set.

Proof. In both cases, Lemma 7.1.2 shows that the two spaces are separable dual spaces and thus have the Radon–Nikodym property. We may now apply Lust-Piquard's characterisation [47]. ■

There are Riesz sets E such that $\mathrm{L}^1_E(\mathbb{T})$ fails (*uap*): indeed, the family of Riesz sets is coanalytic [75] while the second condition is in fact analytic. There are Rosenthal sets that cannot be sup-norm-partitioned [7].
The converse of Proposition 7.1.1(*iii*) does not hold: $\mathrm{L}^1_E(\mathbb{T})$ may have (*uap*) while $\mathscr{C}_E(\mathbb{T})$ fails this property. We have

Proposition 7.1.4 *Let $E \subseteq \mathbb{Z}$.*
(*i*) *The Hardy space $H^1(\mathbb{T}) = \mathrm{L}^1_{\mathbb{N}}(\mathbb{T})$ has (uap).*
(*ii*) *The disc algebra $A(\mathbb{T}) = \mathscr{C}_{\mathbb{N}}(\mathbb{T})$ fails (uap). More generally, if $\mathbb{Z} \setminus E$ is a Riesz set, then $\mathscr{C}_E(\mathbb{T})$ fails (uap).*

Proof. (*i*) Indeed, $H^1(\mathbb{T})$ has an unconditional basis [48]. Note that the first unconditional a.s. for $H^1(\mathbb{T})$ appears in [49, §II, introduction] with the help of Stein's [72, 73] multiplier theorem (see also [77]).
(*ii*) Let $\Delta \subset \mathbb{T}$ be the Cantor set. By Bishop's improvement [4] of Rudin–Carleson's interpolation theorem, every function in $\mathscr{C}(\Delta)$ extends to a function in $\mathscr{C}_E(\mathbb{T})$ if $\mathbb{Z} \setminus E$ is a Riesz set. By [56, main theorem], this implies that $\mathscr{C}(\Delta)$ embeds in $\mathscr{C}_E(\mathbb{T})$. Then $\mathscr{C}_E(\mathbb{T})$ cannot have (*uap*); otherwise $\mathscr{C}(\Delta)$ would embed in a space with an unconditional basis, which is false. ■

Remark 7.1.5 Recent studies of the Daugavet Property by Kadets and Werner generalise Proposition 7.1.4(*ii*). This property of a Banach space X states that for every finite rank operator T on X $\|\mathrm{Id}+T\| = 1+\|T\|$. By [31, Th. 2.1], such an X cannot have (*uap*). Further, by [76, Th. 3.7], $\mathscr{C}_E(\mathbb{T})$ has the Daugavet Property if $\mathbb{Z} \setminus E$ is a so-called semi-Riesz set, that is if all measures with Fourier spectrum in $\mathbb{Z} \setminus E$ are diffuse.

Question 7.1.6 Is there some characterisation of sets $E \subseteq \mathbb{Z}$ such that $\mathscr{C}_E(\mathbb{T})$ has (*uap*)? Only a few classes of such sets are known: Sidon sets and sup-norm-partitioned sets, for which $\mathscr{C}_E(\mathbb{T})$ even has ℓ_1-(*ap*); certain Hilbert sets. Adapting the argument in [63], we get that $\mathscr{C}_E(\mathbb{T})$ fails (*uap*) if E contains the sum of two infinite sets.

7.2 Characterisation of (*umap*) and ℓ_p-(*map*)

Let us introduce

Definition 7.2.1 *Let $E \subseteq \mathbb{Z}$ and X be a homogeneous Banach space on $\mathbb{T}$.*
E enjoys the Fourier block unconditionality property ($\mathscr{U}$) in X whenever, for any $\varepsilon > 0$ and finite $F \subseteq E$, there is a finite $G \subseteq E$ such that for $f \in B_{X_F}$ and $g \in B_{X_{E\setminus G}}$

$$\operatorname*{osc}_{\epsilon \in \mathbb{S}} \|\epsilon f + g\|_X \leq \varepsilon. \tag{28}$$

Lemma 7.2.2 *Let $E \subseteq \mathbb{Z}$ and X be a homogeneous Banach space on $\mathbb{T}$. The following are equivalent.*
(*i*) *X_E has $(u(\tau_f))$, where τ_f is the topology of pointwise convergence of the Fourier coefficients:*

$$x_n \xrightarrow{\tau_f} 0 \iff \forall k\ \widehat{x_n}(k) \to 0.$$

(*ii*) *E enjoys ($\mathscr{U}$) in X.*
(*iii*) *X_E enjoys the property of block unconditionality for any, or equivalently for some, a.s. of multipliers $\{T_k\}$.*

Proof. (*i*) ⇒ (*ii*). Suppose that (*ii*) fails: there are $\varepsilon > 0$ and a finite F such that for each finite G, there are $x_G \in B_{X_F}$ and $y_G \in B_{X_{E\setminus G}}$ such that

$$\operatorname*{osc}_{\epsilon \in \mathbb{S}} \|\epsilon x_G + y_G\| > \varepsilon.$$

As B_{X_F} is compact, we may suppose $x_G = x$. As $y_G \xrightarrow{\tau_f} 0$, $(u(\tau_f))$ fails.

$(ii) \Rightarrow (iii)$. Let C be a uniform bound for $\|T_k\|$. Let $n \geq 1$ and $\varepsilon > 0$. Let F be the finite spectrum of T_n. Let G be such that (28) holds for all $f \in B_{X_F}$ and $g \in B_{X_{E\setminus G}}$. Now there is a term V in de la Vallée-Poussin's a.s. such that $V|_{X_G} = \mathrm{Id}|_{X_G}$ and $\|V\|_{\mathscr{L}(X_E)} \leq 3$. As V has finite rank, we may choose $m > n$ such that $\|(\mathrm{Id} - T_m)V\|_{\mathscr{L}(X_E)} = \|V(\mathrm{Id} - T_m)\|_{\mathscr{L}(X_E)} \leq \varepsilon$. Let then $x \in T_n B_{X_E}$ and $y \in (\mathrm{Id} - T_m)B_{X_E}$. We have

$$\begin{aligned}\|\epsilon x + y\| &\leq \|\epsilon x + (\mathrm{Id} - V)y\| + \varepsilon \overset{(28)}{\leq} \|x + (\mathrm{Id} - V)y\| + 4(C+1)\varepsilon + \varepsilon \\ &\leq \|x + y\| + (4C + 6)\varepsilon.\end{aligned}$$

$(iii) \Rightarrow (i)$ is proved as Lemma 5.2.2 $(ii) \Rightarrow (i)$: note that if $y_j \xrightarrow{\tau_f} 0$, then $\|Ty_j\| \to 0$ for any finite rank multiplier T. ∎

We may now prove the main result of this section.

Theorem 7.2.3 *Let $E \subseteq \mathbb{Z}$ and X be a homogeneous Banach space on $\mathbb{T}$. If X_E has (umap), then E enjoys $(\mathscr{U})$ in X. Conversely, if E enjoys $(\mathscr{U})$ in X and furthermore X_E has (uap) and finite cotype, or simply ℓ_1-(ap), then X_E has (umap). In particular,*
(i) For $1 < p < \infty$, $\mathrm{L}^p_E(\mathbb{T})$ has (umap) if and only if E enjoys $(\mathscr{U})$ in $\mathrm{L}^p(\mathbb{T})$.
(ii) $\mathrm{L}^1_E(\mathbb{T})$ has (umap) if and only if E enjoys $(\mathscr{U})$ in $\mathrm{L}^1(\mathbb{T})$ and $\mathrm{L}^1_E(\mathbb{T})$ has (uap).
(iii) If E enjoys $(\mathscr{U})$ in $\mathscr{C}(\mathbb{T})$ and $\mathscr{C}_E(\mathbb{T})$ has ℓ_1-(ap), in particular if E is a Sidon set, then $\mathscr{C}_E(\mathbb{T})$ has (umap).

Proof. Notice first that (umap) implies $(\mathscr{U})$ by Lemma 7.2.2 $(iii) \Rightarrow (ii)$.
(i) Notice that $\mathrm{L}^p_E(\mathbb{T})$ $(1 < p < \infty)$ has an unconditional (fdd) of multipliers $\{\pi_{E\cap I_k}\}$ and cotype $2 \vee p$. Thus $(\mathscr{U})$ implies (umap) by Theorem 5.3.3$(ii) \Rightarrow (iv)$.
By Lemma 7.2.2, part (ii) and (iii) follow from Theorem 5.3.1$(ii) \Rightarrow (iv)$ and Theorem 5.3.3$(ii) \Rightarrow (iv)$ respectively. ∎

Remark 7.2.4 Consider the special case $E = \{0\} \cup \{j^k\}_{k\geq 0}$, $|j| \geq 2$, and suppose X_E has complex (umap). By Theorem 7.2.3,

$$\operatorname*{osc}_{\epsilon\in\mathbb{T}} \|\epsilon a + b\,\mathrm{e}_{j^k} + c\,\mathrm{e}_{j^{k+1}}\| \xrightarrow[k\to\infty]{} 0.$$

Let us show that then $\{0, 1, j\}$ is a 1-unconditional basic sequence in X. Indeed, for any $\epsilon, \mu, \nu \in \mathbb{T}$, and choosing κ such that $\mu\kappa = \nu\kappa^j$,

$$\begin{aligned}\|\epsilon a + \mu b\,\mathrm{e}_1 + \nu c\,\mathrm{e}_j\| &= \|\epsilon a + \mu\kappa b\,\mathrm{e}_1 + \nu\kappa^j c\,\mathrm{e}_j\| \\ &= \|\epsilon\overline{\mu\kappa} a + b\,\mathrm{e}_1 + c\,\mathrm{e}_j\| = \|\epsilon\overline{\mu\kappa} a + b\,\mathrm{e}_{j^k} + c\,\mathrm{e}_{j^{k+1}}\|\end{aligned}$$

whose oscillation tends to 0 with k. By Proposition 3.2.1(i), X_E fails complex (umap) if X is $\mathrm{L}^p(\mathbb{T})$, p not an even integer, or $\mathscr{C}(\mathbb{T})$. By Proposition 3.2.1(ii), $\mathrm{L}^{2n}_E(\mathbb{T})$, $n \geq 1$ integer, fails complex (umap) if j is positive and $n \geq j$, or if j is negative and $n \geq |j| + 1$.

The study of ℓ_p-(map) in X_E reduces to the trivial case $p = 2$, $X = \mathrm{L}^2(\mathbb{T})$, and to the case $p = 1$, $X = \mathscr{C}(\mathbb{T})$. To see this, note that we have by a repetition of the arguments of Lemma 7.2.2

Lemma 7.2.5 *Let $E \subseteq \mathbb{Z}$ and X be a homogeneous Banach space. The following properties are equivalent.*
(i) X_E has $m_p(\tau_f)$.
(ii) E enjoys the following property $\mathscr{M}_p$ in X: for any $\varepsilon > 0$ and finite $F \subseteq E$, there is a finite $G \subseteq F$ such that for $f \in B_{X_F}$ and $g \in B_{X_{E\setminus G}}$

$$\big|\|f + g\|_X - (\|f\|_X^p + \|g\|_X^p)^{1/p}\big| \leq \varepsilon$$

(iii) X_E enjoys $m_p(T_k)$ for any, or equivalently for some, a.s. of multipliers.

Proposition 7.2.6 *Let $E \subseteq \mathbb{Z}$ and X be a homogeneous Banach space.*
(i) If X_E has ℓ_p-(map), then E enjoys $\mathscr{M}_p$ in X.

(*ii*) $\mathrm{L}^q_E(\mathbb{T})$ *has* ℓ_p*-(map) if and only if* $p = q = 2$.
(*iii*) $\mathscr{C}_E(\mathbb{T})$ *has* ℓ_1*-(map) if and only if it has* ℓ_1*-(ap) and* E *enjoys* $\mathscr{M}_1$ *in* $\mathscr{C}(\mathbb{T})$*: for all* $\varepsilon > 0$ *and finite* $F \subseteq E$*, there is a finite* $G \subseteq E$ *such that*

$$\forall f \in \mathscr{C}_F(\mathbb{T})\ \forall g \in \mathscr{C}_{E\setminus G}(\mathbb{T}) \qquad \|f\|_\infty + \|g\|_\infty \le (1+\varepsilon)\|f+g\|_\infty.$$

Proof. (*i*) Let $\varepsilon > 0$. Let $\{T_k\}$ be an a.s. of multipliers that satisfies (23) with $C < 1+\varepsilon$. By the argument of [43, Lemma 5], we may assume that the T_k's have their range in $\mathscr{P}_E(\mathbb{T})$. Let $n \ge 1$ be such that $\big(\sum_{k>n} \|\Delta T_k f\|_X^p\big)^{1/p} < \varepsilon$ for $f \in B_{X_F}$. Let G be such that $T_k g = 0$ for $k \le n$ and $g \in X_{E\setminus G}$. Then successively

$$\Big|\Big(\sum_{k\le n} \|\Delta T_k(f+g)\|_X^p\Big)^{1/p} - \Big(\sum \|\Delta T_k f\|_X^p\Big)^{1/p}\Big| \le \varepsilon,$$

$$\Big|\Big(\sum_{k>n} \|\Delta T_k(f+g)\|_X^p\Big)^{1/p} - \Big(\sum \|\Delta T_k g\|_X^p\Big)^{1/p}\Big| \le \varepsilon,$$

$$\Big|\Big(\sum \|\Delta T_k(f+g)\|_X^p\Big)^{1/p} - \Big(\sum \|\Delta T_k f\|_X^p + \sum \|\Delta T_k g\|_X^p\Big)^{1/p}\Big| \le 2^{1/p}\varepsilon$$

and

$$\big|\|f+g\|_X - (\|f\|_X^p + \|g\|_X^p)^{1/p}\big| \le 2\varepsilon(1+2^{1/p}).$$

(*ii*) By Corollary 6.2.2, we necessarily have $p = 2$. Furthermore, if $\mathrm{L}^q_E(\mathbb{T})$ has ℓ_2-(*map*), then by property $\mathscr{M}_2$

$$\big|\|\,\mathrm{e}_n + \mathrm{e}_m\,\|_q - \sqrt{2}\big| \xrightarrow[m\to\infty]{} 0.$$

Now $\|\,\mathrm{e}_n + \mathrm{e}_m\,\|_q = \|1 + \mathrm{e}_1\,\|_q$ is constant and differs from $\|1 + \mathrm{e}_1\,\|_2 = \sqrt{2}$ unless $q = 2$: otherwise the only case of equality of the norms $\|\cdot\|_q$ and $\|\cdot\|_2$ occurs for almost everywhere constant functions.
(*iii*) Use Theorem 6.3.3. ∎

8 Property (*umap*) and arithmetical block independence

We may now apply the technique used in the investigation of (*umbs*) in order to obtain arithmetical conditions analogous to $(\mathscr{I}_n)$ (see Def. 3.4.1) for (*umap*). According to Theorem 7.2.3, it suffices to investigate property $(\mathscr{U})$ of block unconditionality: we have to compute an expression of type $\|f + \epsilon g\|_p$, where the spectra of f and g are far apart and $\epsilon \in \mathbb{S}$. As before, $\mathbb{S} = \mathbb{T}$ (*vs.* $\mathbb{S} = \mathbb{D}$) is the complex (*vs.* real) choice of signs.

8.1 Property of block independence

To this end, we return to the notation of Computational lemmas 3.3.1 and 3.3.4. Define

$$\begin{aligned}\Psi_r(\epsilon, z) &= \Theta_r((\overbrace{1,\dots,1}^{j},\overbrace{\epsilon,\dots,\epsilon}^{m-j}), z)\\ &= \int\Big|\mathrm{e}_{r_0}(t) + \sum_{i=1}^{j} z_i\,\mathrm{e}_{r_i}(t) + \epsilon\sum_{i=j+1}^{m} z_i\,\mathrm{e}_{r_i}(t)\Big|^p dm(t)\\ &= \sum_{\alpha\in\mathbb{N}^m}\binom{p/2}{\alpha}^2\prod|z_i|^{2\alpha_i} + \sum_{\substack{\alpha\ne\beta\in\mathbb{N}^m\\ \alpha\sim\beta}}\binom{p/2}{\alpha}\binom{p/2}{\beta}\epsilon^{\sum_{i>j}\alpha_i-\beta_i}\prod z_i^{\alpha_i}\overline{z_i}^{\beta_i}.\end{aligned} \tag{29}$$

As in Computational lemma 3.3.4, we make the following observation:

Computational lemma 8.1.1 *Let* $\zeta_0,\dots,\zeta_m \in \mathbb{Z}^*$ *and* γ,δ *be as in* (11)*. If the arithmetic relation* (12) *holds, then the coefficient of the term* $\epsilon^{\sum_{i>j}\gamma_i-\delta_i}\prod z_i^{\gamma_i}\overline{z_i}^{\delta_i}$ *in* (29) *is* $\binom{p/2}{\gamma}\binom{p/2}{\delta}$ *and thus independent of* r*. If* $\sum|\zeta_i| \le p$ *or* p *is not an even integer, this coefficient is nonzero. If* $\zeta_0+\dots+\zeta_j$ *is nonzero* (*vs. odd*)*, then this term is nonconstant in* $\epsilon \in \mathbb{S}$*.*

Thus the following arithmetical property shows up. It is similar to property $(\mathscr{I}_n)$ of almost independence.

Definition 8.1.2 *Let $E \subseteq \mathbb{Z}$ and $n \geq 1$.*
(*i*) *E enjoys the complex (vs. real) property $(\mathscr{J}_n)$ of block independence if for any $\zeta \in \mathbb{Z}_n^m$ with $\zeta_1+\cdots+\zeta_j$ nonzero (vs. odd) and given $p_1, \ldots, p_j \in E$, there is a finite $G \subseteq E$ such that $\zeta_1 p_1 + \cdots + \zeta_m p_m \neq 0$ for all $p_{j+1}, \ldots, p_m \in E \setminus G$.*
(*ii*) *E enjoys exactly complex (vs. real) $(\mathscr{J}_n)$ if furthermore it fails complex (vs. real) $(\mathscr{J}_{n+1})$.*
(*iii*) *E enjoys complex (vs. real) $(\mathscr{J}_\infty)$ if it enjoys complex (vs. real) $(\mathscr{J}_n)$ for all $n \geq 1$.*

The complex (*vs.* real) property $(\mathscr{J}_n)$ means precisely the following. "For every finite $F \subseteq E$ there is a finite $G \subseteq E$ such that for any two representations of any $k \in \mathbb{Z}$ as a sum of n elements of $F \cup (E \setminus G)$

$$p_1 + \cdots + p_n = k = p'_1 + \cdots + p'_n$$

one necessarily has

$$|\{j : p_j \in F\}| = |\{j : p'_j \in F\}| \text{ in } \mathbb{Z} \ (\textit{vs. } \text{in } \mathbb{Z}/2\mathbb{Z})."$$

Thus property $(\mathscr{J}_n)$ has, unlike $(\mathscr{I}_n)$, a complex and a real version. Real $(\mathscr{J}_n)$ is strictly weaker than complex $(\mathscr{J}_n)$: see Section 9. Notice that $(\mathscr{J}_1)$ is void and $(\mathscr{J}_{n+1}) \Rightarrow (\mathscr{J}_n)$ in both complex and real cases. Also $(\mathscr{I}_n) \not\Rightarrow (\mathscr{J}_n)$: we shall see in the following section that $E = \{0\} \cup \{n^k\}_{k \geq 0}$ provides a counterexample. The property $(\mathscr{J}_2)$ of real block independence appears implicitly in [43, Lemma 12].

Remark 8.1.3 In spite of the intricate form of this arithmetical property, $(\mathscr{J}_n)$ is the "simplest" candidate, in some sense, that reflects the features of $(\mathscr{U})$:

- it must hold for a set E if and only if it holds for a translate $E+k$ of this set: this explains $\sum \zeta_i = 0$ in Definition 8.1.2(*i*);
- as for the property $(\mathscr{U})$ of block independence, it must connect the break of E with its tail;
- Li gives an example of a set E whose pace does not tend to infinity while $\mathscr{C}_E(\mathbb{T})$ has ℓ_1-(*map*). Thus no property $(\mathscr{J}_n)$ should forbid parallelogram relations of the type $p_2 - p_1 = p_4 - p_3$, where p_1, p_2 are in the break of E and p_3, p_4 in its tail. This explains the condition that $\zeta_1 + \cdots + \zeta_j$ be nonzero (*vs.* odd) in Definition 8.1.2(*i*).

We now repeat the argument of Theorem 3.4.2 to obtain an analogous statement which relates property $(\mathscr{U})$ of Definition 7.2.1 with our new arithmetical conditions

Lemma 8.1.4 *Let $E = \{n_k\} \subseteq \mathbb{Z}$ and $1 \leq p < \infty$.*
(*i*) *Suppose p is an even integer. Then E enjoys the complex (vs. real) Fourier block unconditionality property $(\mathscr{U})$ in $\mathrm{L}^p(\mathbb{T})$ if and only if E enjoys complex (vs. real) $(\mathscr{J}_{p/2})$.*
(*ii*) *If p is not an even integer and E enjoys complex (vs. real) $(\mathscr{U})$ in $\mathrm{L}^p(\mathbb{T})$, then E enjoys complex (vs. real) $(\mathscr{J}_\infty)$.*

Proof. Let us first prove the necessity of the arithmetical property and assume E fails $(\mathscr{J}_n)$: then there are $\zeta_0, \ldots, \zeta_m \in \mathbb{Z}^*$ with $\sum \zeta_i = 0$, $\sum |\zeta_i| \leq 2n$ and $\zeta_0 + \cdots + \zeta_j$ nonzero (*vs.* odd); there are $r_0, \ldots, r_j \in E$ and sequences $r^l_{j+1}, \ldots, r^l_m \in E \setminus \{n_1, \ldots, n_l\}$ such that

$$\zeta_0 r_0 + \cdots + \zeta_j r_j + \zeta_{j+1} r^l_{j+1} + \cdots + \zeta_m r^l_m = 0.$$

Assume E enjoys $(\mathscr{U})$ in $\mathrm{L}^p(\mathbb{T})$. Then the oscillation of Ψ_r in (29) satisfies

$$\operatorname*{osc}_{\epsilon \in \mathbb{S}} \Psi_{r^l}(\epsilon, z) \xrightarrow[l \to \infty]{} 0 \tag{30}$$

for each $z \in D^m$. The argument is now exactly the same as in Theorem 3.4.2: we may assume that the sequence of functions Ψ_{r^l} converges in $\mathscr{C}^\infty(\mathbb{S} \times D^m)$ to a function Ψ. Then by (30), $\Psi(\epsilon, z)$ is constant in ϵ for each $z \in D^m$, and this is impossible by Computational lemma 8.1.1 if p is either not an even integer or $p \geq 2n$.

Let us now prove the sufficiency of $(\mathscr{J}_{p/2})$ when p is an even integer. First, let $\mathrm{A}_n^{k,l} = \{\alpha \in \mathrm{A}_n : \alpha_i = 0 \text{ for } k < i \leq l\}$ (A_n is defined before Prop. 3.2.1), and convince yourself that $(\mathscr{J}_{p/2})$ is equivalent to

$$\forall k \ \exists l \geq k \ \forall \alpha, \beta \in \mathrm{A}_{p/2}^{k,l} \quad \sum \alpha_i n_i = \sum \beta_i n_i \ \Rightarrow \ \sum_{i \leq k} \alpha_i = \sum_{i \leq k} \beta_i \ (\textit{vs. } \text{mod } 2). \tag{31}$$

Let $f = \sum a_i \,\mathrm{e}_{n_i} \in \mathscr{P}_E(\mathbb{T})$. Let $k \geq 1$ and $\epsilon \in \mathbb{S}$. By the multinomial formula,

$$\|\epsilon\pi_k f + (f - \pi_l f)\|_p^p = \int \Bigl| \sum_{\alpha \in \mathrm{A}^{k,l}_{p/2}} \binom{p/2}{\alpha} \epsilon^{\sum_{p \leq k} \alpha_i} \Bigl(\prod a_i^{\alpha_i}\Bigr) \mathrm{e}_{\sum \alpha_i n_i} \Bigr|^2 dm$$

$$= \int \Bigl| \sum_{j=0}^{n} \epsilon^j \sum_{\substack{\alpha \in \mathrm{A}^{k,l}_{p/2} \\ \alpha_1 + \cdots + \alpha_k = j}} \binom{p/2}{\alpha} \Bigl(\prod a_i^{\alpha_i}\Bigr) \mathrm{e}_{\sum \alpha_i n_i} \Bigr|^2 dm.$$

(31) now signifies that we may choose $l \geq k$ such that the terms of the above sum over j (*vs.* the terms with j odd and those with j even) have disjoint spectrum. But then $\|\epsilon\pi_k f + (f - \pi_l f)\|_p$ is constant for $\epsilon \in \mathbb{S}$ and E enjoys $(\mathscr{U})$ in $\mathrm{L}^p(\mathbb{T})$. ∎

Note that for even p, we have as in Proposition 3.2.1 a constant $C_p > 1$ such that either (28) holds for $\varepsilon = 0$ or fails for any $\varepsilon \leq C_p$. We thus get

Corollary 8.1.5 *Let $E \subseteq \mathbb{Z}$ and p be an even integer. If E enjoys complex (vs. real) $(\mathscr{U})$ in $\mathrm{L}^p(\mathbb{T})$, then there is a partition $E = \bigcup E_k$ into finite sets such that for any coarser partition $E = \bigcup E'_k$*

$$\forall f \in \mathscr{P}_E(\mathbb{T}) \quad \operatorname*{osc}_{\epsilon_k \in \mathbb{S}} \Bigl\| \sum \epsilon_k \pi_{E'_{2k}} f \Bigr\|_p = 0$$

Among other consequences, $E = E_1 \cup E_2$ where the $\mathrm{L}^p_{E_i}(\mathbb{T})$ have a complex (*vs.* real) 1-unconditional (*fdd*).

Question 8.1.6 Is this rigidity proper to translation invariant subspaces of $\mathrm{L}^p(\mathbb{T})$, p an even integer, or generic for all its subspaces (see [9])?

8.2 Main result

Lemma 8.1.4 and Theorem 7.2.3 yield the main result of this section.

Theorem 8.2.1 *Let $E \subseteq \mathbb{Z}$ and $1 \leq p < \infty$.*
(i) Suppose p is an even integer. Then $\mathrm{L}^p_E(\mathbb{T})$ has complex (vs. real) (umap) if and only if E enjoys complex (vs. real) $(\mathscr{J}_{p/2})$.
(ii) If p is not an even integer and $\mathrm{L}^p_E(\mathbb{T})$ has complex (vs. real) (umap), then E enjoys complex (vs. real) $(\mathscr{J}_\infty)$.

Corollary 8.2.2 *Let $E \subseteq \mathbb{Z}$.*
(i) If $\mathscr{C}_E(\mathbb{T})$ has complex (vs. real) (umap), then E enjoys complex (vs. real) $(\mathscr{J}_\infty)$.
(ii) If any $\mathrm{L}^p_E(\mathbb{T})$, p not an even integer, has complex (vs. real) (umap), then all $\mathrm{L}^p_E(\mathbb{T})$ with p an even integer have complex (vs. real) (umap).

Suppose p is an even integer. Then Section 9 gives various examples of sets such that $\mathrm{L}^p_E(\mathbb{T})$ has complex or real (*umap*). Proposition 10.2.1 gives a general growth condition that ensures (*umap*).
For $X = \mathrm{L}^p(\mathbb{T})$, p not an even integer, and $X = \mathscr{C}(\mathbb{T})$, however, we encounter the same obstacle as for (*umbs*). Section 9 only gives sets E such that X_E fails (*umap*). Thus, we have to prove this property by direct means. This yields four types of examples of sets E such that the space $\mathscr{C}_E(\mathbb{T})$ — and thus by [43, Th. 7] all $\mathrm{L}^p_E(\mathbb{T})$ $(1 \leq p < \infty)$ as well — have (*umap*).

- Sets found by Li [43]: Kronecker's theorem is used to construct a set containing arbitrarily long arithmetic sequences and a set whose pace does not tend to infinity. Meyer's [50, VIII] techniques are used to construct a Hilbert set.
- The sets that satisfy the growth condition of Theorem 10.3.1;
- Sequences $E = \{n_k\} \subseteq \mathbb{Z}$ such that n_{k+1}/n_k is an odd integer: see Proposition 10.1.1.

Question 8.2.3 We know no example of a set E such that some $\mathrm{L}^p_E(\mathbb{T})$, p not an even integer, has (*umap*) while $\mathscr{C}_E(\mathbb{T})$ fails it.

There is also a good arithmetical description of the case where $\{\pi_k\}$ or a subsequence thereof realises (*umap*).

Proposition 8.2.4 *Let $E = \{n_k\}_{k\geq 1} \subseteq \mathbb{Z}$. Consider a partition $E = \bigcup_{k\geq 1} E_k$ into finite sets.*
(i) Suppose p is an even integer. The series $\sum \pi_{E_k}$ realises complex (vs. real) (umap) in $\mathrm{L}^p_E(\mathbb{T})$ if and only if there is an $l \geq 1$ such that

$$\begin{cases} p_1, \ldots, p_m \in E \\ \zeta_1 p_1 + \cdots + \zeta_m p_m = 0 \end{cases} \Rightarrow \quad \forall k \geq l \sum_{p_j \in E_k} \zeta_j = 0 \text{ (vs. is even)} \tag{32}$$

for all $\zeta \in \mathbb{Z}^m_{p/2}$. Then $\mathrm{L}^p_E(\mathbb{T})$ admits the series $\pi_{\cup_{k<l} E_k} + \sum_{k\geq l} \pi_{E_k}$ as 1-unconditional (fdd). In particular, choose $E_k = \{n_k\}$. The sequence $\{\pi_k\}$ realises complex and real (umap) in $\mathrm{L}^p_E(\mathbb{T})$ if and only if there is a finite G such that for $\zeta \in \mathbb{Z}^m_{p/2}$

$$\begin{cases} p_1, \ldots, p_m \in E \\ \zeta_1 p_1 + \cdots + \zeta_m p_m = 0 \end{cases} \Rightarrow \quad p_1, \ldots, p_m \in G. \tag{33}$$

Then $E \setminus G$ is a 1-(ubs) and E enjoys $(\mathscr{I}_{p/2})$.
(ii) Suppose p is not an even integer. If $\sum \pi_{E_k}$ realises complex (vs. real) (umap) in $\mathrm{L}^p_E(\mathbb{T})$, then for each $\zeta \in \mathbb{Z}^m$ there is an $l \geq 1$ such that (32) holds. In particular, if $\{\pi_k\}$ realises either complex or real (umap) in $\mathrm{L}^p_E(\mathbb{T})$, then for all $\zeta \in \mathbb{Z}^m$ there is a finite G such that (33) holds. This is equivalent to $(\mathscr{I}_\infty)$.

Proof. It is analogous to the proof of Lemma 8.1.4: suppose we have $\zeta \in \mathbb{Z}^m_n$ such that (32) fails for any $l \geq 1$. Then there are $\zeta_0, \ldots, \zeta_m \in \mathbb{Z}^*$ with $\sum \zeta_i = 0$, $\sum |\zeta_i| \leq 2n$ and $\zeta_0 + \cdots + \zeta_j$ nonzero (*vs.* odd) for some j; for each l, there are $r^l_0, \ldots, r^l_j \in \cup_{k<l} E_k$ and $r^l_{j+1}, \ldots, r^l_m \in \cup_{k\geq l} E_k$ such that $\zeta_0 r^l_0 + \cdots + \zeta_m r^l_m = 0$.
But then $\sum \pi_{E_k}$ cannot realise complex (*vs.* real) (*umap*): the function Ψ_r in (29) would satisfy (30) and we would obtain a contradiction as in Theorem 3.4.2.
Sufficiency in (i) and (i') is proved exactly as in Lemma 8.1.4(i). ∎

In particular, suppose that the cardinal $|E_k|$ is uniformly bounded by M and $\{\pi_{E_k}\}$ realises (*umap*) in $\mathrm{L}^p_E(\mathbb{T})$. If $p \neq 2$ is an even integer, then E is a $\Lambda(p)$ set as union of a finite set and M $p/2$-independent sets (see Prop. 3.2.1 and [66, Th. 4.5(b)]). If p is not an even integer, then E is a $\Lambda(q)$ set for all q by the same argument.

9 Examples for (*umap*): block independent sets of characters

9.1 General properties

The pairing $\langle \zeta, E \rangle$ underlines the asymptotic nature of property $(\mathscr{J}_n)$. It has been defined before Proposition 4.1.1, whose proof adapts to

Proposition 9.1.1 *Let $E = \{n_k\} \subseteq \mathbb{Z}$.*
(i) If $\langle \zeta, E \rangle < \infty$ for $\zeta_1, \ldots, \zeta_m \in \mathbb{Z}^*$ with $\sum \zeta_i$ nonzero (vs. odd), then E fails complex (vs. real) $(\mathscr{J}_{|\zeta_1|+\cdots+|\zeta_m|})$. Conversely, if E fails complex (vs. real) $(\mathscr{J}_n)$, then there are $\zeta_1, \ldots, \zeta_m \in \mathbb{Z}^*$ with $\sum \zeta_i$ nonzero (vs. odd) and $\sum |\zeta_i| \leq 2n - 1$ such that $\langle \zeta, E \rangle < \infty$.
(ii) Thus E enjoys complex (vs. real) $(\mathscr{J}_\infty)$ if and only if $\langle \zeta, E \rangle = \infty$ for all $\zeta_1, \ldots, \zeta_m \in \mathbb{Z}^*$ with $\sum \zeta_i$ nonzero (vs. odd).

Proof of the converse in (i). If E fails complex (*vs.* real) $(\mathscr{J}_n)$, then there are $\zeta \in \mathbb{Z}^m_n$ with $\zeta_1 + \cdots + \zeta_j$ nonzero (*vs.* odd), $p_1, \ldots, p_j \in E$ and sequences $p^l_{j+1}, \ldots, p^l_m \in \{n_k\}_{k\geq l}$ such that $\sum_{i>j} \zeta_i p^l_i = -\sum_{i\leq j} \zeta_i p_i$. Let $\zeta' = (\zeta_{j+1}, \ldots, \zeta_m)$. Then $\sum |\zeta'_i| \leq 2n - 1$ and $\langle \zeta', E \rangle < \infty$. ∎

An immediate application is, as in Proposition 4.1.1,

Proposition 9.1.2 *Let $E = \{n_k\} \subseteq \mathbb{Z}$.*
(i) Suppose E enjoys $(\mathscr{I}_{2n-1})$. Then E enjoys complex $(\mathscr{J}_n)$ and actually there is a finite set G such that (33) holds for $\zeta \in \mathbb{Z}^m_n$.

(ii) *Suppose E enjoys $(\mathscr{I}_\infty)$. Then E enjoys complex $(\mathscr{J}_\infty)$ and actually for all $\zeta \in \mathrm{Z}^m$ there is a finite G such that (33) holds.*

(iii) *Complex and real $(\mathscr{J}_\infty)$ are stable under bounded perturbations of E.*

(iv) *Suppose there is $h \in \mathbb{Z}$ such that $E \cup \{h\}$ fails complex (vs. real) $(\mathscr{J}_n)$. Then E fails complex (vs. real) $(\mathscr{J}_{2n-1})$. Thus the complex and real properties $(\mathscr{J}_\infty)$ are stable under unions with an element: if E enjoys it, then so does $E \cup \{h\}$.*

(v) *Suppose $jF + s, kF + t \in E$ for an infinite F, $j \neq k \in \mathbb{Z}^*$ and $s, t \in \mathbb{Z}$. Then E fails complex $(\mathscr{J}_{|j|+|k|})$, and also real $(\mathscr{J}_{|j|+|k|})$ if j and k have different parity.*

We now turn to an arithmetical investigation of various sets E.

9.2 Geometric sequences

Let $G = \{j^k\}_{k \geq 0}$ with $j \in \mathbb{Z} \setminus \{-1, 0, 1\}$. We resume Remark 7.2.4.

(1) As $G, jG \subseteq G$, G fails complex $(\mathscr{J}_{|j|+1})$, and also real $(\mathscr{J}_{|j|+1})$ if j is even. The solutions (15) to the Diophantine equation (14) show at once that G enjoys complex $(\mathscr{J}_{|j|})$, since there is no arithmetical relation $\zeta \in \mathrm{Z}^m_{|j|}$ between the break and the tail of G. If j is odd, then G enjoys in fact real $(\mathscr{J}_\infty)$. Indeed, let $\zeta_1, \ldots, \zeta_m \in \mathbb{Z}^*$ and $k_1 < \cdots < k_m$: then $\sum \zeta_i j^{k_i} \in j^{k_1}\mathbb{Z}$ and either $\left|\sum \zeta_i j^{k_i}\right| \geq j^{k_1}$ or $\sum \zeta_i j^{k_i} = 0$. Thus, if $\langle \zeta, E \rangle < \infty$ then $\langle \zeta, E \rangle = 0$ and $\sum \zeta_i$ is even since j is odd. Now apply Proposition 9.1.1*(iii)*. The same argument yields that even $G \cup -G \cup \{0\}$ enjoys real $(\mathscr{J}_\infty)$. Actually much more is true: see Proposition 10.1.1.

(2) $G \cup \{0\}$ may behave differently than G with respect to $(\mathscr{J}_n)$: thus this property is not stable under unions with an element. Indeed, the first solution in (15) may be written as $(-j+1)\cdot 0 + j \cdot j^k + (-1) \cdot j^{k+1} = 0$. If j is positive, $(-j+1) + j + (-1) \leq 2j$ and $G \cup \{0\}$ fails complex $(\mathscr{J}_j)$. A look at (15) shows that it nevertheless enjoys complex $(\mathscr{J}_{j-1})$. On the other hand, $G \cup \{0\}$ still enjoys complex $(\mathscr{J}_{|j|})$ if j is negative. In the real setting, our arguments yield the same if j is even, but we already saw that $G \cup \{0\}$ still enjoys real $(\mathscr{J}_\infty)$ if j is odd.

9.3 Symmetric sets

By Proposition 4.1.1*(iii)* and 9.1.2*(vi)*, they do enjoy neither $(\mathscr{I}_2)$ nor complex $(\mathscr{J}_2)$. They may nevertheless enjoy real $(\mathscr{J}_n)$. Introduce property $(\mathscr{J}_n^{\text{sym}})$ for E: it holds if for all $p_1, \ldots, p_j \in E$ and $\eta \in \mathbb{Z}^{*m}$ with $\sum_1^m \eta_i$ even, $\sum_1^m |\eta_i| \leq 2n$ and $\eta_1 + \cdots + \eta_j$ odd, there is a finite set G such that $\eta_1 p_1 + \cdots + \eta_m p_m \neq 0$ for any $p_{j+1}, \ldots, p_m \in E \setminus G$. Then we obtain

Proposition 9.3.1 *$E \cup -E$ has real $(\mathscr{J}_n)$ if and only if E has $(\mathscr{J}_n^{\text{sym}})$.*

Proof. By definition, $E \cup -E$ has real $(\mathscr{J}_n)$ if and only if for all $p_1, \ldots, p_j \in E$ and $\zeta^1, \zeta^2 \in \mathbb{Z}^m$ with $\zeta^1 + \zeta^2 \in \mathrm{Z}^m_n$ and odd $\sum_{i \leq k} \zeta^1_i - \zeta^2_i$, there is a finite set G such that $\sum(\zeta^1_i - \zeta^2_i)p_i \neq 0$ for any $p_{j+1}, \ldots, p_m \in E \setminus G$ — and thus if and only if E enjoys $(\mathscr{J}_n^{\text{sym}})$: just consider the mappings between arithmetical relations $(\zeta^1, \zeta^2) \mapsto \eta = \zeta^1 - \zeta^2$ and $\eta \mapsto (\zeta^1, \zeta^2)$ such that $\eta = \zeta^1 - \zeta^2$, where $\zeta^1_i = \eta_i/2$ if η_i is even and, noting that the number of odd η_i's must be even, $\zeta^1_i = (\eta_i - 1)/2$ and $\zeta^1_i = (\eta_i + 1)/2$ respectively for each half of them. ■

Consider again a geometric sequence $G = \{j^k\}$ with $j \geq 2$. If j is odd, we saw before that $G \cup -G$ and $G \cup -G \cup \{0\}$ enjoy real $(\mathscr{J}_\infty)$. If j is even, then $G \cup -G$ fails real $(\mathscr{J}_{j+1})$ since G does. $G \cup -G \cup \{0\}$ fails real $(\mathscr{J}_{j/2+1})$ by the arithmetical relation $1 \cdot 0 + j \cdot j^k + (-1) \cdot j^{k+1} = 0$ and Proposition 9.3.1. $G \cup -G$ enjoys real $(\mathscr{J}_j)$ and $G \cup -G \cup \{0\}$ enjoys real $(\mathscr{J}_{j/2})$ as the solutions in (15) show by a simple checking.

9.4 Algebraic and transcendental numbers

The proof of Proposition 4.3.1 adapts to

Proposition 9.4.1 *Let $E = \{n_k\} \subseteq \mathbb{Z}$.*

(i) *If $n_{k+1}/n_k \to \sigma$ where $\sigma > 1$ is transcendental, then E enjoys complex $(\mathscr{J}_\infty)$.*

(ii) *Let $n_k = [\sigma^k]$ with $\sigma > 1$ algebraic. Let $P(x) = \zeta_0 + \cdots + \zeta_d x^d$ be the corresponding polynomial of minimal degree. Then E fails complex $(\mathscr{J}_{|\zeta_0|+\cdots+|\zeta_d|})$, and also real $(\mathscr{J}_{|\zeta_0|+\cdots+|\zeta_d|})$ if $P(1)$ is odd.*

9.5 Polynomial sequences

Let $E = \{P(k)\}$ for a polynomial P of degree d. The arithmetical relation (16) does not adapt to property $(\mathscr{J}_n)$. Notice, though, that $\{\Delta^j P\}_{j=1}^d$ is a basis for the space of polynomials of degree less than d and that $2^d P(k) - P(2k)$ is a polynomial of degree at most $d-1$. Writing it in the basis $\{\Delta^j P\}_1^d$ yields an arithmetical relation $2^d \cdot P(k) - 1 \cdot P(2k) + \sum_{j=0}^d \zeta_j \cdot P(k-j) = 0$ such that $2^d - 1 + \sum \zeta_j$ is odd. By Proposition 9.1.1 (*ii*), E fails real $(\mathscr{J}_n)$ for a certain n. This n may be bounded in certain cases:

▪ The set of squares fails real $(\mathscr{J}_2)$: let F_n be the Fibonacci sequence defined by $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$. As $\{F_{n+1}/F_n\}$ is the sequence of convergents of the continued fraction associated to an irrational (the golden ratio), $F_n \to \infty$ and $F_n F_{n+2} - F_{n+1}^2 = (-1)^n$ (see [17]). Inspired by [53, p. 15], we observe that

$$(F_n F_{n+2} + F_{n+1}^2)^2 + (F_{n+1}^2)^2 = (F_n F_{n+1} + F_{n+1} F_{n+2})^2 + 1^2$$

▪ The set of cubes fails real $(\mathscr{J}_2)$: starting from Binet's [3] simplified solution of Euler's equation [15], we observe that $p_n = 9n^4$, $q_n = 1 + 9n^3$, $r_n = 3n(1+3n^3)$ satisfy $p_n^3 + q_n^3 = r_n^3 + 1^3$ and tend to infinity.

▪ The set of biquadrates fails real $(\mathscr{J}_3)$: by an equality of Ramanujan (see [61, p. 386]),

$$(4n^5 - 5n)^4 + (6n^4 - 3)^4 + (4n^4 + 1)^4 = (4n^5 + n)^4 + (2n^4 - 1)^4 + 3^4.$$

As for $(\mathscr{I}_n)$, a positive answer to Euler's conjecture would imply that the set of kth powers has complex $(\mathscr{J}_2)$ for $k \geq 5$.

Conclusion By Theorem 8.2.1, property $(\mathscr{J}_n)$ yields directly (*umap*) in the space $\mathrm{L}^{2p}(\mathbb{T})$, $p \leq n$ integer. But we do not know whether $(\mathscr{J}_\infty)$ ensures (*umap*) in spaces $\mathrm{L}^p(\mathbb{T})$, p not an even integer, or $\mathscr{C}(\mathbb{T})$.

Nevertheless, the study of property $(\mathscr{J}_3)$ permits us to determine the density of sets such that X_E enjoys (*umap*) for some $X \neq \mathrm{L}^2(\mathbb{T}), \mathrm{L}^4(\mathbb{T})$: see Proposition 11.2. Other applications are given in Section 13.

10 Positive results: parity and a sufficient growth condition

10.1 $\mathscr{C}_{\{3^k\}}(\mathbb{T})$ has real (*umap*) because 3 is odd

In the real case, parity plays an unexpected rôle.

Proposition 10.1.1 *Let $E = \{n_k\} \subseteq \mathbb{Z}$ and suppose that n_{k+1}/n_k is an odd integer for all sufficiently large k. Then $\mathscr{C}_E(\mathbb{T})$ has real (umap).*

Then X_E also has real (*umap*) for every homogeneous Banach space X on $\mathbb{T}$.

Proof. Let us verify that real $(\mathscr{U})$ holds. Let $\varepsilon > 0$ and $F \subseteq E \cap [-n, n]$. Let l, to be chosen later, such that n_{k+1}/n_k is an odd integer for $k \geq l$. Take $G \supseteq \{n_1, \ldots, n_l\}$ finite. Let $f \in B_{\mathscr{C}_F}$ and $g \in B_{\mathscr{C}_{E \setminus G}}$. Then $g(u \exp \mathrm{i}\pi/n_l) = -g(u)$ and

$$|f(u \exp \mathrm{i}\pi/n_l) - f(u)| \leq \pi/|n_l| \cdot \|f'\|_\infty \leq \pi n/|n_l| \leq \varepsilon$$

by Bernstein's inequality and for l large enough. Thus, for some $u \in \mathbb{T}$,

$$\begin{aligned} \|f - g\|_\infty &= |f(u) + g(u \exp \mathrm{i}\pi/n_l)| \\ &\leq |f(u \exp \mathrm{i}\pi/n_l) + g(u \exp \mathrm{i}\pi/n_l)| + \varepsilon \\ &\leq \|f + g\|_\infty + \varepsilon. \end{aligned}$$

As E is a Sidon set, we may apply Theorem 7.2.3(*iii*). ∎

Furthermore, if E satisfies the hypothesis of Proposition 10.1.1, so does $E \cup -E = \{n_1, -n_1, n_2, -n_2, \ldots\}$. But $E \cup -E$ fails even complex $(\mathscr{J}_2)$ and no $X_{E\cup-E} \neq \mathrm{L}^2_{E\cup-E}(\mathbb{T})$ has complex (*umap*). On the other hand, if there is an even integer h such that $n_{k+1}/n_k = h$ infinitely often, then E fails real $(\mathscr{J}_{|h|+1})$ by Proposition 9.1.2(vi).

Remark 10.1.2 Note that if n_{k+1}/n_k is furthermore uniformly bounded, then the a.s. that realises (*umap*) cannot be too simple. In particular, it cannot be a (*fdd*) in translation invariant spaces $\mathscr{C}_{E_i}(\mathbb{T})$: let k be such that n_k and n_{k+1} are in distinct E_i; then $n_{k+1} + (-n_{k+1}/n_k)\cdot n_k = 0$ and we may apply Proposition 8.2.4(ii). This justifies the use of Theorem 7.2.3(iii).

10.2 Growth conditions: the case $\mathrm{L}^p(\mathbb{T})$, p an even integer

For $X = \mathrm{L}^p(\mathbb{T})$ with p an even integer, a look at $(\mathscr{I}_n)$ and $(\mathscr{J}_n)$ gives by Theorems 3.4.2 and 8.2.1 the following general growth condition:

Proposition 10.2.1 *Let $E = \{n_k\} \subseteq \mathbb{Z}$ and $p \geq 1$ an integer. If*

$$\liminf |n_{k+1}/n_k| \geq p + 1, \tag{34}$$

then $\{\pi_k\}$ realises the complex (umap) in $\mathrm{L}^{2p}_E(\mathbb{T})$ and there is a finite $G \subseteq E$ such that $E \setminus G$ is a 1-unconditional basic sequence in $\mathrm{L}^{2p}(\mathbb{T})$.

Proof. Suppose we have an arithmetical relation

$$\zeta_1 n_{k_1} + \cdots + \zeta_m n_{k_m} = 0 \quad \text{with} \quad \zeta \in \mathrm{Z}_p^m \text{ and } |n_{k_1}| < \cdots < |n_{k_m}|. \tag{35}$$

Then $|\zeta_m n_{k_m}| \leq |\zeta_1 n_{k_1}| + \cdots + |\zeta_{m-1} n_{k_{m-1}}|$. The left hand side is smallest when $|\zeta_m| = 1$. As $|\zeta_1| + \cdots + |\zeta_m| \leq 2p$ and necessarily $|\zeta_i| \leq p$, the right hand side is largest when $|\zeta_{m-1}| = p$ and $|\zeta_{m-2}| = p - 1$. Furthermore, it is largest when $k_m = k_{m-1} + 1 = k_{m-2} + 2$. Thus, if (35) holds, then

$$|n_{k_m}| \leq p|n_{k_{m-1}}| + (p-1)|n_{k_{m-2}}|.$$

By (34), this is impossible as soon as m is chosen sufficiently large, because $p+1 > p + (p-1)/(p+1)$. ∎

Note that Proposition 10.2.1 is best possible: if j is negative, then $\{j^k\}$ fails $(\mathscr{I}_{|j|})$. If j is positive, then $\{j^k\} \cup \{0\}$ fails complex $(\mathscr{J}_j)$.

10.3 A general growth condition

Although we could prove that E enjoys $(\mathscr{I}_\infty)$ and $(\mathscr{J}_\infty)$ when $n_{k+1}/n_k \to \infty$, we need a direct argument in order to get the corresponding functional properties: we have

Theorem 10.3.1 *Let $E = \{n_k\} \subseteq \mathbb{Z}$ such that $n_{k+1}/n_k \to \infty$. Then $\mathscr{C}_E(\mathbb{T})$ has ℓ_1-(map) with $\{\pi_k\}$ and E is a Sidon set with constant asymptotically 1. If the ratios n_{k+1}/n_k are all integers, then the converse holds.*

Note that by Proposition 3.1.3(ii), E is a metric unconditional basic sequence in every homogeneous Banach space X on $\mathbb{T}$. Further X_E has complex (*umap*) since $\mathscr{C}_E(\mathbb{T})$ does.

Proof. Suppose $|n_{j+1}/n_j| \geq q$ for $j \geq l$ and some $q > 1$ to be fixed later. Let $f = \sum a_j \,\mathrm{e}_{n_j} \in \mathscr{P}_E(\mathbb{T})$ and $k \geq l$. We show by induction that for all $p \geq k$

$$\|\pi_p f\|_\infty \geq \left(1 - \frac{\pi^2}{2}\frac{1 - q^{2(k-p)}}{q^2 - 1}\right)\|\pi_k f\|_\infty + \sum_{j=k+1}^{p}\left(1 - \frac{\pi^2}{2}\frac{1 - q^{2(j-p)}}{q^2 - 1}\right)|a_j|. \tag{36}$$

- There is nothing to show for $p = k$.
- By Bernstein's inequality applied to $\pi_k f''$ and separately to each $a_j \,\mathrm{e}''_{n_j}$, $j > k$,

$$\|\pi_p f''\|_\infty \leq n_k^2 \|\pi_k f\|_\infty + \sum_{j=k+1}^{p} n_j^2 |a_j|. \tag{37}$$

Furthermore, by Lemmas 1 and 2 of [50, §VIII.4.2],

$$\|\pi_{p+1}f\|_\infty \geq \|\pi_p f\|_\infty + |a_{p+1}| - \pi^2/(2n_{p+1}^2)\|\pi_p f''\|_\infty. \tag{38}$$

(38) together with (36) and (37) yield (36) with p replaced by $p+1$. Therefore

$$\|f\|_\infty = \lim_{p\to\infty} \|\pi_p f\|_\infty \geq \Big(1 - \frac{\pi^2}{2}\frac{1}{q^2-1}\Big)\Big(\|\pi_k f\|_\infty + \sum_{j=k+1}^{\infty} |a_j|\Big). \tag{39}$$

Thus $\{\pi_j\}_{j\geq k}$ realises ℓ_1-*(ap)* with constant $1+\pi^2/(2q^2-2-\pi^2)$. As q may be chosen arbitrarily large, E has ℓ_1-*(map)* with $\{\pi_j\}$. Additionally (39) shows by choosing $\pi_k f = 0$ that E is a *(umbs)* in $\mathscr{C}(\mathbb{T})$. Finally, the converse holds by Proposition 9.1.2(vi): if n_{k+1}/n_k does not tend to infinity while being integer, then there are $h \in \mathbb{Z}\setminus\{0,1\}$ and an infinite F such that $F, hF \subseteq E$. ∎

Remark 10.3.2 The technique of Riesz products as exposed in [33, Appendix V, §1.II] would have sufficed to prove Theorem 10.3.1.

Remark 10.3.3 Suppose still that $E = \{n_k\} \subseteq \mathbb{Z}$ with $n_{k+1}/n_k \to \infty$. A variation of the above argument yields that the space of *real* functions with spectrum in $E \cup -E$ has ℓ_1-*(ap)*.

Remark 10.3.4 Note however that there are sets E that satisfy $n_{k+1}/n_k \to 1$ and nevertheless enjoy $(\mathscr{I}_\infty)$ (see end of Section 11): they might be *(umbs)* in $\mathscr{C}(\mathbb{T})$, but this is unknown.

10.4 Sidon constant of Hadamard sets

Recall that $E = \{n_k\} \subseteq \mathbb{Z}$ is a Hadamard set if there is a $q > 1$ such that $n_{k+1}/n_k \geq q$ for all k. It is a classical fact that then E is a Sidon set: Riesz products (see [46, Chapter 2]) even yield effective bounds for its Sidon constant. In particular, if $q \geq 3$, then E's Sidon constant is at most 2. Our computations provide an alternative proof for $q > \sqrt{\pi^2/2+1} \approx 2.44$ and give a better bound for $q > \sqrt{\pi^2+1} \approx 3.30$. Putting $k = 1$ in (39), we obtain

Corollary 10.4.1 *Let* $E = \{n_k\} \subseteq \mathbb{Z}$.
(i) *Let* $q > \sqrt{\pi^2/2+1}$. *If* $|n_{k+1}| \geq q|n_k|$ *for all* k, *then the Sidon constant of* E *is at most* $1+\pi^2/(2q^2-2-\pi^2)$.
(ii) *[55, Cor. 5.2] Let* $q \geq 2$ *be an integer. If* $E \supseteq \{n, n+k, n+qk\}$ *for some* n *and* k, *then the Sidon constant of* E *is at least* $\big(\cos(\pi/2q)\big)^{-1} \geq 1+\pi^2/(8q^2)$.

In particular, we have the following bounds for the Sidon constant C of $G = \{j^k\}$, $j \in \mathbb{Z}\setminus\{-1,0,1\}$:

$$1+\pi^2/(8(j+1)^2) \leq C \leq 1+\pi^2/(2j^2-2-\pi^2).$$

11 Density conditions

We apply combinatorial tools to find out how "big" a set E may be while enjoying $(\mathscr{I}_n)$ or $(\mathscr{J}_n)$, and how "small" it must be.
The coarsest notion of largeness is that of density. Recall that the maximal density of $E \subseteq \mathbb{Z}$ is defined by

$$d^*(E) = \lim_{h\to\infty} \max_{a\in\mathbb{Z}} \frac{\big|E \cap \{a+1,\dots,a+h\}\big|}{h}.$$

Suppose E enjoys $(\mathscr{I}_n)$ with $n \geq 2$. Then E is a $\Lambda(2n)$ set by Theorem 3.4.2(i). By [66, Th. 3.5] (see also [51, §1, Cor. 2]), $d^*(E) = 0$. Now suppose E enjoys complex or real $(\mathscr{J}_n)$ with $n \geq 2$. As Li [43, Th. 2] shows, there are sets E such that $\mathscr{C}_E(\mathbb{T})$ has ℓ_1-*(map)* while E contains arbitrarily long arithmetic sequences: we cannot apply Szemeredi's Theorem.
Kazhdan (see [28, Th. 3.1]) proved that if $d^*(E) > 1/n$, then there is a $t \in \{1,\dots,n-1\}$ such that $d^*(E \cap E+t) > 0$. One might hope that it should in fact suffice to choose t in any interval of length n. However, Hindman [28, Th. 3.2] exhibits a counterexample: given $s \in \mathbb{Z}$ and positive ε, there is a set E with $d^*(E) > 1/2-\varepsilon$ and there are arbitrarily large a such that $E \cap E-t = \emptyset$ for all $t \in \{a+1,\dots,a+s\}$. Thus, we have to be satisfied with

Lemma 11.1 *Let $E \subseteq \mathbb{Z}$ with positive maximal density. Then there is a $t \geq 1$ such that the following holds: for any $s \in \mathbb{Z}$ we have some a, $|a| \leq t$, such that $d^*(E + a \cap E + s) > 0$.*

Proof. By a result of Erdős (see [28, Th. 3.8]), there is a $t \geq 1$ such that $F = E + 1 \cup \cdots \cup E + t$ satisfies $d^*(F) > 1/2$. But then, by [28, Th. 3.4], $d^*(F \cap F + s) > 0$ for any $s \in \mathbb{Z}$. This means that for any s there are $1 \leq a, b \leq t$ such that $d^*(E + a \cap E + s + b) > 0$. ■

We are now able to prove

Proposition 11.2 *Let $E \subseteq \mathbb{Z}$.*
(i) If E has positive maximal density, then there is an $a \in \mathbb{Z}$ such that $E \cup \{a\}$ fails real ($\mathscr{J}_2$). Therefore E fails real ($\mathscr{J}_3$).
(ii) If $d^*(E) > 1/2$, then E fails real ($\mathscr{J}_2$).

Proof. (ii) is established in [43, Prop. 14]. (i) is a consequence of Lemma 11.1: indeed, if E has positive maximal density, then this lemma yields some $a \in \mathbb{Z}$ and an infinite $F \subseteq E$ such that for all $s \in F$ there are arbitrarily large $k, l \in E$ such that $k + a = l + s$. Thus $E \cup \{a\}$ fails real ($\mathscr{J}_2$). Furthermore, E fails real ($\mathscr{J}_3$) by Proposition 9.1.2(iv). ■

Remark 11.3 We may reformulate the remaining open case of ($\mathscr{J}_2$). Let us introduce the infinite difference set of E: $\Delta E = \{t : |E \cap (E - t)| = \infty\}$ (see [74] and [68]). Then E has real ($\mathscr{J}_2$) if and only if, for any $a \in E$, ΔE meets $E - a$ finitely many times only. Thus our question is: are there sets with positive maximal density such that $E - a \cap \Delta E$ is finite for all $a \in E$?

Proposition 10.2.1 and Theorem 10.3.1 show that there is only one general condition of lacunarity on E that ensures properties ($\mathscr{I}_n$), ($\mathscr{J}_n$) or ($\mathscr{I}_\infty$), ($\mathscr{J}_\infty$): E must grow exponentially or superexponentially. One may nevertheless construct inductively "large" sets that enjoy these properties: they must only be sufficiently irregular to avoid all arithmetical relations. Thus there are sequences with growth slower than k^{2n-1} which nevertheless enjoy both ($\mathscr{I}_n$) and complex and real ($\mathscr{J}_n$). See [25, §II, (3.52)] for a proof in the case $n = 2$: it can be easily adapted to $n \geq 2$ and shows also the way to construct, for any sequence $n_k \to \infty$, sets that satisfy ($\mathscr{I}_\infty$) and ($\mathscr{J}_\infty$) and grow more slowly than k^{n_k}.

12 Unconditionality vs. probabilistic independence

12.1 Cantor group

Let us first show how simple the problems of (*umbs*) and (*umap*) become when considered for independent uniformly distributed random variables and their span in some space.
Let $\mathbb{D}^\infty$ be the Cantor group and Γ its dual group of Walsh functions. Consider the set $R = \{r_i\} \subseteq \Gamma$ of Rademacher functions, *i. e.* of the coordinate functions on $\mathbb{D}^\infty$: they form a family of independent random variables that take values -1 and 1 with equal probability $\frac{1}{2}$: Thus $\|\sum \epsilon_i a_i r_i\|_X$ does not depend on the choice of signs $\epsilon_i = \pm 1$ for any homogeneous Banach space X on $\mathbb{D}^\infty$ and R is a real 1-(*ubs*) in X.
Clearly, R is also a complex (*ubs*) in all such X. But its complex unconditionality constant is $\pi/2$ [69] and $\mathrm{L}^p_W(\mathbb{D}^\infty)$ has complex (*umap*) if and only if $p = 2$ or $W = \{w_i\} \subseteq \Gamma$ is finite. Indeed, W would have an analogue property ($\mathscr{U}$) of block unconditionality in $\mathrm{L}^p(\mathbb{D}^\infty)$: for any $\varepsilon > 0$ there would be n such that

$$\max_{w \in \mathbb{T}} \|\epsilon a w_1 + b w_n\|_p \leq (1 + \varepsilon) \|a w_1 + b w_n\|_p.$$

But this is false: for $1 \leq p < 2$, take $a = b = 1$, $\epsilon = \mathrm{i}$:

$$\max_{\epsilon \in \mathbb{T}} \|\epsilon w_1 + w_n\|_p \geq \left(\tfrac{1}{2}(|\mathrm{i} + 1|^p + |\mathrm{i} - 1|^p)\right)^{1/p} = \sqrt{2} > \|w_1 + w_n\|_p = 2^{1-1/p};$$

for $2 < p \leq \infty$, take $a = 1$, $b = \mathrm{i}$, $\epsilon = \mathrm{i}$:

$$\max_{\epsilon \in \mathbb{T}} \|\epsilon w_1 + \mathrm{i} w_n\|_p \geq \left(\tfrac{1}{2}(|\mathrm{i} + \mathrm{i}|^p + |\mathrm{i} - \mathrm{i}|^p)\right)^{1/p} = 2^{1-1/p} > \|w_1 + \mathrm{i} w_n\|_p = \sqrt{2}.$$

This is simply due to the fact that the image domain of the characters on $\mathbb{D}^\infty$ is too small. Take now the infinite torus $\mathbb{T}^\infty$ and consider the set $S = \{s_i\}$ of Steinhaus functions, *i. e.* the coordinate functions on $\mathbb{T}^\infty$: they form again a family of independent random variables with values uniformly distributed in $\mathbb{T}$. Then S is clearly a complex 1-(*ubs*) in any homogeneous Banach space X on $\mathbb{T}^\infty$.

12.2 Two notions of approximate probabilistic independence

As the random variables $\{\mathrm{e}_n\}$ also have their values uniformly distributed in $\mathbb{T}$, some sort of approximate independence should suffice to draw the same conclusions as in the case of S.
A first possibility is to look at the joint distribution of $(\mathrm{e}_{p_1}, \ldots, \mathrm{e}_{p_n})$, $p_1, \ldots, p_n \in E$, and to ask it to be close to the product of the distributions of the e_{p_i}. For example, Pisier [58, Lemma 2.7] gives the following characterisation: E is a Sidon set if and only if there are a neighbourhood V of 1 in $\mathbb{T}$ and $0 < \varrho < 1$ such that for any finite $F \subseteq E$

$$m[\mathrm{e}_p \in V : p \in F] \le \varrho^{|F|}. \tag{40}$$

Murai [54, §4.2] calls $E \subseteq \mathbb{Z}$ pseudo-independent if for all $A_1, \ldots, A_n \subseteq \mathbb{T}$

$$m[\mathrm{e}_{p_i} \in A_i : 1 \le i \le n] \xrightarrow[\substack{p_i \in E \\ p_i \to \infty}]{} \prod_{i=1}^{n} m[\mathrm{e}_{p_i} \in A_i] = \prod_{i=1}^{n} m[A_i]. \tag{41}$$

We have

Proposition 12.2.1 *Let $E \subseteq \mathbb{Z}$. The following are equivalent.*
(i) E is pseudo-independent,
(ii) E enjoys $(\mathscr{I}_\infty)$,
(iii) For every $\varepsilon > 0$ and $m \ge 1$, there is a finite subset $G \subseteq E$ such that the Sidon constant of any subset of $E \setminus G$ with m elements is less than $1 + \varepsilon$.

Note that by Corollary 3.4.3, (41) does not imply (40).
Proof. $(i) \Leftrightarrow (ii)$ follows by Proposition 9.1.1(iii) and [54, Lemma 30]. $(iii) \Rightarrow (ii)$ is true because (iii) is just what is needed to draw our conclusion in Corollary 3.4.3. Let us prove $(i) \Rightarrow (iii)$. Let $\varepsilon > 0$, $m \ge 1$ and $\mathscr{A}$ be a covering of $\mathbb{T}$ with intervals of length ε. By (41), there is a finite set $G \subseteq E$ such that for $p_1, \ldots, p_m \in E \setminus G$ and $A_i \in \mathscr{A}$ we have $m[\mathrm{e}_{p_i} \in A_i : A_i \in \mathscr{A}] > 0$. But then

$$\left\| \sum a_i \, \mathrm{e}_{p_i} \right\|_\infty \ge \sum |a_i| \cdot (1 - \varepsilon). \qquad \blacksquare$$

Remark 12.2.2 $(ii) \Rightarrow (iii)$ may be proved directly by the technique of Riesz products: see [33, Appendix V, §1.II].

Another possibility is to define some notion of almost independence. Berkes [1] introduces the following notion: let us call a sequence of random variables $\{X_n\}$ almost i.i.d. (independent and identically distributed) if, after enlarging the probability space, there is an i.i.d. sequence $\{Y_n\}$ such that $\|X_n - Y_n\|_\infty \to 0$. We have the straightforward

Proposition 12.2.3 *Let $E = \{n_k\} \subseteq \mathbb{Z}$. If E is almost i.i.d., then E is a Sidon set with constant asymptotically 1.*

Proof. Let $\{Y_j\}$ be an i.i.d sequence and suppose $\|\mathrm{e}_{n_j} - Y_j\|_\infty \le \varepsilon$ for $j \ge k$. Then

$$\sum_{j \ge k} |a_j| = \left\| \sum_{j \ge k} a_j Y_j \right\|_\infty \le \left\| \sum_{j \ge k} a_j \, \mathrm{e}_{n_j} \right\|_\infty + \varepsilon \sum_{j \ge k} |a_j|$$

and the unconditionality constant of $\{n_k, n_{k+1}, \ldots\}$ is less than $(1 - \varepsilon)^{-1}$. ■

Suppose $E = \{n_k\} \subseteq \mathbb{Z}$ is such that n_{k+1}/n_k is an integer for all k. In that case, Berkes [1] proves that E is almost i.i.d. if and only if $n_{k+1}/n_k \to \infty$. We thus recover a part of Theorem 10.3.1.
Question 12.2.4 What about the converse in Proposition 12.2.3?

13 Summary of results. Remarks and questions

For the convenience of the reader, we now reorder our results by putting together those which are relevant to a given class of Banach spaces.

Let us first summarise our arithmetical results on the geometric sequence $G = \{j^k\}_{k\geq 0}$ $(j \in \mathbb{Z} \setminus \{-1,0,1\})$. The number given in the first (*vs.* second, third) column is the value $n \geq 1$ for which the set in the corresponding row achieves exactly $(\mathscr{I}_n)$ (*vs.* complex $(\mathscr{J}_n)$, real $(\mathscr{J}_n)$).

$G = \{j^k\}_{k\geq 0}$ with $\|j\| \geq 2$	$(\mathscr{I}_n)$	$\mathbb{C}$-$(\mathscr{J}_n)$	$\mathbb{R}$-$(\mathscr{J}_n)$
G, $j > 0$ odd	$\|j\|$	$\|j\|$	∞
G, $j > 0$ even	$\|j\|$	$\|j\|$	$\|j\|$
$G \cup \{0\}$, $j > 0$ odd	$\|j\|$	$\|j\|-1$	∞
$G \cup \{0\}$, $j > 0$ even	$\|j\|$	$\|j\|-1$	$\|j\|-1$
G, $G \cup \{0\}$, $j < 0$ odd	$\|j\|-1$	$\|j\|$	∞
G, $G \cup \{0\}$, $j < 0$ even	$\|j\|-1$	$\|j\|$	$\|j\|$
$G \cup -G$, $G \cup -G \cup \{0\}$, j odd	1	1	∞
$G \cup -G$, j even	1	1	$\|j\|$
$G \cup -G \cup \{0\}$, j even	1	1	$\|j\|/2$

Table 13.1

13.1 The case $X = \mathrm{L}^p(\mathbb{T})$ with p an even integer

Let $p \geq 4$ be an even integer. We observed the following facts.

- Real and complex (*umap*) differ among subspaces $\mathrm{L}^p_E(\mathbb{T})$ for each p: consider Proposition 10.1.1 or $\mathrm{L}^p_E(\mathbb{T})$ with $E = \{\pm(p/2)^k\}$.
- By Theorem 8.2.1, $\mathrm{L}^p_E(\mathbb{T})$ has complex (*vs.* real) (*umap*) if so does $\mathrm{L}^{p+2}_E(\mathbb{T})$;
- The converse is false for any p. In the complex case, $E = \{(p/2)^k\}$ is a counterexample. In the real case, take $E = \{0\} \cup \{\pm p^k\}$.
- Property (*umap*) is not stable under unions with an element: for each p, there is a set E such that $\mathrm{L}^p_E(\mathbb{T})$ has complex (*vs.* real) (*umap*), but $\mathrm{L}^p_{E\cup\{0\}}(\mathbb{T})$ does not. In the complex case, consider $E = \{(p/2)^k\}$. In the real case, consider $E = \{\pm(2\lceil p/4\rceil)^k\}$.
- If E is a symmetric set and $p \neq 2$, then $\mathrm{L}^p_E(\mathbb{T})$ fails complex (*umap*). Proposition 9.3.1 gives a criterion for real (*umap*).

What is the relationship between (*umbs*) and complex (*umap*)? We have by Proposition 9.1.2(*i*) and 8.2.4(*i*)

Proposition 13.1.1 *Let $E = \{n_k\} \subseteq \mathbb{Z}$ and $n \geq 1$.*
(i) If E is a (umbs) in $\mathrm{L}^{4n-2}(\mathbb{T})$, then $\mathrm{L}^{2n}_E(\mathbb{T})$ has complex (umap).
(ii) If $\{\pi_k\}$ realises complex (umap) in $\mathrm{L}^{2n}_E(\mathbb{T})$, then E is a (umbs) in $\mathrm{L}^{2n}(\mathbb{T})$.

We also have, by Proposition 11.2(*i*)

Proposition 13.1.2 *Let $E \subseteq \mathbb{Z}$ and $p \neq 2, 4$ an even integer. If $\mathrm{L}^p_E(\mathbb{T})$ has real (umap), then $d^*(E) = 0$.*

Note also this consequence of Propositions 4.3.1, 9.4.1, 12.2.1 and Theorems 3.4.2, 8.2.1

Proposition 13.1.3 *Let $\sigma > 1$ and $E = \{[\sigma^k]\}$. Then the following properties are equivalent:*
(i) σ is transcendental;
(ii) $\mathrm{L}^p_E(\mathbb{T})$ has complex (umap) for any even integer p;
(iii) E is a (umbs) in any $\mathrm{L}^p(\mathbb{T})$, p an even integer;
(iv) E is pseudo-independent.
(v) For every $\varepsilon > 0$ and $m \geq 1$, there is an l such that for $k_1, \ldots, k_m \geq l$ the Sidon constant of $\{[\sigma^{k_1}], \ldots, [\sigma^{k_m}]\}$ is less than $1 + \varepsilon$.

13.2 Cases $X = \mathrm{L}^p(\mathbb{T})$ with p not an even integer and $X = \mathscr{C}(\mathbb{T})$

In this section, X denotes either $\mathrm{L}^p(\mathbb{T})$, p not an even integer, or $\mathscr{C}(\mathbb{T})$.
Theorems 3.4.2 and 8.2.1 only permit us to use the negative results of Section 9: thus, we can just gather negative results about the functional properties of E. For example, we know by Proposition 9.1.2(*iv*) that $(\mathscr{I}_\infty)$ and $(\mathscr{J}_\infty)$ are stable under union with an element. Nevertheless, we cannot conclude that the same holds for (*umap*). The negative results are (by Section 9):

- for any infinite $E \subseteq \mathbb{Z}$, $X_{E\cup 2E}$ fails real (*umap*). Thus (*umap*) is not stable under unions;
- if E is a polynomial sequence (see Section 9), then E is not a (*umbs*) in X and X_E fails real (*umap*);
- if E is a symmetric set, then E is not a (*umbs*) in X and X_E fails complex (*umap*). Proposition 9.3.1 gives a criterion for real (*umap*);
- if $E = \{[\sigma^k]\}$ with $\sigma > 1$ an algebraic number — in particular if E is a geometric sequence —, then E is not a (*umbs*) in X and X_E fails complex (*umap*).

Furthermore, by Proposition 10.1.1, real and complex (*umap*) differ in X.
Theorem 10.3.1 is the only but general positive result on (*umbs*) and complex (*umap*) in X. Proposition 10.1.1 yields further examples for real (*umap*).
What about the sets that satisfy $(\mathscr{I}_\infty)$ or $(\mathscr{J}_\infty)$? We only know that $(\mathscr{I}_\infty)$ does not even ensure Sidonicity by Corollary 3.4.3.
One might wonder whether for some reasonable class of sets E, E is a finite union of sets that enjoy $(\mathscr{I}_\infty)$ or $(\mathscr{J}_\infty)$. This is false even for Sidon sets: for example, let E be the geometric sequence $\{j^k\}_{k\geq 0}$ with $j \in \mathbb{Z} \setminus \{-1, 0, 1\}$ and suppose $E = E_1 \cup \cdots \cup E_n$. Then $E_i = \{j^k\}_{k\in A_i}$, where the A_i's are a partition of the set of positive integers. But then one of the A_i contains arbitrarily large a and b such that $|a - b| \leq n$. This means that there is an infinite subset $B \subseteq A_i$ and an h, $1 \leq h \leq n$, such that $h + B \subseteq A_i$. We may apply Proposition 9.1.2(*vi*): E_i enjoys neither $(\mathscr{I}_{j^h+1})$ nor complex $(\mathscr{J}_{j^h+1})$ — nor real $(\mathscr{J}_{j^h+1})$ if furthermore j is even.
Does Proposition 13.1.1(*ii*) remain true for general X? We do not know this. Suppose however that we know that $\{\pi_k\}$ realises (*umap*) in the following strong manner: for any $\varepsilon > 0$, a tail $\{\pi_k\}_{k\geq l}$ is a $(1+\varepsilon)$-unconditional a.s. in X_E. Then E is trivially a (*umbs*) in X. In particular, this is the case if

$$1 + \varepsilon_n = \sup_{\epsilon\in\mathbb{S}} \|\mathrm{Id} - (1+\epsilon)\pi_n\|_{\mathscr{L}(X)}$$

converges so rapidly to 1 that $\sum \varepsilon_n < \infty$. Indeed,

$$\sup_{\epsilon_k\in\mathbb{S}} \|\pi_{n-1} + \sum_{k\geq n} \epsilon_k \Delta\pi_k\| \leq (1+\varepsilon_n) \sup_{\epsilon_k\in\mathbb{S}} \|\pi_n + \sum_{k>n} \epsilon_k \Delta\pi_k\|.$$

and thus, for all $f \in \mathscr{P}_E(\mathbb{T})$,

$$\sup_{\epsilon_k\in\mathbb{S}} \|\pi_l f + \sum_{k>l} \epsilon_k \Delta\pi_k f\| \leq \prod_{k>l} (1+\varepsilon_k)\, \|f\|.$$

Let us finally state

Proposition 13.2.1 *Let $E \subseteq \mathbb{Z}$. If X_E has real (umap), then $d^*(E) = 0$.*

13.3 Questions

The following questions remain open:

Combinatorics Regarding Proposition 11.2(*i*), is there a set E enjoying $(\mathscr{J}_2)$ with positive maximal density, or even with a uniformly bounded pace? Furthermore, may a set E with positive maximal density admit a partition $E = \bigcup E_i$ in finite sets such that all $E_i + E_j$, $i \leq j$, are pairwise disjoint? Then $\mathrm{L}^4_E(\mathbb{T})$ would admit a 1-unconditional (*fdd*) by Proposition 8.2.4(*i*).

Functional analysis Let $X \in \{\mathrm{L}^1(\mathbb{T}), \mathscr{C}(\mathbb{T})\}$ and consider Theorem 7.2.3. Is $(\mathscr{U})$ sufficient for X_E to share (*umap*)? Is there a set $E \subseteq \mathbb{Z}$ such that some space $\mathrm{L}^p_E(\mathbb{T})$, p not an even integer, has (*umap*), while $\mathscr{C}_E(\mathbb{T})$ fails it?

Harmonic analysis Is there a Sidon set $E = \{n_k\} \subseteq \mathbb{Z}$ of constant asymptotically 1 such that n_{k+1}/n_k is uniformly bounded? What about the case $E = [\sigma^k]$ for a transcendental $\sigma > 1$? If E enjoys $(\mathscr{I}_\infty)$, is E a (*umbs*) in $\mathrm{L}^p(\mathbb{T})$ $(1 \leq p < \infty)$? What about $(\mathscr{J}_\infty)$?

References

[1] I. Berkes. On almost i. i. d. subsequences of the trigonometric system. In E. W. Odell and H. P. Rosenthal, editors, *Functional analysis (Austin, 1986–87)*, Lect. Notes Math. 1332, pages 54–63. Springer, 1988. (pp. 12, 13, and 41).

[2] I. Berkes. Probability theory of the trigonometric system. In I. Berkes, E. Csáki, and P. Révész, editors, *Limit theorems in probability and statistics (Pécs, 1989)*, Coll. Math. Soc. János Bolyai 57, pages 35–58. North-Holland, 1990. (p. 12).

[3] J. P. M. Binet. Note sur une question relative à la théorie des nombres. *C. R. Acad. Sci. Paris*, 12:248–250, 1841. (p. 37).

[4] Errett Bishop. A general Rudin–Carleson theorem. *Proc. A. M. S.*, 13:140–143, 1962. (p. 30).

[5] J. Bourgain and H. P. Rosenthal. Geometrical implications of certain finite dimensional decompositions. *Bull. Soc. Math. Belg. (B)*, 32:57–82, 1980. (p. 12).

[6] V. Brouncker. Lettre X. Vicomte Brouncker à John Wallis. In *Œuvres de Fermat 3*, pages 419–420. Gauthier-Villars, 1896. (p. 21).

[7] Blei Ron C. A simple Diophantine condition in harmonic analysis. *Studia Math.*, 52:195–202, 1974/75. (pp. 27 and 30).

[8] P. G. Casazza and N. J. Kalton. Notes on approximation properties in separable Banach spaces. In P. F. X. Müller and W. Schachermayer, editors, *Geometry of Banach spaces (Strobl, 1989)*, London Math. Soc. Lect. Notes 158, pages 49–63. Cambridge Univ. Press, 1991. (pp. 11 and 22).

[9] F. Delbaen, H. Jarchow, and A. Pełczyński. Subspaces of L_p isometric to subspaces of l_p. *Positivity*, 2:339–367, 1998. (p. 34).

[10] R. Deville, G. Godefroy, and V. Zizler. *Smoothness and renormings in Banach spaces*. Pitman Monographs and Surveys 64. Longman, 1993. (p. 12).

[11] Diophantus of Alexandria. *Les six livres arithmétiques et le livre des nombres polygones*. Blanchard, 1959. (p. 21).

[12] S. W. Drury. Birelations and Sidon sets. *Proc. Amer. Math. Soc.*, 53:123–128, 1975. (p. 15).

[13] Randy L. Ekl. Equal sums of four seventh powers. *Math. Comp.*, 65:1755–1756, 1996. (p. 21).

[14] Randy L. Ekl. New results in equal sums of like powers. *Math. Comp.*, 67(223):1309–1315, 1998. (p. 21).

[15] L. Euler. Solutio generalis quorundam problematum Diophanteorum, quae vulgo nonnisi solutiones speciales admittere videntur. In *Op. Omnia (I) II*, pages 428–458. Teubner, 1915. (p. 37).

[16] L. Euler. Observationes circa bina biquadrata, quorum summam in duo alia biquadrata resolvere liceat. In *Op. Omnia (I) III*, pages 211–217. Teubner, 1917. (p. 21).

[17] L. Euler. Specimen algorithmi singularis. In *Op. Omnia (I) XV*, pages 31–49. Teubner, 1927. (p. 37).

[18] Moshe Feder. On subspaces of spaces with an unconditional basis and spaces of operators. *Illinois J. Math.*, 24(2):196–205, 1980. (pp. 11, 22, and 29).

[19] F. Forelli. The isometries of H^p. *Can. J. Math.*, 16:721–728, 1964. (pp. 11 and 17).

[20] J. J. F. Fournier. Two UC-sets whose union is not a UC-set. *Proc. Amer. Math. Soc.*, 84:69–72, 1982. (p. 14).

[21] G. Godefroy and N. J. Kalton. Approximating sequences and bidual projections. *Quart. J. Math. Oxford (2)*, 48:179–202, 1997. (pp. 22, 23, and 29).

[22] G. Godefroy, N. J. Kalton, and D. Li. On subspaces of L^1 which embed into ℓ_1. *J. reine angew. Math.*, 471:43–75, 1996. (pp. 12, 22, and 28).

[23] G. Godefroy, N. J. Kalton, and P. D. Saphar. Unconditional ideals in Banach spaces. *Studia Math.*, 104:13–59, 1993. (p. 22).

[24] G. Godefroy and P. D. Saphar. Duality in spaces of operators and smooth norms on Banach spaces. *Ill. J. Math.*, 32:672–695, 1988. (p. 28).

[25] H. Halberstam and K. F. Roth. *Sequences*. Springer, second edition, 1983. (p. 40).

[26] P. Harmand, D. Werner, and W. Werner. *M-ideals in Banach spaces and Banach algebras*. Springer, 1993. (p. 11).

[27] S. Hartman. Some problems and remarks on relative multipliers. *Colloq. Math.*, 54:103–111, 1987. Erratum dans Colloq. Math. 57 (1989), 189. (p. 14).

[28] N. Hindman. On density, translates, and pairwise sums of integers. *J. Combin. Theory (A)*, 33:147–157, 1982. (pp. 39 and 40).

[29] B. Host, J.-F. Méla, and F. Parreau. *Analyse harmonique des mesures*. Astérisque 135–136. Société mathématique de France, 1986. (p. 27).

[30] W. B. Johnson, H. P. Rosenthal, and M. Zippin. On bases, finite dimensional decompositions and weaker structures in Banach spaces. *Israel J. Math.*, 9:488–506, 1971. (p. 26).

[31] Vladimir M. Kadets. Some remarks concerning the Daugavet equation. *Quaestiones Math.*, 19(1-2), 1996. (p. 30).

[32] J.-P. Kahane. Sur les fonctions moyenne-périodiques bornées. *Ann. Inst. Fourier*, 7:293–314, 1957. (p. 15).

[33] Jean-Pierre Kahane and Raphaël Salem. *Ensembles parfaits et séries trigonométriques*. Hermann, 1963. (pp. 39 and 41).

[34] N. J. Kalton. Spaces of compact operators. *Math. Ann.*, 208:267–278, 1974. (p. 28).

[35] N. J. Kalton. M-ideals of compact operators. *Illinois J. Math.*, 37(1):147–169, 1993. (p. 11).

[36] N. J. Kalton and Dirk Werner. Property (M), M-ideals, and almost isometric structure of Banach spaces. *J. reine angew. Math.*, 461:137–178, 1995. (pp. 12, 22, 25, 27, and 28).

[37] S. Karlin. Bases in Banach spaces. *Duke Math. J.*, 15:971–985, 1948. (p. 14).

[38] Yitzhak Katznelson. *An introduction to harmonic analysis*. John Wiley & Sons Inc., New York, 1968. (pp. 13 and 14).

[39] Timo Ketonen. On unconditionality in L_p spaces. *Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes*, 35, 1981. (p. 14).

[40] A. L. Koldobsky. Isometries of $L_p(X; L_q)$ and equimeasurability. *Indiana Math. J.*, 40:677–705, 1991. (p. 12).

[41] L. J. Lander, T. R. Parkin, and J. L. Selfridge. A survey of equal sums of like powers. *Math. Comp.*, 21:446–459, 1967. (p. 21).

[42] Daniel Li. On Hilbert sets and $C_\Lambda(G)$-spaces with no subspace isomorphic to c_0. *Coll. Math.*, 68:67–77, 1995. Addendum, ibid., p. 79. (p. 27).

[43] Daniel Li. Complex unconditional metric approximation property for $\mathscr{C}_\Lambda(\mathbb{T})$ spaces. *Studia Math.*, 121(3):231–247, 1996. (pp. 11, 13, 19, 27, 29, 32, 33, 34, 39, and 40).

[44] Joram Lindenstrauss and Lior Tzafriri. *Classical Banach spaces I. Sequence spaces*. Springer, 1977. (p. 22).

[45] J. E. Littlewood and R. E. A. C. Paley. Theorems on Fourier series and power series. *J. London Math. Soc.*, 6:230–233, 1931. (p. 29).

[46] Jorge M. López and Kenneth A. Ross. *Sidon sets*. Lecture Notes Pure Appl. Math. 13. Marcel Dekker Inc., New York, 1975. (p. 39).

[47] Françoise Lust-Piquard. Ensembles de Rosenthal et ensembles de Riesz. *C. R. Acad. Sci. Paris (A)*, 282:833–835, 1976. (p. 30).

[48] Bernard Maurey. Isomorphismes entre espaces H_1. *Acta Math.*, 145:79–120, 1980. (p. 30).

[49] Yves Meyer. Endomorphismes des idéaux fermés de $L^1(G)$, classes de Hardy et séries de Fourier lacunaires. *Ann. sci. École Norm. Sup. (4)*, 1:499–580, 1968. (p. 30).

[50] Yves Meyer. *Algebraic numbers and harmonic analysis*. North-Holland, 1972. (pp. 34 and 39).

[51] I. M. Mikheev. On lacunary series. *Math. USSR–Sb.*, 27:481–502, 1975. (p. 39).

[52] A. Moessner. Einige numerische Identitäten. *Proc. Nat. Acad. Sci. India (A)*, 10:296–306, 1939. (p. 21).

[53] L. J. Mordell. *Diophantine equations*. Academic Press, 1969. (p. 37).

[54] T. Murai. On lacunary series. *Nagoya Math. J.*, 85:87–154, 1982. (pp. 12, 13, 20, and 41).

[55] Stefan Neuwirth. The size of bipartite graphs with girth eight. `http://arxiv.org/math/0102210`, 2001. (p. 39).

[56] A. Pełczyński. On simultaneous extension of continuous functions. A generalization of theorems of Rudin–Carleson and Bishop. *Studia Math.*, 24:285–304, 1964. Supplement, ibid. 25 (1965), 157–161. (p. 30).

[57] A. Pełczyński and P. Wojtaszczyk. Banach spaces with finite dimensional expansions of identity and universal bases of finite dimensional subspaces. *Studia Math.*, 40:91–108, 1971. (p. 22).

[58] G. Pisier. Conditions d'entropie et caractérisations arithmétiques des ensembles de Sidon. In L. De Michele and F. Ricci, editors, *Topics in modern harmonic analysis II (Torino/Milano 1982)*, pages 911–944. Ist. Naz. Alta Mat. Francesco Severi, 1983. (p. 41).

[59] Gilles Pisier. De nouvelles caractérisations des ensembles de Sidon. In *Mathematical analysis and applications, part B*, Advances in Math. Suppl. Series 7B, pages 685–726. Academic Press, 1981. (p. 11).

[60] A. I. Plotkin. Continuation of L^p-isometries. *J. Sov. Math.*, 2:143–165, 1974. (pp. 11 and 17).

[61] S. Ramanujan. *Notebooks*. Tata, 1957. (p. 37).

[62] S. K. Rao. On sums of sixth powers. *J. London Math. Soc.*, 9:172–173, 1934. (p. 21).

[63] Éric Ricard. *Décompositions de H^1, multiplicateurs de Schur et espaces d'opérateurs*. PhD thesis, Université Paris 6, 2001. `www.institut.math.jussieu.fr/theses/2001/ricard`. (p. 30).

[64] M. Rosenblatt. A central limit theorem and a strong mixing condition. *Proc. Nat. Acad. Sci. U. S. A.*, 42:43–47, 1956. (p. 12).

[65] Haskell P. Rosenthal. *Sous-espaces de L^1. Cours de troisième cycle*. Université Paris 6, 1979. Unpublished. (p. 28).

[66] Walter Rudin. Trigonometric series with gaps. *J. Math. Mech.*, 9:203–228, 1960. (pp. 15, 17, 19, 21, 35, and 39).

[67] Walter Rudin. L^p-isometries and equimeasurability. *Indiana Univ. Math. J.*, 25:215–228, 1976. (p. 16).

[68] I. Z. Ruzsa. On difference sets. *Studia Sci. Math. Hung.*, 13:319–326, 1978. (p. 40).

[69] Josef A. Seigner. Rademacher variables in connection with complex scalars. *Acta Math. Univ. Comenian. (N.S.)*, 66:329–336, 1997. (pp. 14, 17, and 40).

[70] I. Singer. *Bases in Banach spaces II*. Springer, 1981. (p. 24).

[71] C. M. Skinner and T. D. Wooley. On equal sums of two powers. *J. reine angew. Math.*, 462:57–68, 1995. (p. 21).

[72] Elias Stein. Classes H^p, multiplicateurs et fonctions de Littlewood-Paley. *C. R. Acad. Sci. Paris Sér. A-B*, 263:A716–A719, 1966. (p. 30).

[73] Elias Stein. Classes H^p, multiplicateurs et fonctions de Littlewood-Paley. Applications de résultats antérieurs. *C. R. Acad. Sci. Paris Sér. A-B*, 263:A780–A781, 1966. (p. 30).

[74] C. L. Stewart and R. Tijdeman. On infinite-difference sets. *Can. J. Math.*, 31:897–910, 1979. (p. 40).

[75] Valérie Tardivel. Ensembles de Riesz. *Trans. Amer. Math. Soc.*, 305(1):167–174, 1988. (p. 30).

[76] Dirk Werner. The Daugavet equation for operators on function spaces. *J. Funct. Anal.*, 143(1):117–128, 1997. (p. 30).

[77] P. Wojtaszczyk. The Banach space H_1. In *Functional analysis: surveys and recent results III (Paderborn, 1983)*, pages 1–33. North-Holland, 1984. (p. 30).

Index of notation

$\lvert B \rvert$	cardinal of B
X_E	space of X-functions with spectrum in E
$\widehat{f}$	Fourier transform of f: $\widehat{f}(n) = \int f(t)\,\mathrm{e}_{-n}(t)dm(t)$
$\binom{x}{\alpha}$	multinomial number, §3.3
$\langle \zeta, E \rangle$	pairing of the arithmetical relation ζ against the spectrum E, §4.1
$u_n \preccurlyeq v_n$	$\lvert u_n \rvert$ is bounded by $C \lvert v_n \rvert$ for some C
1-(*ubs*)	1-unconditional basic sequence of characters, Def. 3.1.1(i)
$A(\mathbb{T})$	disc algebra $\mathscr{C}_{\mathbb{N}}(\mathbb{T})$
$\mathrm{A}_n, \mathrm{A}_n^m$	sets of multi-indices viewed as arithmetic relations, §3.2
a.s.	approximating sequence, Def. 5.1.1
B_X	unit ball of the Banach space X
$\mathscr{C}(\mathbb{T})$	space of continuous functions on $\mathbb{T}$
$\mathbb{D}$	set of real signs $\{-1, 1\}$
ΔT_k	difference sequence of the T_k: $\Delta T_k = T_k - T_{k-1}$ $(T_0 = 0)$
e_n	character of $\mathbb{T}$: $\mathrm{e}_n(z) = z^n$ for $z \in \mathbb{T}$, $n \in \mathbb{Z}$
(*fdd*)	finite dimensional decomposition, Def. 5.1.1
$H^1(\mathbb{T})$	Hardy space $\mathrm{L}^1_{\mathbb{N}}(\mathbb{T})$
$(\mathscr{I}_n)$	arithmetical property of almost independence, Def. 3.4.1
Id	identity
i.i.d.	independent identically distributed, §12.2
$(\mathscr{J}_n)$	arithmetical property of block independence, Def. 8.1.2
$\mathscr{L}(X)$	space of bounded linear operators on the Banach space X
$\mathrm{L}^p(\mathbb{T})$	Lebesgue space of p-integrable functions on $\mathbb{T}$
ℓ_p-(*ap*)	p-additive approximation property, Def. 6.1.1
ℓ_p-(*map*)	metric p-additive approximation property, Def. 6.1.1
$\Lambda(p)$	Rudin's class of lacunary sets, Def. 3.1.6
$\mathscr{M}_p$	functional property of Fourier block p-additivity, Lemma 7.2.5(ii)
$\mathscr{M}(\mathbb{T})$	space of Radon measures on $\mathbb{T}$
$m[A]$	measure of $A \subseteq \mathbb{T}$
$(m_p(\tau))$	functional property of τ-p-additivity, Def. 6.3.1(i)
$(m_p(T_k))$	functional property of commuting block p-additivity, Def. 6.3.1(ii)
$\operatorname{osc} f$	oscillation of f
$\mathscr{P}(\mathbb{T})$	space of trigonometric polynomials on $\mathbb{T}$
π_j	projection of X_E, $E = \{n_k\}$, onto $X_{\{n_1,\dots,n_j\}}$
π_F	projection of X_E onto X_F
$\mathbb{S}$	real $(\mathbb{S} = \mathbb{D})$ or complex $(\mathbb{S} = \mathbb{T})$ choice of signs
$(\mathbb{T}, dm)$	unit circle in $\mathbb{C}$ with its normalised Haar measure
τ_f	topology of pointwise convergence of the Fourier coefficients, Lemma 7.2.2(i)
$(\mathscr{U})$	functional property of Fourier block unconditionality, Def. 7.2.1
$(u(\tau))$	functional property of τ-unconditionality, Def. 5.2.1(i)
$(u(T_k))$	functional property of commuting block unconditionality, Def. 5.2.1(ii)
(*uap*)	unconditional approximation property, Def. 5.1.1
(*ubs*)	unconditional basic sequence, Def. 3.1.1
(*umap*)	metric unconditional approximation property, Def. 5.1.1
(*umbs*)	metric unconditional basic sequence, Def. 3.1.1
$\mathrm{Z}^m, \mathrm{Z}_n^m$	sets of multi-indices viewed as arithmetic relations, §3.2

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