Cycles and 1-unconditional matrices

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Abstract

We characterise the 1-unconditional subsets $(\mathrm{e}_{rc})_{(r,c)\in I}$ of the set of elementary matrices in the Schatten-von-Neumann class S^p. The set of couples I must be the set of edges of a bipartite graph without cycles of even length $4 \leqslant p$ if p is an even integer, and without cycles at all if p is a positive real number that is not an even integer. In the latter case, I is even a Varopoulos set of V-interpolation of constant 1. We also study the metric unconditional approximation property for the space S^p_I spanned by $(\mathrm{e}_{rc})_{(r,c)\in I}$ in S^p.

Résumé en français

Je caractérise les sous-suites 1-inconditionnelles $(\mathrm{e}_{rc})_{(r,c)\in I}$ de la suite des matrices élémentaires dans la classe de Schatten-von-Neumann S^p. L'ensemble de couples I doit être l'ensemble des arêtes d'un graphe biparti sans cycle de longueur paire $l \in \{4, 6, \ldots, p\}$ si p est un entier pair, et sans cycle du tout si p est un réel positif qui n'est pas un entier pair. Dans ce dernier cas, I est même un ensemble de Varopoulos de V-interpolation de constante 1. J'étudie aussi la propriété d'approximation inconditionnelle métrique pour le sous-espace vectoriel fermé S^p_I engendré par $(\mathrm{e}_{rc})_{(r,c)\in I}$ dans S^p.

1 Introduction

The starting point for this investigation has been the following isometric question on the Schatten-von-Neumann class S^p.

Question 1.1. Which matrix coefficients of an operator $x \in \mathrm{S}^p$ must vanish so that the norm of x does not depend on the argument, or on the sign, of the remaining nonzero matrix coefficients?

Let C be the set of columns and R be the set of rows for coordinates in the matrix. Let $I \subseteq R \times C$ be the set of matrix coordinates of the nonzero matrix coefficients of x (the *pattern*.) Question 1.1 describes the notion of a complex, or real, 1-unconditional basic sequence $(\mathrm{e}_{rc})_{(r,c)\in I}$ of elementary matrices in S^p (see Definition 4.1.)

By a convexity argument, Question 1.1 is equivalent to the following question on Schur multiplication.

Question 1.2. Which matrix coefficients of an operator $x \in \mathrm{S}^p$ must vanish so that for all matrices φ of complex, or real, numbers

$$\|\varphi * x\| \leqslant \sup |\varphi_{rc}| \; \|x\|,$$

where $\varphi * x$ is the Schur (or Hadamard or entrywise) product defined by

$$(\varphi * x)_{rc} = \varphi_{rc} x_{rc}?$$

In the case $p = \infty$, Grothendieck's inequality yields an estimation for the norm of Schur multiplication by φ in terms of the projective tensor product $\ell^\infty_C \mathbin{\hat{\otimes}} \ell^\infty_R$: this norm is equivalent to the supremum of the norm of those elements of $\ell^\infty_C \mathbin{\hat{\otimes}} \ell^\infty_R$ whose coefficient matrices are finite submatrices of φ. In the framework of tensor algebras over discrete spaces, Question 1.2 turns out to describe as well the isometric counterpart to Varopoulos' V-Sidon sets as well as to his sets of V-interpolation. The following isometric question has however a different answer.

Question 1.3. Which coefficients of a tensor $u \in \ell^\infty_C \overset{\wedge}{\otimes} \ell^\infty_R$ must vanish so that the norm of u is the maximal modulus of its coefficients?

In our answer to Question 1.2, S^p and Schur multiplication are treated as a noncommutative analogue to L^p and convolution. The main step is a careful study of the Schatten-von-Neumann norm $\|x\| = \bigl(\mathrm{tr}(x^*x)^{p/2}\bigr)^{1/p}$ for p an even integer. The rule of matrix multiplication provides an expression for this norm as a series in the matrix coefficients of x and their complex conjugate, indexed by the $p$uples $(v_1, v_2, \ldots, v_p)$ satisfying $(v_{2i-1}, v_{2i}), (v_{2i+1}, v_{2i}) \in I$, where $v_{p+1} = v_1$: see the computation in Eq. (10). These are best understood as closed walks of length p on the bipartite graph G canonically associated to I: its vertex classes are C and R and its edges are given by the couples in I. A structure theorem for closed walks and a detailed study of the particular case in which G is a cycle yield the two following theorems that answer Questions 1.1 and 1.2.

Theorem 1.4. *Let $p \in (0, \infty] \setminus \{2, 4, 6, \ldots\}$. If the sequence of elementary matrices $(\mathrm{e}_{rc})_{(r,c)\in I}$ is a real 1-unconditional basic sequence in S^p, then the graph G associated to I contains no cycle. In this case, I is even a set of V-interpolation with constant 1: every sequence $\varphi \in \ell^\infty_I$ may be interpolated by a tensor $u \in \ell^\infty_C \overset{\wedge}{\otimes} \ell^\infty_R$ such that $\|u\| = \|\varphi\|$.*

Theorem 1.5. *Let $p \in \{2, 4, 6, \ldots\}$. The sequence $(\mathrm{e}_{rc})_{(r,c)\in I}$ is a complex, or real, 1-unconditional basic sequence in S^p if and only if G contains no cycle of length $4, 6, \ldots, p$.*

These theorems hold also for the complete counterparts to 1-unconditional basic sequences in the sense of Def. 4.1(c).

In particular, if we denote by U_p the property that $(\mathrm{e}_{rc})_{(r,c)\in I}$ is a 1-unconditional basic sequence in S^p, then we obtain the following hierarchy:

$$\mathrm{U}_p \text{ for a } p \in (0,\infty] \setminus \{2,4,6,\ldots\} \;\Rightarrow\; \cdots \;\Rightarrow\; \mathrm{U}_{2n+2} \;\Rightarrow\; \mathrm{U}_{2n} \;\Rightarrow\; \cdots \;\Rightarrow\; \mathrm{U}_2\,.$$

If C and R are finite, extremal graphs without cycles of given lengths remain an ongoing area of research in graph theory. Finite geometries seem to provide all known examples of such graphs when C and R become large. Proposition 11.6 and Remark 11.7 gather up known facts on this issue.

One may also avoid the terminology of graph theory and give an answer in terms of polygons drawn in a matrix by joining matrix coordinates with sides that follow alternately the row (horizontal) and the column (vertical) direction of the matrix:

- Suppose that p is not an even integer. If a pattern I contains the vertices of such a polygon, then there is an operator $x \in \mathrm{S}^p$ whose matrix coefficients vanish outside I and whose norm depends on the sign of its matrix coefficients. This condition is also necessary.

- If matrix coordinates of nonzero matrix coefficients of x are the vertices of such a polygon with n sides, then the norm of x in S^p depends on the argument of its matrix coefficients for every even integer $p \geqslant n$; if the matrix coefficients of x are real, then the norm of x even depends on the sign of its matrix coefficients. These conditions are also necessary.

An elementary example is given by the set

$$I = \bigl\{(r,c) \in \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z} : r + c \in \{0,1,3\}\bigr\}. \tag{1}$$

The associated bipartite graph is known as the Heawood graph (Fig. 1:) it is the incidence graph of the Fano plane (the finite projective plane $\mathrm{PG}(2,2)$,) which is the smallest generalised triangle, and corresponds to the Steiner system $\mathrm{S}(2,3;7)$. It contains no cycle of length 4, but every pair of vertices is contained in a cycle of length 6.

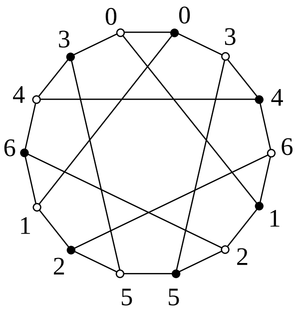

Figure 1: The Heawood graph

Thus the p-trace norm of every matrix with pattern

$$\begin{array}{c} \\ 0\\1\\2\\3\\4\\5\\6 \end{array}\begin{array}{c} \begin{array}{ccccccc} 0&1&2&3&4&5&6 \end{array} \\ \begin{pmatrix} * & * & 0 & * & 0 & 0 & 0 \\ * & 0 & * & 0 & 0 & 0 & * \\ 0 & * & 0 & 0 & 0 & * & * \\ * & 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & * & * & 0 & * & 0 \\ 0 & * & * & 0 & * & 0 & 0 \end{pmatrix} \end{array}$$

does not depend on the sign of its coefficients if and only if $p \in \{2, 4\}$.

These results give a complete description of the situation in which $(\mathrm{e}_{rc})_{(r,c)\in I}$ is a 1-unconditional basis of the space S^p_I it spans in S^p. If this is not the case, S^p_I might still admit some other 1-unconditional basis. This leads to the following more general question.

Question 1.6. For which sets I does S^p_I admit some kind of almost 1-unconditional finite dimensional expansion of the identity?

The metric unconditional approximation property (*muap*) provides a formal definition for the object of Question 1.6: see Def. 10.1. We obtain the following results.

Theorem 1.7. *Let $p \in [1, \infty] \setminus \{2, 4, 6, \ldots\}$. If S^p_I has real (muap), then the distance of any two vertices that are not in the same vertex class is asymptotically infinite in G: their distance becomes arbitrarily large by deleting a finite number of edges from G.*

Theorem 1.8. *Let $p \in \{2, 4, 6, \ldots\}$. The space S^p_I has complex, or real, (muap) if and only if any two vertices at distance $2j + 1 \leqslant p/2$ are asymptotically at distance at least $p - 2j + 1$.*

We now turn to a detailed description of this article. In Section 2, we provide tools for the computation of Schur multiplier norms. Section 3 characterises idempotent Schur multipliers and 0, 1-tensors in $\ell^\infty_C \overset{\wedge}{\otimes} \ell^\infty_R$ of norm 1. In Section 4, we define the complex and real unconditional constants of basic sequences of elementary matrices and show that they are not equal in general. Section 5 looks back on Varopoulos' results about tensor algebras over discrete spaces. Section 6 puts the connection between p-trace norm and closed walks of length p in the concrete form of closed walk relations. In Section 7, we compute the norm of relative Schur multipliers by signs in the case that G is a cycle, and estimate the corresponding unconditional constants. Section 8 is dedicated to a proof of Th. 1.4 and an answer to Question 1.3. Section 9 establishes Th. 1.5. In Section 10, we study the metric unconditional approximation property for spaces S^p_I. The final section provides four kinds of examples: sets obtained by a transfer of n-independent subsets of a discrete abelian group, Hankel sets, Steiner systems and Tits' generalised polygons.

Terminology. C is the set of *columns* and R is the set of *rows*, both finite or countable and if necessary indexed by natural numbers. V, the set of *vertices*, is their disjoint union $C \amalg R$: if there is a risk of confusion, an element $n \in V$ that is a column (vs. a row) will be referred to as "col n"

(vs. "row n".) An *edge* on V is a pair $\{v, w\} \subseteq V$. A *graph* on V is given by a set of edges E. A *bipartite* graph on V with vertex classes C and R has only edges $\{r, c\}$ such that $c \in C$ and $r \in R$ and may therefore be given alternatively by the set of couples $I = \bigl\{(r, c) \in R \times C : \{r, c\} \in E\bigr\}$: this will be our custom throughout the article. A bipartite graph on V is *complete* if its set of couples I is the whole of $R \times C$. Two graphs are *disjoint* if so are the sets of vertices of their edges. I is a *column section* if $(r, c), (r', c) \in I \Rightarrow r = r'$, and a *row section* if $(r, c), (r, c') \in I \Rightarrow c = c'$.

A *walk* of length $s \geqslant 0$ in a graph is a sequence $(v_0, \ldots, v_s)$ of $s+1$ vertices such that $\{v_0, v_1\}$, $\ldots$, $\{v_{s-1}, v_s\}$ are edges of the graph. A walk is a *path* if its vertices are pairwise distinct. The *distance* of two vertices in a graph is the minimal length of a path in the graph that joins the two vertices; it is infinite if no such path exists. A *closed* walk of length $p \geqslant 0$ in a graph is a sequence $(v_1, \ldots, v_p)$ of p vertices such that $\{v_1, v_2\}, \ldots, \{v_{p-1}, v_p\}, \{v_p, v_1\}$ are edges of the graph. Note that p is necessarily even if the graph is bipartite. A closed walk is a *cycle* if its vertices are pairwise distinct. We take the convention that if a closed walk in a bipartite graph on $V = C \amalg R$ is nonempty, then its first vertex is a column vertex: $v_1 \in C$. We shall identify a path and a cycle with its set of edges $\{r, c\}$ or the corresponding set of couples (r, c).

A bipartite graph on V is a *tree* if there is exactly one path between any two of its vertices. In this case, its vertices may be indexed by finite words over its set of vertices in the following way. Choose any row vertex r as *root* and index it by $\emptyset$. If v is a vertex and $(r, c, \ldots, v)$ is the unique path from r to v, let the word $c^\frown \cdots {}^\frown v$ index v. Let W be the set of all words thus formed. Then

- $\emptyset \in W$ and every beginning of a word in W is also in W: if $w \in W \setminus \{\emptyset\}$, then w is the concatenation $w'^\frown v$ of a word $w' \in W$ with a letter v;
- words of even length index row vertices;
- words of odd length index column vertices;
- a pair of vertices is an edge exactly if their indices have the form w and $w^\frown v$, where w is a word and v is a letter.

A *forest* is a union of pairwise disjoint trees; equivalently, it is a cycle free graph.

Notation. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

The unit ball of a Banach space X is denoted by B_X.

Given an index set I and $q \in I$, e_q is the sequence defined on I as the indicator function $\chi_{\{q\}}$ of the singleton $\{q\}$.

Let $I = R \times C$ and $q = (r, c)$. Then $\mathrm{e}_q = \mathrm{e}_{rc}$ is the *elementary matrix* identified with the operator from ℓ^2_C to ℓ^2_R that maps e_c on e_r and all other basis vectors on 0. The *matrix coefficient* at coordinate q of an operator x from ℓ^2_C to ℓ^2_R is $x_q = \operatorname{tr} \mathrm{e}_q^* x$ and its *matrix representation* is $(x_q)_{q \in R \times C} = \sum_{q \in R \times C} x_q \mathrm{e}_q$. The *support* of x is $\{q \in R \times C : x_q \neq 0\}$.

The *Schatten-von-Neumann class* S^p, $0 < p < \infty$, is the space of those compact operators x from ℓ^2_C to ℓ^2_R such that $\|x\|_p^p = \operatorname{tr} |x|^p = \operatorname{tr}(x^* x)^{p/2} < \infty$. S^∞ is the space of compact operators with the operator norm. S^p is a quasi-normed space, and a Banach space if $p \geqslant 1$. Let $(R_n \times C_n)_{n \geqslant 0}$ be a sequence of finite sets that tends to $R \times C$. Then the sequence of operators $P_n \colon x \mapsto \sum_{q \in R_n \times C_n} x_q \mathrm{e}_q$ tends pointwise to the identity on S^p if $p \geqslant 1$.

For $I \subseteq R \times C$, the *entry space* S^p_I is the subspace of those $x \in \mathrm{S}^p$ whose support is a subset of I. S^p_I is also the closed subspace of S^p spanned by $(\mathrm{e}_q)_{q \in I}$.

The S^p-valued Schatten-von-Neumann class $\mathrm{S}^p(\mathrm{S}^p)$ is the space of those compact operators x from ℓ^2_C to $\ell^2_R(\mathrm{S}^p)$ such that $\|x\|_p^p = \operatorname{tr}(\operatorname{tr} |x|^p) < \infty$, where the inner trace is the S^p-valued analogue of the usual trace: such operators have an S^p-valued matrix representation and their support is defined as in the scalar case. An element $x \in \mathrm{S}^p(\mathrm{S}^p)$ can also be considered as a compact operator from $\ell^2_C(\ell_2) = \ell_2 \otimes_2 \ell^2_C$ to $\ell^2_R(\ell_2) = \ell_2 \otimes_2 \ell^2_R$ such that $\|x\|_p^p = \operatorname{tr} \otimes \operatorname{tr} |x|^p < \infty$; the matrix coefficient of x at q is then $x_q = (\mathrm{Id}_{\mathrm{S}^p} \otimes \operatorname{tr})\bigl((\mathrm{Id}_{\ell_2} \otimes \mathrm{e}_q^*) x\bigr)$ and its matrix representation is $\sum_{q \in R \times C} x_q \otimes \mathrm{e}_q$. The entry space $\mathrm{S}^p_I(\mathrm{S}^p)$ is defined in the same way as S^p_I.

A *relative Schur multiplier* on S^p_I is a sequence $\varphi = (\varphi_q)_{q \in I} \in \mathbb{C}^I$ such that the associated *Schur multiplication operator* M_φ defined by $\mathrm{e}_q \mapsto \varphi_q \mathrm{e}_q$ for $q \in I$ is bounded on S^p_I. The Schur multiplier φ is furthermore *completely bounded* (c.b. for short) on S^p_I if $\mathrm{Id}_{\mathrm{S}^p} \otimes \mathrm{M}_\varphi$, the operator defined by $x_q \mathrm{e}_q \mapsto \varphi_q x_q \mathrm{e}_q$ for $x_q \in \mathrm{S}^p$ and $q \in I$, is bounded on $\mathrm{S}^p_I(\mathrm{S}^p)$ (see [21, Lemma 1.7].) The norm of

φ is the norm of M_φ and its *complete norm* is the norm of $\mathrm{Id}_{\mathrm{S}^p} \otimes \mathrm{M}_\varphi$. This norm is the supremum of the norm of its restrictions to finite rectangle sets $R' \times C'$. Note that φ is a Schur multiplier on S^∞ if and only if, for every bounded operator $x\colon \ell^2_C \to \ell^2_R$, $(\varphi_q x_q)$ is the matrix representation of a bounded operator; also φ is automatically c.b. on S^∞ [22, Th. 5.1]. We used [21, 22] as a reference.

Let G be a compact abelian group endowed with its normalised Haar measure. Let $\Gamma = \hat{G}$ be the dual group of characters on G. The *spectrum* of an integrable function f on G is $\{\gamma \in \Gamma : \hat{f}(\gamma) \neq 0\}$. Let $\Lambda \subseteq \Gamma$. If X is a space of integrable functions on G, then X_Λ is the *translation invariant* subspace of those $f \in X$ whose spectrum is a subset of Λ.

Let X be the space of continuous functions $\mathrm{C}(G)$ or the Lebesgue space $\mathrm{L}^p(G)$ with $0 < p < \infty$. Then X_Λ is also the closed subspace of X spanned by Λ. A *relative Fourier multiplier* on X_Λ is a sequence $\mu = (\mu_\gamma)_{\gamma\in\Lambda} \in \mathbb{C}^\Lambda$ such that the associated convolution operator M_μ defined by $\gamma \mapsto \mu_\gamma \gamma$ for $\gamma \in \Lambda$ is bounded on X_Λ. The Fourier multiplier μ is furthermore c.b. if $\mathrm{Id}_{\mathrm{S}^p} \otimes \mathrm{M}_\mu$, the operator defined by $a_\gamma \gamma \mapsto \mu_\gamma a_\gamma \gamma$ for $a_\gamma \in \mathrm{S}^p$ and $\gamma \in \Lambda$, is bounded on the S^p-valued space $X_\Lambda(\mathrm{S}^p)$ (where $p = \infty$ if $X = \mathrm{C}(G)$.) The norm of μ is the norm of M_μ and its complete norm is the norm of $\mathrm{Id}_{\mathrm{S}^p} \otimes \mathrm{M}_\mu$. Note that μ is a Fourier multiplier on $\mathrm{C}_\Lambda(G)$ if and only if, for every $f \in \mathrm{L}^\infty_\Lambda(G)$, $\sum \mu_\gamma \hat{f}(\gamma)\gamma$ is the Fourier series of an element of $\mathrm{L}^\infty_\Lambda(G)$: μ is a *relative Fourier multiplier on* $\mathrm{L}^\infty(G)$; also μ is automatically c.b. on $\mathrm{C}_\Lambda(G)$ [22, Cor. 3.18].

Let X, Y be Banach spaces and $u \in X \otimes Y$. Its *projective* tensor norm is

$$\|u\|_{X \overset{\wedge}{\otimes} Y} = \inf\Bigg\{\sum_{j=1}^n \|x_j\|\,\|y_j\| : u = \sum_{j=1}^n x_j \otimes y_j\Bigg\}$$

and $X \overset{\wedge}{\otimes} Y$ is the completion of $X \otimes Y$ with respect to this norm. Note that $\ell^n_\infty \overset{\wedge}{\otimes} \ell^m_\infty \subset \mathrm{c}_0 \overset{\wedge}{\otimes} \mathrm{c}_0$ because ℓ^n_∞ and ℓ^m_∞ are 1-complemented in c_0, and that $\mathrm{c}_0 \overset{\wedge}{\otimes} \mathrm{c}_0 \subset \ell_\infty \overset{\wedge}{\otimes} \ell_\infty$ because ℓ_∞ is the bidual of c_0.

Let $\sum x_j \otimes y_j$ be any representation of the tensor u. If $\xi \otimes \eta \in X^* \otimes Y^*$, we define $\langle \xi \otimes \eta, u\rangle = \sum \langle \xi, x_j\rangle\langle \eta, y_j\rangle$. The *injective* tensor norm of u is

$$\|u\|_{X \overset{\vee}{\otimes} Y} = \sup_{(\xi,\eta)\in B_{X^*}\times B_{Y^*}} |\langle \xi \otimes \eta, u\rangle|$$

and $X \overset{\vee}{\otimes} Y$ is the completion of $X \otimes Y$ with respect to this norm.

If X and Y are both finite dimensional, then

$$\big(X \overset{\vee}{\otimes} Y\big)^* = X^* \overset{\wedge}{\otimes} Y^* \quad \text{and} \quad \big(X \overset{\wedge}{\otimes} Y\big)^* = X^* \overset{\vee}{\otimes} Y^*.$$

Further $\big(\mathrm{c}_0 \overset{\wedge}{\otimes} \mathrm{c}_0\big)^* = \ell_1 \overset{\vee}{\otimes} \ell_1$: in fact, $\big(\mathrm{c}_0 \overset{\wedge}{\otimes} \mathrm{c}_0\big)^*$ may be identified with the space of bounded operators from c_0 to ℓ_1 and $\ell_1 \overset{\vee}{\otimes} \ell_1$ may be identified with the closure of finite rank operators in that space, and they are the same because every bounded operator from c_0 to ℓ_1 is compact and ℓ_1 has the approximation property.

If X is a sequence space on C and Y is a sequence space on R, then the *coefficient* of the tensor u at (r, c) is $\langle \mathrm{e}_c \otimes \mathrm{e}_r, u\rangle$. Its *support* is the set of coordinates (r, c) of its nonvanishing coefficients. One may use [26] as a reference.

2 Relative Schur multipliers

The following proposition is a straightforward consequence of [17].

Proposition 2.1. *Let $I \subseteq R \times C$ and φ be a Schur multiplier on S^∞_I with norm D. Then φ is also a c.b. Schur multiplier on S^p_I for every $p \in (0, \infty]$, with complete norm bounded by D.*

Proof. We may assume that $D = 1$. Let $R' \times C'$ be any finite subset of $R \times C$. By [17, Th. 3.2], there exist vectors w_c and v_r of norm at most 1 in a Hilbert space H such that $\varphi_{rc} = \langle w_c, v_r\rangle$ for every $(r, c) \in I \cap R' \times C'$. If we define $W\colon \ell^2_{C'} \to \ell^2_{C'}(H)$ and $V\colon \ell^2_{R'} \to \ell^2_{R'}(H)$ by $W\zeta = (\zeta_c w_c)_{c\in C'}$

and $V\eta = (\eta_r v_r)_{r\in R'}$, then V and W have norm at most 1, and the proposition follows from the factorisation
$$\mathrm{M}_\varphi x = V^*(x\otimes \mathrm{Id}_H)W$$
for every x with support in $I\cap R'\times C'$. ☐

Remark 2.2. Éric Ricard showed us an elementary proof that a Schur multiplier on S^∞_I is automatically c.b., included here by his kind permission. A Schur multiplier φ is bounded on S^∞_I by a constant D if and only if
$$\forall\,\xi\in B_{\mathrm{S}^\infty_I}\ \forall\,\eta\in B_{\ell^2_R}\ \forall\,\zeta\in B_{\ell^2_C}\quad \Bigl|\sum_{(r,c)\in I}\eta_r\varphi_{rc}\xi_{rc}\zeta_c\Bigr|\leqslant D. \tag{2}$$
It is furthermore completely bounded on S^∞_I by D if
$$\forall\,x\in B_{\mathrm{S}^\infty_I(\mathrm{S}^\infty)}\ \forall\,y\in B_{\ell^2_R(\ell_2)}\ \forall\,z\in B_{\ell^2_C(\ell_2)}\quad \Bigl|\sum_{(r,c)\in I}\varphi_{rc}\langle y_r,x_{rc}z_c\rangle\Bigr|\leqslant D. \tag{3}$$
Suppose that x,y,z are as quantified in Ineq. (3). Let
$$\xi_{rc}=\langle y_r/\|y_r\|,x_{rc}z_c/\|z_c\|\rangle,\ \eta_r=\|y_r\|_{\ell_2}\text{ and }\zeta_c=\|z_c\|_{\ell_2}.$$
Then $\|\eta\|_{\ell^2_R},\|\zeta\|_{\ell^2_C}\leqslant 1$ and
$$\begin{aligned}\|\xi\|=\sup\Bigl\{\Bigl|\sum_{(r,c)\in I}\langle\alpha_r y_r/\|y_r\|_{\ell_2},x_{rc}\beta_c z_c/\|z_c\|_{\ell_2}\rangle\Bigr| : \alpha\in B_{\ell^2_R},\ \beta\in B_{\ell^2_C}\Bigr\}\\ \leqslant\|x\|\sup_{\alpha\in B_{\ell^2_R}}\bigl\|\bigl(\alpha_r y_r/\|y_r\|_{\ell_2}\bigr)\bigr\|_{\ell^2_R(\ell_2)}\sup_{\beta\in B_{\ell^2_C}}\bigl\|\bigl(\beta_c z_c/\|z_c\|_{\ell_2}\bigr)\bigr\|_{\ell^2_C(\ell_2)}\leqslant 1,\end{aligned}$$
so that Ineq. (2) implies Ineq. (3).

The fact that the canonical basis of an ℓ^2 space is 1-unconditional yields that Schatten-von-Neumann norms are *matrix unconditional* in the terminology of [27]:
$$\forall\,\zeta\in\mathbb{T}^C\ \forall\,\eta\in\mathbb{T}^R\quad \Bigl\|\sum_{(r,c)\in R\times C}\zeta_c\eta_r a_{rc}\mathrm{e}_{rc}\Bigr\|_p=\Bigl\|\sum_{(r,c)\in R\times C}a_{rc}\mathrm{e}_{rc}\Bigr\|_p \tag{4}$$
for every finitely supported sequence of complex or S^p-valued coefficients a_{rc}. Let $\zeta\otimes\eta$ denote the *elementary* Schur multiplier $(\zeta_c\eta_r)_{(r,c)\in R\times C}$. Equation (4) shows that if $\zeta\in\mathbb{T}^C$ and $\eta\in\mathbb{T}^R$, then $\mathrm{M}_{\zeta\otimes\eta}$ is an isometry on every S^p. This yields that if $\zeta\in\ell^\infty_C$, $\eta\in\ell^\infty_R$, then the complete norm of $\mathrm{M}_{\zeta\otimes\eta}$ is $\|\zeta\|_{\ell^\infty_C}\|\eta\|_{\ell^\infty_R}$ on every S^p.

Relative Schur multipliers also have a central place among operators on S^p_I because they appear as the range of a contractive projection defined by the following averaging scheme.

Definition 2.3. Let $T\colon\mathrm{S}^p_J\to\mathrm{S}^p_I$ be an operator. Let $R'\times C'$ be a finite subset of $R\times C$ and let $P_{R'\times C'}$ be the contractive projection onto $\mathrm{S}^p_{R'\times C'}$ defined by the Schur multiplier $\chi_{C'}\otimes\chi_{R'}$. Then the *average of T with respect to* $R'\times C'$ is given by
$$[T]_{R'\times C'}(x)=\int_{\mathbb{T}^R}\mathrm{d}\eta\int_{\mathbb{T}^C}\mathrm{d}\zeta\,\mathrm{M}_{\zeta^*\otimes\eta^*}P_{R'\times C'}T(\mathrm{M}_{\zeta\otimes\eta}x), \tag{5}$$
where $\zeta^*=(\overline{\zeta_c})_{c\in C}$ and $\eta^*=(\overline{\eta_r})_{r\in R}$.

Proposition 2.4. *Let $T\colon\mathrm{S}^p_J\to\mathrm{S}^p_I$ be an operator and $R'\times C'$ a finite subset of $R\times C$. Then $[T]_{R'\times C'}$ is a Schur multiplication operator from S^p_J to $\mathrm{S}^p_{I\cap R'\times C'}$ such that $\|[T]_{R'\times C'}\|\leqslant\|T\|$. In fact, $[T]_{R'\times C'}=M_{\varphi^{R'\times C'}}$ with*
$$\varphi^{R'\times C'}_{rc}=\begin{cases}\operatorname{tr}\mathrm{e}^*_{rc}T(\mathrm{e}_{rc}) & \text{if } (r,c)\in J\cap R'\times C'\\ 0 & \text{if } (r,c)\in J\setminus R'\times C'.\end{cases}$$
If T is a projection onto S^p_I, then $\varphi^{R'\times C'}=\chi_{I\cap R'\times C'}$, so that $[T]_{R'\times C'}$ is a projection onto $\mathrm{S}^p_{I\cap R'\times C'}$. Let $\varphi=\bigl(\operatorname{tr}\mathrm{e}^*_qT(\mathrm{e}_q)\bigr)_{q\in J}$. Then $\|\mathrm{M}_\varphi\|\leqslant\|T\|$ and we define the average of T *by* $[T]=\mathrm{M}_\varphi$.

Proof. Formula (5) shows that $\|[T]_{R'\times C'}(x)\| \leqslant \|T\|\,\|x\|$. We have

$$\begin{aligned}[T]_{R'\times C'}(\mathrm{e}_{rc}) &= \int_{\mathbb{T}^R} \mathrm{d}\eta \int_{\mathbb{T}^C} \mathrm{d}\zeta\, \mathrm{M}_{\zeta^*\otimes\eta^*} P_{R'\times C'} T(\zeta_c\eta_r\mathrm{e}_{rc}) \\ &= \int_{\mathbb{T}^R} \mathrm{d}\eta \int_{\mathbb{T}^C} \mathrm{d}\zeta\, \zeta_c\eta_r \mathrm{M}_{\zeta^*\otimes\eta^*} \sum_{(r',c')\in R'\times C'} \mathrm{tr}\big(\mathrm{e}^*_{r'c'}T(\mathrm{e}_{rc})\big)\mathrm{e}_{r'c'} \\ &= \sum_{(r',c')\in R'\times C'} \int_{\mathbb{T}^R} \mathrm{d}\eta \int_{\mathbb{T}^C} \mathrm{d}\zeta\, \zeta_c\eta_r\, \mathrm{tr}\big(\mathrm{e}^*_{r'c'}T(\mathrm{e}_{rc})\big)\zeta_{c'}^{-1}\eta_{r'}^{-1}\mathrm{e}_{r'c'} = \varphi^{R'\times C'}_{rc}\mathrm{e}_{rc}.\end{aligned}$$

As the norm of a Schur multiplier is the supremum of the norm of its restrictions to finite rectangle sets, this shows that φ is a Schur multiplier on S^p_J and $\|\mathrm{M}_\varphi\| \leqslant \|T\|$. If T is a projection onto S^p_I, note that $\mathrm{tr}\, \mathrm{e}^*_{rc}T(\mathrm{e}_{rc}) = \chi_I(r,c)$. ☐

The following proposition relates Fourier multipliers to Herz-Schur multipliers in the fashion of [22, Th. 6.4] and will be very useful in the exact computation of the norm of certain relative Schur multipliers.

Proposition 2.5. *Let Γ be a countable discrete abelian group and $\Lambda \subseteq \Gamma$. Let R and C be two copies of Γ and consider $I = \{(r,c) \in R \times C : r - c \in \Lambda\}$. Let $\varphi \in \mathbb{C}^I$ such that there is $\mu \in \mathbb{C}^\Lambda$ with $\varphi(r,c) = \mu(r-c)$ for all $(r,c) \in I$. Let $G = \hat{\Gamma}$, so that Γ is the group of characters on the compact group G. Let $p \in (0,\infty]$.*

(*a*) *The complete norm of the relative Schur multiplier φ on S^p_I is bounded by the complete norm of the relative Fourier multiplier μ on $\mathrm{L}^p_\Lambda(G)$.*

(*b*) *Suppose that Γ is finite. The norm of the relative Fourier multiplier μ on $\mathrm{L}^p_\Lambda(G)$ is bounded by the norm of the relative Schur multiplier φ on S^p_I. The same holds for complete norms.*

Remark 2.6. Part (*b*) is just an abstract counterpart to [20, Chapter 6, Lemma 3.8], where the case of the finite cyclic group $\Gamma = \mathbb{Z}/n\mathbb{Z}$ is treated.

Proof. (*a*) is [21, Lemma 8.1.4]: for all $a_q \in \mathrm{S}^p$, of which only a finite number are nonzero, and all $g \in G$, we have by matrix unconditionality (Eq. (4))

$$\begin{aligned}\Big\|\sum_{q\in I} a_q\mathrm{e}_q\Big\|_{\mathrm{S}^p_I(\mathrm{S}^p)} &= \Big\|\sum_{(r,c)\in I} r(g)c(g)^{-1}a_{rc}\mathrm{e}_{rc}\Big\|_{\mathrm{S}^p_I(\mathrm{S}^p)} \\ &= \Big\|\sum_{\gamma\in\Lambda}\Big(\sum_{\substack{(r,c)\in I\\ r-c=\gamma}} a_{rc}\mathrm{e}_{rc}\Big)\gamma(g)\Big\|_{\mathrm{S}^p_I(\mathrm{S}^p)} = \Big\|\sum_{\gamma\in\Lambda}\Big(\sum_{\substack{(r,c)\in I\\ r-c=\gamma}} a_{rc}\mathrm{e}_{rc}\Big)\gamma\Big\|_{\mathrm{L}^p_\Lambda(G,\mathrm{S}^p(\mathrm{S}^p))}.\end{aligned} \tag{6}$$

This yields an isometric embedding of $\mathrm{S}^p_I(\mathrm{S}^p)$ in $\mathrm{L}^p_\Lambda(G, \mathrm{S}^p_I(\mathrm{S}^p))$. As $\mathrm{S}^p(\mathrm{S}^p)$ may be identified with $\mathrm{S}^p(\ell^2_\Gamma(\ell^2))$,

$$\Big\|\sum_{q\in I}\varphi_q a_q\mathrm{e}_q\Big\|_{\mathrm{S}^p_I(\mathrm{S}^p)} = \Big\|\sum_{\gamma\in\Lambda}\mu_\gamma\Big(\sum_{\substack{(r,c)\in I\\ r-c=\gamma}} a_{rc}\mathrm{e}_{rc}\Big)\gamma\Big\|_{\mathrm{L}^p_\Lambda(G,\mathrm{S}^p(\mathrm{S}^p))} \leqslant \|\mathrm{Id}\otimes\mathrm{M}_\mu\|\Big\|\sum_{q\in I}a_q\mathrm{e}_q\Big\|_{\mathrm{S}^p_I(\mathrm{S}^p)}.$$

(*b*). Let us embed $\mathrm{L}^p_\Lambda(G)$ into S^p_I by $f \mapsto \mathrm{m}_{\hat{f}}$, where $\mathrm{m}_{\hat{f}} \colon \ell^2_C \to \ell^2_R$ is the convolution operator defined by

$$\mathrm{m}_{\hat{f}}\mathrm{e}_c = \hat{f} * \mathrm{e}_c = \sum_{\gamma\in\Lambda}\hat{f}(\gamma)\mathrm{e}_\gamma * \mathrm{e}_c = \sum_{r-c\in\Lambda}\hat{f}(r-c)\mathrm{e}_r :$$

$\mathrm{m}_{\hat{f}}$ has the matrix representation $\sum_{(r,c)\in I}\hat{f}(r-c)\mathrm{e}_{rc}$. The characters $g \in G$ form an orthonormal basis for ℓ^2_C such that $\mathrm{m}_{\hat{f}}g = f(g)g$: therefore

$$\|\mathrm{m}_{\hat{f}}\|_p = \Big(\sum_{g\in G}|f(g)|^p\Big)^{1/p} = (\#\, G)^{1/p}\|f\|_{\mathrm{L}^p(G)}.$$

As $\mathrm{M}_\varphi\mathrm{m}_{\hat{f}} = \mathrm{m}_{\widehat{\mathrm{M}_\mu f}}$, this shows that the norm of μ on $\mathrm{L}^p_\Lambda(G)$ is the norm of φ on the subspace of circulant matrices in S^p_I. The same holds for complete norms. ☐

3 Idempotent Schur multipliers of norm 1

A Schur multiplier is *idempotent* if it is the indicator function χ_I of some set $I \subseteq R \times C$; if χ_I is a Schur multiplier on S^p, then it is a projection of S^p onto S^p_I. Idempotent Schur multipliers on S^p and tensors in $\ell^\infty_C \overset{\wedge}{\otimes} \ell^\infty_R$ with $0, 1$ coefficients of norm 1 may be characterised by the combinatorics of I.

Proposition 3.1. *Let $I \subseteq R \times C$ be nonempty and $0 < p \neq 2 < \infty$. The following are equivalent.*

(*a*) *For every finite rectangle set $R' \times C'$ intersecting I*

$$\Big\| \sum_{(r,c)\in I\cap R'\times C'} \mathrm{e}_c \otimes \mathrm{e}_r \Big\|_{\ell^\infty_C \overset{\wedge}{\otimes} \ell^\infty_R} = 1.$$

(*b*) *S^p_I is completely 1-complemented in S^p.*

(*c*) *S^p_I is 1-complemented in S^p.*

(*d*) *I is a union of pairwise disjoint complete bipartite graphs: there are pairwise disjoint sets $R_j \subseteq R$ and pairwise disjoint sets $C_j \subseteq C$ such that $I = \bigcup R_j \times C_j$.*

Property (d) means that the pattern I is, up to a permutation of columns and rows, block-diagonal:

$$\begin{array}{c|c} & \begin{array}{cccc} C_1 & C_2 & C_3 & \cdots \end{array} \\ \begin{array}{c} R_1 \\ R_2 \\ R_3 \\ \vdots \end{array} & \begin{pmatrix} * & 0 & 0 & \cdots \\ 0 & * & 0 & \ddots \\ 0 & 0 & * & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}. \end{array}$$

Proof. (b) $\Rightarrow$ (c) is trivial.

(a) $\Rightarrow$ (b). The complete norm of a Schur multiplier φ on S^p is the supremum of the complete norm of its restrictions $\varphi' = (\varphi_q)_{q\in R'\times C'}$ to finite rectangle sets $R' \times C'$. Furthermore, the complete norm of an elementary Schur multiplier $(\eta_c\zeta_r)_{(r,c)\in R\times C} = \eta \otimes \zeta$ on S^p equals $\|\eta\|_{\ell^\infty_C}\|\zeta\|_{\ell^\infty_R}$.

(c) $\Rightarrow$ (d). If I is not a union of pairwise disjoint complete bipartite graphs, then there are $r_0, r_1 \in R$ and $c_0, c_1 \in C$ such that

$$I' = I \cap \{r_0, r_1\} \times \{c_0, c_1\} = \{(r_0, c_0), (r_1, c_0), (r_0, c_1)\}.$$

By Proposition 2.4, the average of a contractive projection of S^p onto S^p_I with respect to $\{r_0, r_1\} \times \{c_0, c_1\}$ would be the contractive projection associated to the Schur multiplier $\chi_{I'}$. Let $x(t)$, $t \in \mathbb{R}$, be the operator from ℓ^2_C to ℓ^2_R whose matrix coefficients vanish except for its $\{r_0, r_1\} \times \{c_0, c_1\}$ submatrix, which is $\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & t \end{pmatrix}$. Its eigenvalues are

$$\frac{1 + t + \sqrt{9 - 2t + t^2}}{2} = 2 + \frac{t}{3} + o(t), \quad \frac{1 + t - \sqrt{9 - 2t + t^2}}{2} = -1 + \frac{2t}{3} + o(t),$$

so that

$$\begin{cases} \|x(t)\|_\infty = 2 + t/3 + o(t) \\ \|x(t)\|_p^p = 2^p + 1 + p(2^p - 4)t/6 + o(t) \quad \text{for } 0 < p < \infty \end{cases}$$

and therefore $\|\chi_{I'} * x(t)\|_p = \|x(0)\|_p > \|x(t)\|_p$ for some $t \neq 0$ if $p \neq 2$.

(d) $\Rightarrow$ (a). Suppose (d) and let $R' \times C'$ intersect I. Then there are pairwise disjoint sets R'_j and pairwise disjoint sets C'_j such that $I \cap R' \times C' = R'_1 \times C'_1 \cup \cdots \cup R'_n \times C'_n$ and

$$\sum_{(r,c)\in I\cap R'\times C'} \mathrm{e}_c \otimes \mathrm{e}_r = \sum_{j=1}^n \chi_{C'_j} \otimes \chi_{R'_j} = \operatorname*{average}_{\epsilon_j=\pm1} \Big(\sum_{j=1}^n \epsilon_j \chi_{C'_j}\Big) \otimes \Big(\sum_{j=1}^n \epsilon_j \chi_{R'_j}\Big)$$

which is an average of elementary tensors of norm 1, so that its projective tensor norm is bounded by 1, and actually is equal to 1. □

Remark 3.2. Note that the proof of Prop. 3.1 shows that the norm of a projection $\mathrm{M}_{\chi_I}: \mathrm{S}^\infty \to \mathrm{S}^\infty_I$ is either 1 or at least $2/\sqrt{3}$, as

$$\left\|\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}\right\|_\infty = \sqrt{3}, \qquad \left\|\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix}\right\|_\infty = 2.$$

This is a noncommutative analogue to the fact that an idempotent measure on a locally compact abelian group G has either norm 1 or at least $\sqrt{5}/2$ [25, Th. 3.7.2]. The norm of M_{χ_I} actually equals $2/\sqrt{3}$ for $I = \{(0,0),(0,1),(1,0)\}$, as shown in [13, Lemma 3]. In fact, the following decomposition holds:

$$\mathrm{e}_0 \otimes \mathrm{e}_0 + \mathrm{e}_0 \otimes \mathrm{e}_1 + \mathrm{e}_1 \otimes \mathrm{e}_0 = \\ \big((\mathrm{e}^{-\mathrm{i}\pi/12}, \mathrm{e}^{\mathrm{i}\pi/4}) \otimes (\mathrm{e}^{-\mathrm{i}\pi/12}, \mathrm{e}^{\mathrm{i}\pi/4}) + (\mathrm{e}^{\mathrm{i}\pi/12}, \mathrm{e}^{-\mathrm{i}\pi/4}) \otimes (\mathrm{e}^{\mathrm{i}\pi/12}, \mathrm{e}^{-\mathrm{i}\pi/4})\big)/\sqrt{3}.$$

Remark 3.3. Results related to the equivalence of (c) with (d) have been obtained independently by Banks and Harcharras [1].

4 Unconditional basic sequences in S^p

Definition 4.1. Let $0 < p \leqslant \infty$ and $I \subseteq R \times C$. Let $\mathbb{S} = \mathbb{T}$ (vs. $\mathbb{S} = \{-1, 1\}$.)

(a) I is an *unconditional basic sequence* in S^p if there is a constant D such that

$$\Big\|\sum_{q\in I} \epsilon_q a_q \mathrm{e}_q\Big\|_p \leqslant D\Big\|\sum_{q\in I} a_q \mathrm{e}_q\Big\|_p \tag{7}$$

for every choice of signs $\epsilon_q \in \mathbb{S}$ and every finitely supported sequence of complex coefficients a_q. Its *complex (vs. real) unconditional constant* is the least such constant D.

(b) I is a *completely* unconditional basic sequence in S^p if there is a constant D such that (7) holds for every choice of signs $\epsilon_q \in \mathbb{S}$ and every finitely supported sequence of operator coefficients $a_q \in \mathrm{S}^p$. Its complex (vs. real) *complete* unconditional constant is the least such constant D.

(c) I is a complex (vs. real, complex completely, real completely) *1-unconditional* basic sequence in S^p if its complex (vs. real, complex complete, real complete) unconditional constant is 1: Inequality (7) turns into the equality

$$\Big\|\sum_{q\in I} \epsilon_q a_q \mathrm{e}_q\Big\|_p = \Big\|\sum_{q\in I} a_q \mathrm{e}_q\Big\|_p.$$

If Inequality (7) holds for every real choice of signs, then it also holds for every complex choice of signs at the cost of replacing D by $D\pi/2$ (see [28],) so that there is no need to distinguish between complex and real unconditional basic sequences.

The notions defined in (a) and (b) are called $\sigma(p)$ *sets* and *complete* $\sigma(p)$ *sets* in [8, §4] and [9] (see also the survey [23, § 9].) The notions defined in (c) are their isometric counterparts.

By [27, proof of Cor. 4], the real unconditional constant of any basis of S^p_I cannot be lower than a fourth of the real unconditional constant of I in S^p.

Example 4.2. A single column $R \times \{c\}$, a single row $\{r\} \times C$, the diagonal set $\{(\mathrm{row}\, n, \mathrm{col}\, n)\}_{n\in\mathbb{N}}$ if R and C are copies of $\mathbb{N}$, are 1-unconditional basic sequences in all S^p. In fact, every column section and every row section (this is the terminology of [32, Def. 4.3]) is a 1-unconditional basic sequence; note that the length of every path in the corresponding graph is at most 2.

Note that the set I is a (completely) 1-unconditional basic sequence in S^p if and only if the relative Schur multipliers by signs on S^p_I define (complete) isometries. This yields by Prop. 2.1:

Proposition 4.3. *Let $I \subseteq R \times C$ and $0 < p \leqslant \infty$. If I is a real (vs. complex) 1-unconditional basic sequence in S^∞, then I is also a real (vs. complex) completely 1-unconditional basic sequence in S^p.*

Example 4.4. If $R = C = \{0, \ldots, n-1\}$, $1 \leqslant p \leqslant \infty$ and $I = R \times C$, then the complex unconditional constant of the basis of elementary matrices in S^p is $n^{|1/2-1/p|}$ and coincides with its complete unconditional constant (see [21, Lemma 8.1.5].) This is also the real unconditional constant if $n = 2^k$ is a power of 2 as the norm of Schur multiplication by the kth tensor power $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{\otimes k}$ (the kth Walsh matrix) on S^p is $\left(2^{|1/2-1/p|}\right)^k = n^{|1/2-1/p|}$. Let us now show that if $n = 3$, the real unconditional constant of the basis of elementary matrices in S^∞ is $5/3$ and differs from its complex unconditional constant, $\sqrt{3}$. In fact, because the canonical bases of ℓ^2_C and ℓ^2_R are symmetric, the norm of a Schur multiplier by real signs turns out to equal the norm of one of the following three Schur multipliers:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \text{ or } \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

The first one has norm 1: it defines the identity. The second one has the same norm as the Schur multiplier $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, which is $\sqrt{2}$, because the norm of that multiplier equals the norm of its tensor product by $\mathrm{Id}_{\ell^2_2}$, which is $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$. By Prop. 2.5 for $\Gamma = \mathbb{Z}/3\mathbb{Z}$, the third one has the same norm as the Fourier multiplier $\varphi = (-1, 1, 1)$ on $\mathrm{L}^\infty(G)$, where $G = \{z \in \mathbb{C} : z^3 = 1\}$: as this multiplier acts by convolution with $f = -1 + z + z^2$, its norm is $\|f\|_{\mathrm{L}^1(G)}$, that is

$$\left(|-1+1+1| + |-1+\mathrm{e}^{2\mathrm{i}\pi/3}+\mathrm{e}^{4\mathrm{i}\pi/3}| + |-1+\mathrm{e}^{4\mathrm{i}\pi/3}+\mathrm{e}^{2\mathrm{i}\pi/3}|\right)/3 = 5/3.$$

Complex interpolation yields that the real unconditional constant of the basis of elementary matrices is in fact strictly less than its complex counterpart in all S^p with $p \neq 2$.

5 Varopoulos' characterisation of unconditional matrices in S^∞

Our results may be seen as the isometric counterpart to results by Varopoulos [32] on tensor algebras over discrete spaces and their generalisation to S^p. He characterised unconditional basic sequences of elementary matrices in S^∞ in his study of the projective tensor product $\mathrm{c}_0 \overset{\wedge}{\otimes} \mathrm{c}_0$. We gather up his results in the next theorem, as they are difficult to extract from the literature.

Theorem 5.1. *Let $I \subseteq R \times C$. The following are equivalent.*

(*a*) *I is an unconditional basic sequence in S^∞.*

(*b*) *I is an* interpolation set *for Schur multipliers on S^∞: every bounded sequence on I is the restriction of a Schur multiplier on S^∞.*

(*c*) *I is a* V-Sidon set *as defined in [32, Def. 4.1]: every null sequence on I is the restriction of the sequence of coefficients of a tensor in $\mathrm{c}_0(C) \overset{\wedge}{\otimes} \mathrm{c}_0(R)$.*

(*d*) *The coefficients of every tensor in $\ell^1_C \overset{\vee}{\otimes} \ell^1_R$ with support in I form an absolutely convergent series.*

(*e*) *$(z_c z'_r)_{(r,c)\in I}$ is a* Sidon set *in the dual of $\mathbb{T}^C \times \mathbb{T}^R$, that is, an unconditional basic sequence in $\mathrm{C}(\mathbb{T}^C \times \mathbb{T}^R)$.*

(*f*) *There is a constant λ such that for all $R' \subseteq R$ and $C' \subseteq C$ with n elements $\#[I \cap R' \times C'] \leqslant \lambda n$.*

(*g*) *I is a finite union of forests.*

(*h*) *I is a finite union of row sections and column sections.*

(*i*) *Every bounded sequence supported by I is a Schur multiplier on* S^∞.

Sketch of proof. $(a) \Rightarrow (b)$. If (a) holds, every sequence of signs $\epsilon \in \{-1,1\}^I$ is a Schur multiplier on S^∞_I. By a convexity argument, this implies that every bounded sequence is a Schur multiplier on S^∞_I, which may be extended to a Schur multiplier on S^∞ with the same norm by [17, Cor. 3.3].

$(b) \Rightarrow (c)$ holds by Grothendieck's inequality (see [22, § 5]) and an approximation argument.

(d) is but the formulation dual to (c) (see [31, § 6.2].)

$(d) \Rightarrow (e)$. A computation yields

$$\Big\|\sum_{(r,c)\in I} a_{rc}\mathrm{e}_c \otimes \mathrm{e}_r\Big\|_{\ell^1_C \overset{\vee}{\otimes} \ell^1_R} = \sup_{|z_c|,|z'_r|=1}\Big|\sum_{(r,c)\in I} a_{rc} z_c z'_r\Big|. \tag{8}$$

$(e) \Rightarrow (f)$ is [32, Th. 4.2]. (The proof can be found in [31, § 6.3] and in [30, § 5].)

$(f) \Rightarrow (g)$, $(f) \Rightarrow (h)$ can be found in [30, Th. 6.1].

$(g) \Rightarrow (h)$. In fact, a forest is the union of a column section I' with a row section I'' (a *bisection* in the terminology of [32, Def. 4.3].) It suffices to prove this for a tree. Let its vertices be indexed by words as described in the Terminology. Then the set I' of couples of the form $(w, w^\frown c)$ with w a word and c a letter is a column section; the set I'' of couples of the form $(w^\frown r, w)$ with w a word and r a letter is a row section.

$(h) \Rightarrow (i)$ is [30, Th. 4.5]. Note that row sections and column sections form 1-unconditional basic sequences in S^∞ and are 1-complemented in S^∞ by Prop. 3.1.

$(i) \Rightarrow (a)$ follows from the open mapping theorem.

6 Closed walk relations

We now introduce and study the combinatorial objects that we need in order to analyse the expansion of the function defined by

$$\Phi_I(\epsilon, a) = \mathrm{tr}\Big|\sum_{q\in I} \epsilon_q a_q \mathrm{e}_q\Big|^p \tag{9}$$

for $I \subseteq R \times C$, a positive even integer $p = 2k$, signs $\epsilon_q \in \mathbb{T}$ and coefficients $a_q \in \mathbb{C}$, of which only a finite number are nonzero. In fact,

$$\begin{aligned}
\Phi_I(\epsilon, a) &= \mathrm{tr}\Big(\sum_{(r,c),(r',c')\in I} (\epsilon_{rc} a_{rc} \mathrm{e}_{rc})^* (\epsilon_{r'c'} a_{r'c'} \mathrm{e}_{r'c'})\Big)^k \\
&= \mathrm{tr} \sum_{\substack{(r_1,c_1),(r'_1,c'_1),\ldots,\\ (r_k,c_k),(r'_k,c'_k)\in I}} \prod_{i=1}^k (\epsilon^{-1}_{r_ic_i} \overline{a_{r_ic_i}} \mathrm{e}_{c_ir_i})(\epsilon_{r'_ic'_i} a_{r'_ic'_i} \mathrm{e}_{r'_ic'_i}) \\
&= \sum_{\substack{(r_1,c_1),(r_1,c_2),\ldots,\\ (r_k,c_k),(r_k,c_{k+1})\in I}} \prod_{i=1}^k \epsilon^{-1}_{r_ic_i}\epsilon_{r_ic_{i+1}} \overline{a_{r_ic_i}} a_{r_ic_{i+1}} \quad (\text{where } c_{k+1} = c_1.)
\end{aligned} \tag{10}$$

The latter sum runs over all closed walks $(c_1, r_1, c_2, \ldots, c_k, r_k)$ of length p in the graph I. With multinomial notation, its terms have the form $\epsilon^{\beta-\alpha}\overline{a}^\alpha a^\beta$ with $|\alpha| = |\beta| = k$. The attempt to describe those couples (α, β) that effectively arise in this expansion yields the following definition.

Definition 6.1. Let $p = 2k \geqslant 0$ be an even integer and $I \subseteq R \times C$.

(*a*) Let $\mathrm{A}^I_k = \{\alpha \in \mathbb{N}^I : \sum_{q\in I} \alpha_q = k\}$ and set

$$\mathrm{B}^I_k = \big\{(\alpha, \beta) \in \mathrm{A}^I_k \times \mathrm{A}^I_k : \forall\, r \textstyle\sum_c \alpha_{rc} = \sum_c \beta_{rc} \text{ and } \forall\, c \sum_r \alpha_{rc} = \sum_r \beta_{rc}\big\}.$$

(*b*) Two couples $(\alpha^1, \beta^1) \in \mathrm{B}^I_{k_1}$, $(\alpha^2, \beta^2) \in \mathrm{B}^I_{k_2}$ are *disjoint* if $k_1, k_2 \geqslant 1$ and

$$\alpha^1_{rc} \geqslant 1 \quad\Rightarrow\quad \forall\,(r', c) \in I \quad \alpha^2_{r'c} = 0 \quad \text{and} \quad \forall\,(r, c') \in I \quad \alpha^2_{rc'} = 0. \tag{11}$$

(*c*) The set $\mathscr{W}_k^I$ of *closed walk relations* of length p in I is the subset of those $(\alpha,\beta)\in \mathrm{B}_k^I$ that cannot be decomposed into the sum of two disjoint couples.

(*d*) Let W_k^I be the set of closed walks of length p in the graph I. To every closed walk $P = (c_1,r_1,c_2,r_2,\dots,c_k,r_k)$ of length p we associate the couple $(\alpha,\beta)\in \mathrm{A}_k^I\times \mathrm{A}_k^I$ defined by

$$\alpha_q = \#\big[i\in\{1,\dots,k\} : (r_i,c_i)=q\big]$$
$$\beta_q = \#\big[i\in\{1,\dots,k\} : (r_i,c_{i+1})=q\big] \quad (\text{where } c_{k+1}=c_1.)$$

We shall write $P\sim(\alpha,\beta)$ and call $n_{\alpha\beta}$ the number of elements of W_k^I mapped onto (α,β).

Note that the conditions in Eq. (11) is in fact symmetric and that it may be stated with β^1 and β^2 instead of α^1 and α^2.

Example 6.2. Let $R=C=\{0,1,2,3\}$ and $I=R\times C$. The couple $(\mathrm{e}_{00}+\mathrm{e}_{11}+\mathrm{e}_{22}+\mathrm{e}_{33}, \mathrm{e}_{01}+\mathrm{e}_{10}+\mathrm{e}_{23}+\mathrm{e}_{32})$ is an element of $\mathrm{B}_4^I\setminus\mathscr{W}_4^I$: it is the sum of the two disjoint closed walk relations $(\mathrm{e}_{00}+\mathrm{e}_{11}, \mathrm{e}_{01}+\mathrm{e}_{10})$ and $(\mathrm{e}_{22}+\mathrm{e}_{33}, \mathrm{e}_{23}+\mathrm{e}_{32})$.

Example 6.3. Let $I=R\times C=\{0,1\}\times\{0,1\}$. Two closed walks are associated with the closed walk relation $(\mathrm{e}_{00}+\mathrm{e}_{11}, \mathrm{e}_{01}+\mathrm{e}_{10})\in\mathscr{W}_2^I$: the two cycles $(\mathrm{col}\,0,\mathrm{row}\,0,\mathrm{col}\,1,\mathrm{row}\,1)$ and $(\mathrm{col}\,1,\mathrm{row}\,1,\mathrm{col}\,0,\mathrm{row}\,0)$. Six closed walks are mapped onto the closed walk relation $(2\mathrm{e}_{00}+2\mathrm{e}_{01}, 2\mathrm{e}_{00}+2\mathrm{e}_{01})$: the $\dfrac{4!}{2!\,2!}$ concatenations of a permutation of $(\mathrm{col}\,1,\mathrm{row}\,0)$, $(\mathrm{col}\,1,\mathrm{row}\,0)$, $(\mathrm{col}\,0,\mathrm{row}\,0)$, $(\mathrm{col}\,0,\mathrm{row}\,0)$.

The next proposition shows that, for our purpose, closed walk relations describe entirely closed walks.

Proposition 6.4. *Let $p=2k\geqslant 0$ be an even integer and $I\subseteq R\times C$. The image of the mapping in Def. 6.1(d) is $\mathscr{W}_k^I$:*

(*a*) *if $P\in\mathrm{W}_k^I$ and $P\sim(\alpha,\beta)$, then $(\alpha,\beta)\in\mathscr{W}_k^I$;*

(*b*) *if $(\alpha,\beta)\in\mathscr{W}_k^I$, then there is a $P\in\mathrm{W}_k^I$ such that $P\sim(\alpha,\beta)$, so that $n_{\alpha\beta}\geqslant 1$.*

Proof. (*a*). Let $P=(c_1,r_1,c_2,r_2,\dots,c_k,r_k)$. In fact,

$$\sum\nolimits_c \alpha_{rc} = \#[i\in\{1,\dots,k\} : r_i = r] = \sum\nolimits_c \beta_{rc}$$
$$\sum\nolimits_r \alpha_{rc} = \#[i\in\{1,\dots,k\} : c_i = c] = \sum\nolimits_r \beta_{rc}$$

and $(\alpha,\beta)\in\mathrm{B}_k^I$. If $(\alpha,\beta)=(\alpha^1,\beta^1)+(\alpha^2,\beta^2)$ with $(\alpha^i,\beta^i)\in\mathrm{B}_{k_i}^I$ and $k_i\geqslant 1$, there is an i such that $\alpha^1_{r_ic_i}\geqslant 1$ and $\alpha^2_{r_{i+1}c_{i+1}}\geqslant 1$ (where $(r_{i+1},c_{i+1})=(r_1,c_1)$ if $i=k$.) If $\beta^1_{r_ic_{i+1}}\geqslant 1$, then $\sum_r\alpha^1_{rc_{i+1}}\geqslant 1$, so that there is an r such that $\alpha^1_{rc_{i+1}}\geqslant 1$. Otherwise $\beta^2_{r_ic_{i+1}}\geqslant 1$, so that $\sum_c\alpha^2_{r_ic}\geqslant 1$ and there is a c such that $\alpha^2_{r_ic}\geqslant 1$. Therefore (α^1,β^1) and (α^2,β^2) are not disjoint and $(\alpha,\beta)\in\mathscr{W}_k^I$.

(*b*). We have to find a closed walk of length p that is mapped onto (α,β). If $k=0$, the empty closed walk suits. Suppose that $k\geqslant 1$; Consider a walk $(c_1,r_1,c_2,r_2,\dots,c_j,r_j,c_{j+1})$ in I such that $\alpha^1_q=\#[i:(r_i,c_i)=q]\leqslant\alpha_q$ and $\beta^1_q=\#[i:(r_i,c_{i+1})=q]\leqslant\beta_q$ for every $q\in R\times C$, and furthermore j is maximal. We claim (A) that $c_{j+1}=c_1$ and (B) that $j=k$. Let $(\alpha^2,\beta^2)=(\alpha,\beta)-(\alpha^1,\beta^1)$.

(A). If $c_{j+1}\neq c_1$, then

$$\sum\nolimits_r \alpha^1_{rc_{j+1}} = \#[i\in\{1,\dots,j\} : c_i = c_{j+1}]$$
$$\sum\nolimits_r \beta^1_{rc_{j+1}} = \#[i\in\{1,\dots,j+1\} : c_i = c_{j+1}] = 1+\sum\nolimits_r \alpha^1_{rc_{j+1}},$$

so that there must be r_{j+1} with $\alpha^2_{r_{j+1}c_{j+1}}\geqslant 1$. But then

$$\sum\nolimits_c \beta^2_{r_{j+1}c} = \sum\nolimits_c \alpha^2_{r_{j+1}c}\geqslant 1$$

and there must be c_{j+2} such that $\beta^2_{r_{j+1}c_{j+2}}\geqslant 1$: j is not maximal.

(B). Suppose that $j < k$. Then $(\alpha^1, \beta^1) \in \mathrm{B}^I_j$ and $(\alpha^2, \beta^2) \in \mathrm{B}^I_{k-j}$. By hypothesis, they are not disjoint: there are r, c, c' such that $\alpha^1_{rc}\alpha^2_{rc'} \geqslant 1$ or r, r', c such that $\alpha^1_{rc}\alpha^2_{r'c} \geqslant 1$. By interchanging R and C and by relabelling the vertices if necessary, we may suppose without loss of generality that for $r'_1 = r_j$ there is c'_1 such that $\alpha^2_{r'_1c'_1} \geqslant 1$. Then there is c'_2 such that $\beta^2_{r'_1c'_2} \geqslant 1$. By the argument used in Claim (A), there is a closed walk $(c'_1, r'_1, c'_2, \ldots, c'_{j'}, r'_{j'})$ such that $\#[i : (r'_i, c'_i) = q] \leqslant \alpha^2_q$ and $\#[i : (r'_i, c'_{i+1}) = q] \leqslant \beta^2_q$ (where $c'_{j'+1} = c'_1$.) Then the closed walk

$$(c_1, r_1, c_2, r_2, \ldots, c_j, r_j, c'_2, r'_2, \ldots, c'_{j'}, r'_{j'}, c'_1, r'_1)$$

shows that j is not maximal. □

We are now in position to state the following theorem, a matrix counterpart to the computation presented in [14, Prop. 2.5(ii)].

Theorem 6.5. *Let $p = 2k$ be a positive even integer and $I \subseteq R \times C$.*

(a) *The function Φ_I in Eq. (9) has the expansion*

$$\Phi_I(\epsilon, a) = \sum_{(\alpha,\beta)\in\mathscr{W}^I_k} n_{\alpha\beta}\epsilon^{\beta-\alpha}\overline{a}^\alpha a^\beta, \tag{12}$$

where $n_{\alpha\beta} \geqslant 1$ for every $(\alpha, \beta) \in \mathscr{W}^I_k$.

(b) *If $\epsilon \in \mathbb{T}^I$ and $a \in (\mathrm{S}^p)^I$ is finitely supported, then the function*

$$\Psi_I(\epsilon, a) = \mathrm{tr}\Big|\sum_{q\in I} \epsilon_q a_q \mathrm{e}_q\Big|^p \tag{13}$$

has the expansion

$$\sum_{(\alpha,\beta)\in\mathscr{W}^I_k} \epsilon^{\beta-\alpha} \sum_{(c_1,r_1,\ldots,c_k,r_k)\sim(\alpha,\beta)} \prod_{i=1}^{k} a^*_{r_ic_i} a_{r_ic_{i+1}} \text{ (with } c_{k+1} = c_1.) \tag{14}$$

Proof. This follows from Def. 6.1 and Prop. 6.4. □

Note that the edges of a closed walk $P \sim (\alpha, \beta)$ are precisely those $\{r, c\}$ such that $\alpha_{rc} + \beta_{rc} \geqslant 1$. P is a cycle if and only if P does not have length 0 or 2 and $\sum_r \alpha_{rc} \leqslant 1$ for all c and $\sum_c \alpha_{rc} \leqslant 1$ for all r. We now show how to decompose closed walks into cycles.

Proposition 6.6. *Let $P = (c_1, r_1, c_2, r_2, \ldots, c_k, r_k) \sim (\alpha, \beta)$ be a closed walk.*

(a) *If $r_i = r_j$ (vs. $c_i = c_j$) for some $i \neq j$, then P is the juxtaposition of two nonempty closed walks $P_1 \sim (\alpha^1, \beta^1)$ and $P_2 \sim (\alpha^2, \beta^2)$ such that $(\alpha, \beta) = (\alpha^1, \beta^1) + (\alpha^2, \beta^2)$ and $\sum_c \alpha^1_{r_ic}, \sum_c \alpha^2_{r_ic} \geqslant 1$ (vs. $\sum_r \alpha^1_{rc_i}, \sum_r \alpha^2_{rc_i} \geqslant 1$.)*

(b) *P is the juxtaposition of nonempty closed walks $P_j \sim (\alpha^j, \beta^j)$ such that $\sum_r \alpha^j_{rc} \leqslant 1$ for all c, $\sum_c \alpha^j_{rc} \leqslant 1$ for all r and $(\alpha, \beta) = \sum(\alpha^j, \beta^j)$.*

(c) *There are cycles $P_j \sim (\alpha^j, \beta^j)$ and a γ such that $(\alpha, \beta) = (\gamma, \gamma) + \sum(\alpha^j, \beta^j)$.*

Proof. (a). If $r_i = r_j$ for $i < j$, we may suppose that $j = k$: consider the closed walks $P_1 = (c_1, r_1, \ldots, c_i, r_i)$ and $P_2 = (c_{i+1}, r_{i+1}, \ldots, c_k, r_k)$. If $c_i = c_j$ for $i < j$, we may suppose that $i = 1$: consider then $P_1 = (c_1, r_1, \ldots, c_{j-1}, r_{j-1})$ and $P_2 = (c_j, r_j, \ldots, c_k, r_k)$.

(b). Use (a) in a maximality argument.

(c). Note that the closed walks P_j in (b) are either cycles or have length 2; in the latter case $P_j = q \sim (\mathrm{e}_q, \mathrm{e}_q)$ for some $q \in I$. □

7 Schur multipliers on a cycle

We can realise a cycle of even length $2s$, $s \geqslant 2$, in the following convenient way. Let $\Gamma = \mathbb{Z}/s\mathbb{Z}$. Then the adjacency relation of integers modulo s turns Γ into the cycle $(0, 1, \dots, s-1)$ of length s. We double this cycle into the bipartite cycle $(\operatorname{col} 0, \operatorname{row} 0, \operatorname{col} 1, \operatorname{row} 1, \dots, \operatorname{col} s-1, \operatorname{row} s-1)$ on $\Gamma \amalg \Gamma$, corresponding to the set of couples $I = \{(i,i), (i,i+1) : i \in \Gamma\} \subseteq \Gamma \times \Gamma$: I is the pattern

$$\begin{array}{c c}
 & \begin{array}{cccccc} \scriptstyle 0 & \scriptstyle 1 & \scriptstyle 2 & \scriptstyle \cdots & \scriptstyle s-2 & \scriptstyle s-1 \end{array} \\
\begin{array}{c} \scriptstyle 0 \\ \scriptstyle 1 \\ \scriptstyle 2 \\ \vdots \\ \scriptstyle s-2 \\ \scriptstyle s-1 \end{array} &
\begin{pmatrix}
* & * & 0 & \ddots & 0 & 0 \\
0 & * & * & \ddots & 0 & 0 \\
0 & 0 & * & \ddots & 0 & 0 \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ddots & * & * \\
* & 0 & 0 & \ddots & 0 & *
\end{pmatrix}
\end{array}.$$

Γ is the group dual to $G = \hat{\Gamma} = \{z \in \mathbb{C} : z^s = 1\}$. We shall consider the space $\mathrm{L}^p_\Lambda(G)$ spanned by $\Lambda = \{1, z\}$ in $\mathrm{L}^p(G)$, where z is the identical function on G: its norm is given by $\|a + bz\|_{\mathrm{L}^p(G)} = \bigl(s^{-1} \sum_{z^s=1} |a+bz|^p\bigr)^{1/p}$.

Proposition 7.1. *Let $0 < p \leqslant \infty$, $s \geqslant 2$ and $I = \{(i,i),(i,i+1) : i \in \mathbb{Z}/s\mathbb{Z}\}$. Let $\epsilon \in \mathbb{T}^I$ be a Schur multiplier by signs on S^p_I.*

- *(a) The Schur multiplier ϵ has the same norm as the Schur multiplier $\hat\epsilon$ given by $\hat\epsilon_q = 1$ for $q \neq (s-1,0)$ and $\hat\epsilon_{s-1,0} = \overline{\epsilon_{00}}\epsilon_{01} \dots \overline{\epsilon_{s-1,s-1}}\epsilon_{s-1,0}$.*
- *(b) The Schur multiplier ϵ has the same norm as $\check\epsilon$ given by $\check\epsilon_{ii} = 1$ and $\check\epsilon_{i,i+1} = \vartheta$ with ϑ any sth root of $\hat\epsilon_{s-1,0}$ or its complex conjugate: without loss of generality, $\vartheta = \mathrm{e}^{\mathrm{i}\alpha}$ with $\alpha \in [0, \pi/s]$.*
- *(c) The norm of ϵ on S^p_I is bounded below by the norm of the relative Fourier multiplier $\mu : a + bz \mapsto a + \vartheta bz$ on $\mathrm{L}^p_\Lambda(G)$; their complete norms are equal.*
- *(d) The norm of ϵ on S^1_I and on S^∞_I is equal to the norm of μ on $\mathrm{L}^1_\Lambda(G)$ and on $\mathrm{L}^\infty_\Lambda(G)$: this norm is*
$$\frac{\cos(\alpha/2 - \pi/2s)}{\cos \pi/2s} = \frac{\max_{z^s=-1} |\vartheta + z|}{|1 + \mathrm{e}^{\mathrm{i}\pi/s}|}.$$
- *(e) The Schur multiplication operator M_ϵ is an isometry on S^p_I if and only if $p/2 \in \{1, 2, \dots, s-1\}$ or $\overline{\epsilon_{00}}\epsilon_{01} \dots \overline{\epsilon_{s-1,s-1}}\epsilon_{s-1,0} = 1$.*

Proof. (a) and (b) follow from the matrix unconditionality of Schatten-von-Neumann norms (see Eq. (4)) and from the fact that the Schur multipliers ϵ and $\bar\epsilon = (\overline{\epsilon_q})_{q \in I}$ have the same norm on S^p_I.

(c) follows from Prop. 2.5.

(d). Let us compute $f(\beta) = \|1 + \mathrm{e}^{\mathrm{i}\beta} z\|_{\mathrm{L}^1(G)}$. As $f(\beta) = f(\beta + 2\pi/s) = f(-\beta)$, we may suppose without loss of generality that $\beta \in [0, \pi/s]$. Then $|\beta/2 + k\pi/s| \leqslant \pi/2$ if $-\lfloor s/2 \rfloor \leqslant k \leqslant \lceil s/2 \rceil - 1$, so that

$$\begin{aligned}
f(\beta) &= \frac{1}{s} \sum_{k=-\lfloor s/2 \rfloor}^{\lceil s/2 \rceil - 1} \bigl|1 + \mathrm{e}^{\mathrm{i}\beta} \mathrm{e}^{2\mathrm{i}k\pi/s}\bigr| \\
&= \frac{2}{s} \sum_{k=-\lfloor s/2 \rfloor}^{\lceil s/2 \rceil - 1} \cos(\beta/2 + k\pi/s) \\
&= \frac{2}{s} \Re\biggl(\mathrm{e}^{\mathrm{i}\beta/2} \frac{\mathrm{e}^{\mathrm{i}\lceil s/2 \rceil \pi/s} - \mathrm{e}^{-\mathrm{i}\lfloor s/2 \rfloor/s}}{\mathrm{e}^{\mathrm{i}\pi/s} - 1} \biggr) \\
&= \frac{2}{s \sin(\pi/2s)} \cdot \begin{cases} \cos(\beta/2 - \pi/2s) & \text{if } s \text{ is even} \\ \cos(\beta/2) & \text{if } s \text{ is odd.} \end{cases}
\end{aligned}$$

This shows in both cases that the norm of μ on $\mathrm{L}^1_\Lambda(G)$ is bounded below by $\cos(\alpha/2-\pi/2s)/\cos(\pi/2s)$. The complete norm of μ on $\mathrm{L}^\infty_\Lambda(G)$ is equal to its norm and thus to the maximum of $g(w)=\|w+\vartheta z\|_{\mathrm{L}^\infty(G)}/\|w+z\|_{\mathrm{L}^\infty(G)}$ for $w\in\mathbb{C}$. Let $w=r\mathrm{e}^{\mathrm{i}\beta}$ with $r\geqslant 0$ and $\beta\in\mathbb{R}$. Note that

$$\|w+z\|_{\mathrm{L}^\infty(G)}=\left|r+\mathrm{e}^{\mathrm{i}\mathrm{d}(\beta,(2\pi/s)\mathbb{Z})}\right|$$

is a decreasing function of $\mathrm{d}(\beta,(2\pi/s)\mathbb{Z})$ and that

$$\mathrm{d}(\alpha-\beta,(2\pi/s)\mathbb{Z})<\mathrm{d}(\beta,(2\pi/s)\mathbb{Z})\ \Leftrightarrow\ \beta\in\,]\alpha/2,\pi/s+\alpha/2[\ \mathrm{mod}\ 2\pi/s.$$

As $g(w)=g(wz)$ if $z^s=1$, we may suppose without loss of generality that $\beta\in\,]\alpha/2,\pi/s+\alpha/2[$. Therefore

$$g(w)=\begin{cases}\left|\dfrac{w+\mathrm{e}^{\mathrm{i}\alpha}}{w+1}\right| & \text{if } \beta\in\,]\alpha/2,\pi/s]\\ \left|\dfrac{w+\mathrm{e}^{\mathrm{i}\alpha}}{w+\mathrm{e}^{2\mathrm{i}\pi/s}}\right| & \text{if } \beta\in[\pi/s,\pi/s+\alpha/2[.\end{cases}$$

As g tends to 1 at infinity and $g(w)=1$ if $\beta\in\{\alpha/2,\pi/s+\alpha/2\}$, the maximum principle shows that g attains its maximum with $\beta=\pi/s$. Finally,

$$\begin{aligned}g(r\mathrm{e}^{\mathrm{i}\pi/s})^2&=\frac{1+2r\cos(\pi/s-\alpha)+r^2}{1+2r\cos(\pi/s)+r^2}\\&=1+\frac{\cos(\pi/s-\alpha)-\cos\pi/s}{\cos(\pi/s)+(r+1/r)/2}\leqslant g(\mathrm{e}^{\mathrm{i}\pi/s})^2=\left(\frac{\cos(\pi/2s-\alpha/2)}{\cos\pi/2s}\right)^2.\end{aligned}$$

(e). If p is not an even integer and $\vartheta^s\neq 1$, then μ is not an isometry on $\mathrm{L}^p_\Lambda(G)$: otherwise the functions z and ϑz would have the same distribution by the Plotkin-Rudin Equimeasurability Theorem (see [11, Th. 2]). If $p\in\{2,4,\dots,2s-2\}$, then I contains no cycle of length $4,6,\dots,p$, so that by Prop. 6.6(c) every closed walk $P\sim(\alpha,\beta)$ satisfies $\alpha=\beta$. The function $\Phi_I(\epsilon,a)$ in Eq. (9) is therefore constant in ϵ by Th. 6.5(a). If $p\in\{2s,2s+2,\dots\}$, the closed walk relation

$$(\alpha,\beta)=\Big(\sum_{i\in\Gamma}\mathrm{e}_{ii},\sum_{i\in\Gamma}\mathrm{e}_{i,i+1}\Big)+(p/2-s)(\mathrm{e}_{00},\mathrm{e}_{00})$$

satisfies $n_{\alpha\beta}\geqslant 1$ by Prop. 6.4. Then the coefficient of $\Phi_I(\epsilon,a)$ in $\bar a^\alpha a^\beta$ equals

$$n_{\alpha\beta}\overline{\epsilon_{00}}\epsilon_{01}\dots\overline{\epsilon_{s-1,s-1}}\epsilon_{s-1,0}$$

and must equal the same quantity with ϵ replaced by 1 if ϵ defines an isometry on S^p_I. ☐

Remark 7.2. See [12, p. 245] for a similar application of the Plotkin-Rudin Equimeasurability Theorem in (e).

The real unconditional constant of I is therefore the norm of $\check\epsilon$ with $\alpha=\pi/s$, and the complex unconditional constant is the maximum of the norm of $\check\epsilon$ for $\alpha\in[0,\pi/s]$. This yields

Corollary 7.3. *Let $0<p\leqslant\infty$ and $s\geqslant 2$. Let I be the cycle of length $2s$.*

- (a) *I is a real 1-unconditional basic sequence in S^p if and only if $p\in\{2,4,\dots,2s-2\}$.*
- (b) *The real and complex unconditional constants of I in the spaces S^1 and S^∞ equal $\sec\pi/2s$.*

8 1-unconditional matrices in S^p, p not an even integer

We now state the announced isometric counterpart to Varopoulos' characterisation of unconditional matrices in S^∞ (Section 5) and its generalisation to S^p for p not an even integer.

Theorem 8.1. *Let $I\subseteq R\times C$ be nonempty and $p\in(0,\infty]\setminus 2\mathbb{N}$. The following are equivalent.*

- (a) *I is a complex completely 1-unconditional basic sequence in S^p.*

(*b*) *I is a complex 1-unconditional basic sequence in* S^p.

(*c*) *I is a real 1-unconditional basic sequence in* S^p.

(*d*) *I is a forest.*

(*e*) *For each $\epsilon \in \mathbb{T}^I$ there are $\zeta \in \mathbb{T}^C$ and $\eta \in \mathbb{T}^R$ such that $\epsilon_{rc} = \zeta(c)\eta(r)$ for all $(r, c) \in I$.*

(*f*) *For each $\epsilon \in \{-1, 1\}^I$ there are $\zeta \in \{-1, 1\}^C$ and $\eta \in \{-1, 1\}^R$ such that $\epsilon_{rc} = \zeta(c)\eta(r)$ for all $(r, c) \in I$.*

(*g*) *I is a* set of V-interpolation of constant 1*: for all $\varphi \in \ell^\infty_I$*

$$\inf\Big\{\Big\|\sum_{(r,c)\in R\times C} \tilde\varphi_{rc}\mathrm{e}_c \otimes \mathrm{e}_r\Big\|_{\ell^\infty_C \overset{\wedge}{\otimes} \ell^\infty_R} : \tilde\varphi|_I = \varphi\Big\} = \sup_{q\in I} |\varphi_q|. \tag{15}$$

(*h*) *I is a* V-Sidon set of constant 1*: for all $\varphi \in \mathrm{c}_0(I)$*

$$\inf\Big\{\Big\|\sum_{(r,c)\in R\times C} \tilde\varphi_{rc}\mathrm{e}_c \otimes \mathrm{e}_r\Big\|_{\mathrm{c}_0(C) \overset{\wedge}{\otimes} \mathrm{c}_0(R)} : \tilde\varphi|_I = \varphi\Big\} = \sup_{q\in I} |\varphi_q|. \tag{16}$$

(*i*) *For every tensor $u = \sum_{(r,c)\in I} a_{rc}\mathrm{e}_c \otimes \mathrm{e}_r$ in $\ell^1_C \overset{\vee}{\otimes} \ell^1_R$ with support in I we have $\|u\|_{\ell^1_C \overset{\vee}{\otimes} \ell^1_R} = \sum_{(r,c)\in I} |a_{rc}|$.*

(*j*) *$(z_c z'_r)_{(r,c)\in I}$ is a* Sidon set of constant 1 *in the dual of $\mathbb{T}^C \times \mathbb{T}^R$, that is, a 1-unconditional basic sequence in $\mathrm{C}(\mathbb{T}^C \times \mathbb{T}^R)$: if (a_{rc}) is finitely supported,*

$$\sup_{(z,z')\in\mathbb{T}^C\times\mathbb{T}^R}\Big|\sum_{(r,c)\in I} a_{rc} z_c z'_r\Big| = \sum_{(r,c)\in I} |a_{rc}| \ .$$

(*k*) *For all $R' \subseteq R$ and $C' \subseteq C$ with $k \geqslant 1$ elements $\#[I \cap R' \times C'] \leqslant 2k - 1$.*

(*l*) *I is an* isometric interpolation set *for Schur multipliers on S^∞: every $\varphi \in \ell^\infty_I$ is the restriction of a Schur multiplier on S^∞ with norm $\|\mathrm{M}_\varphi\| = \|\varphi\|_{\ell^\infty_I}$.*

Proof. $(a) \Rightarrow (b) \Rightarrow (c)$ is trivial.

$(c) \Rightarrow (d)$. Suppose that I contains a cycle $(c_0, r_0, \ldots, c_{s-1}, r_{s-1})$ with $s \geqslant 2$. Cor. 7.3(a) shows that I is not a real 1-unconditional basic sequence in S^p.

$(d) \Leftrightarrow (k)$. A tree on $2k$ vertices has exactly $2k-1$ edges, so that a forest I satisfies (k). Conversely, a cycle of length $2s$ is a graph with s row vertices, s column vertices and $2s$ edges.

$(d) \Rightarrow (e)$. Suppose first that I is a tree and index the vertices of its edges by words $w \in W$ as described in the Terminology. Let us define η and ζ inductively. If r is the root of the tree, indexed by $\emptyset$, let $\eta(r) = 1$. Suppose that η and ζ have been defined for all vertices indexed by words of length at most $2n$. If c is indexed by a word w of length $2n+1$, let r be the vertex indexed by the word of length $2n$ with which w begins and let $\zeta(c) = \epsilon(r, c)/\eta(r)$. If r is indexed by a word w of length $2n+2$, let c be the vertex indexed by the word of length $2n+1$ with which w begins and let $\eta(r) = \epsilon(r, c)/\zeta(c)$. If I is a union of pairwise disjoint trees, we may define η and ζ on each tree separately. We may finally extend η to R and ζ to C in an arbitrary manner.

$(d) \Rightarrow (f)$ may be proved as $(d) \Rightarrow (e)$.

$(f) \Rightarrow (c)$. If (f) holds, then every Schur multiplier by signs $\epsilon \in \{-1, 1\}^I$ is elementary in the sense that $\epsilon = \zeta \otimes \eta$. The complete norm of M_ϵ on any S^p_I is therefore $\|\zeta\|_{\ell^\infty_C}\|\eta\|_{\ell^\infty_R} = 1$.

$(e) \Rightarrow (g)$. If (e) holds, every $\varphi \in \mathbb{T}^I \subseteq \ell^\infty_I$ may be extended to an elementary tensor $\zeta \otimes \eta$ of norm 1. (g) follows because every element of ℓ^∞_I with norm 1 is the half sum of two elements of $\mathbb{T}^I$: note that $\mathrm{e}^{\mathrm{i}t}\cos u = \big(\mathrm{e}^{\mathrm{i}(t+u)} + \mathrm{e}^{\mathrm{i}(t-u)}\big)/2$.

$(g) \Rightarrow (h)$. It suffices to check Equality (16) for φ with support contained in a finite rectangle set $R' \times C'$. As $\ell^\infty_{C'} \overset{\wedge}{\otimes} \ell^\infty_{R'}$ is a subspace of $\ell^\infty_C \overset{\wedge}{\otimes} \ell^\infty_R$, Eq. (15) yields Eq. (16).

$(h) \Leftrightarrow (i)$ because they are dual statements.

$(i) \Leftrightarrow (j)$. Use Equality (8).

$(h) \Rightarrow (l)$ may be deduced by the argument of Prop. 3.1$(a) \Rightarrow (b)$.

$(l) \Rightarrow (a)$. Taking sign sequences $\varphi \in \mathbb{T}^I$ in (l) shows that all relative Schur multipliers by signs on S^∞_I define isometries. Apply Prop. 4.3. □

Remark 8.2. The equivalence of (e) with (j) may also be shown as a consequence of the characterisation of Sidon sets of constant 1 in [4].

Let us now answer Question 1.3.

Corollary 8.3. *Let $I \subseteq R \times C$. The following are equivalent.*

- (a) *For all $\varphi \in \mathrm{c}_0(I)$ one has $\left\|\sum_{(r,c)\in I} \varphi_{rc}\mathrm{e}_c \otimes \mathrm{e}_r\right\|_{\mathrm{c}_0(C)\,\hat{\otimes}\,\mathrm{c}_0(R)} = \sup_{q\in I} |\varphi_q|$.*
- (b) *There are pairwise disjoint sets $R_j \subseteq R$ and pairwise disjoint sets $C_j \subseteq C$ such that R_j or C_j is a singleton for each j and $I = \bigcup R_j \times C_j$: I is the union of the column section $\bigcup_{\#\,R_j=1} R_j \times C_j$ with the disjoint row section $\bigcup_{\#\,R_j>1} R_j \times C_j$.*
- (c) *I is a union of pairwise disjoint star graphs: every path in I has length at most 2.*

Proof. $(a) \Rightarrow (b)$ follows from Prop. 3.1$(a) \Rightarrow (d)$ and Th. 8.1$(g) \Rightarrow (d)$.

$(b) \Leftrightarrow (c)$. (b) holds if and only if $(r, c), (r', c), (r, c') \in I \Rightarrow (r = r'$ or $c = c')$ and therefore if and only if (c) holds.

$(b) \Rightarrow (a)$. Suppose (b) and let $\varphi \in \mathrm{c}_0(I)$. Let $\alpha_j = \sup_{(r,c)\in R_j\times C_j} |\varphi_{rc}|^{1/2}$. If $\alpha_j = 0$, let us define $\varrho^j = 0$ and $\gamma^j = 0$. Otherwise, if R_j is a singleton $\{r\}$, let us define $\varrho^j = \alpha_j\mathrm{e}_r$ and γ^j by $\gamma^j_c = \varphi_{rc}/\alpha_j$ if $c \in C_j$ and $\gamma^j_c = 0$ otherwise. Otherwise, C_j is a singleton $\{c\}$ and we define $\gamma^j = \alpha_j\mathrm{e}_c$ and ϱ^j by $\varrho^j_r = \varphi_{rc}/\alpha_j$ if $r \in R_j$ and $\varrho^j_r = 0$ otherwise. Note that the γ^j have pairwise disjoint support and are null sequences, as well as the ϱ^j. Then

$$\sum_{(r,c)\in I} \varphi_{rc}\mathrm{e}_c \otimes \mathrm{e}_r = \sum_j \gamma^j \otimes \varrho^j = \operatorname*{average}_{\epsilon_j=\pm1}\Bigl(\sum_j \epsilon_j\gamma^j\Bigr) \otimes \Bigl(\sum_j \epsilon_j\varrho^j\Bigr)$$

is an average of elementary tensors in $\mathrm{c}_0(C)\,\hat{\otimes}\,\mathrm{c}_0(R)$ of norm $\sup_{q\in I} |\varphi_q|$, so that this average is also bounded by this norm, which obviously is a lower bound. □

9 1-unconditional matrices in S^p, p an even integer

Let us now prove Theorem 1.5 as a consequence of Theorem 6.5 together with Proposition 6.6(c).

Theorem 9.1. *Let $I \subseteq R \times C$ and $p = 2k$ a positive even integer. The following assertions are equivalent.*

- (a) *I is a complex completely 1-unconditional basic sequence in S^p.*
- (b) *I is a complex 1-unconditional basic sequence in S^p.*
- (c) *For every finite subset $F \subseteq I$ there is an operator $x \in \mathrm{S}^p$, whose support S contains F, such that $\left\|\sum \epsilon_q x_q \mathrm{e}_q\right\|_p$ does not depend on the complex choice of signs $\epsilon \in \mathbb{T}^S$.*
- (d) *I is a real 1-unconditional basic sequence in S^p.*
- (e) *For every finite subset $F \subseteq I$ there is an operator $x \in \mathrm{S}^p$ with real matrix coefficients, whose support S contains F, such that $\left\|\sum \epsilon_q x_q \mathrm{e}_q\right\|_p$ does not depend on the real choice of signs $\epsilon \in \{-1, 1\}^S$.*
- (f) *Every closed walk $P \sim (\alpha, \beta)$ of length $2s \leqslant 2k$ in I satisfies $\alpha = \beta$.*
- (g) *I does not contain any cycle of length $2s \leqslant 2k$ as a subgraph.*

(h) *For each $v, w \in V$ there is at most one path in I of length $l \leqslant k$ that joins v to w.*

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c), (b) $\Rightarrow$ (d) $\Rightarrow$ (e) are trivial.

(c) $\Rightarrow$ (g). Suppose that I contains a cycle $P \sim (\gamma, \delta)$ of length $2s \leqslant 2k$: the corresponding set of couples is $F = \{q : \gamma_q + \delta_q = 1\}$. Let x be as in (c) and let $(\alpha, \beta) = (\gamma, \delta) + (k-s)(\mathrm{e}_q, \mathrm{e}_q)$ for some arbitrary $q \in F$. Then $(\alpha, \beta) \in \mathscr{W}_k^S$. Consider $f(\epsilon) = \left\|\sum \epsilon_q x_q \mathrm{e}_q\right\|_p^p$ as a function on the group $\mathbb{T}^S$. Then the Fourier coefficient $\widehat{f}(\epsilon^{\beta-\alpha})$ of f at the Steinhaus character $\epsilon^{\beta-\alpha}$ is, by Th. 6.5(a),

$$\sum\left\{n_{\varepsilon\zeta} \overline{x}^{\varepsilon} x^{\zeta} : (\varepsilon, \zeta) \in \mathscr{W}_k^S \text{ and } \zeta - \varepsilon = \beta - \alpha\right\} \\ = \overline{x}^{\gamma} x^{\delta} \sum\left\{n_{\varepsilon\zeta} \overline{x}^{\varepsilon-\gamma} x^{\zeta-\delta} : (\varepsilon, \zeta) \in \mathscr{W}_k^S \text{ and } \zeta - \delta = \varepsilon - \gamma\right\}.$$

(Note that $\beta - \alpha = \delta - \gamma$.) As this last sum has only positive terms and contains at least the term corresponding to (α, β), f cannot be constant.

(e) $\Rightarrow$ (g). Let $P \sim (\gamma, \delta)$, $F = \{q : \gamma_q + \delta_q = 1\}$ and (α, β) be as in the proof of the implication (c) $\Rightarrow$ (h). Let x be as in (e). Consider $f(\epsilon) = \left\|\sum \epsilon_q x_q \mathrm{e}_q\right\|_p^p$ as a function on the group $\{-1, 1\}^S$. Then the Fourier coefficient $\widehat{f}(\epsilon^{\beta-\alpha})$ of f at the Walsh character $\epsilon^{\beta-\alpha}$ is, by Th. 6.5(a),

$$\sum\left\{n_{\varepsilon\zeta} x^{\varepsilon+\zeta} : (\varepsilon, \zeta) \in \mathscr{W}_k^S \text{ and } \zeta - \varepsilon \equiv \beta - \alpha \pmod 2\right\} \\ = x^{\gamma+\delta} \sum\left\{n_{\varepsilon\zeta} x^{\varepsilon+\zeta-\gamma-\delta} : (\varepsilon, \zeta) \in \mathscr{W}_k^S \text{ and } \zeta - \varepsilon \equiv \delta - \gamma \pmod 2\right\}.$$

As this last sum has only positive terms and contains at least the term corresponding to (α, β), f cannot be constant.

(f) $\Leftrightarrow$ (g). Apply Prop. 6.6(c).

(g) $\Leftrightarrow$ (h). If I contains a cycle $(v_0, \ldots, v_{2s-1})$, then I contains two distinct paths $(v_0, \ldots, v_s)$, $(v_0, v_{2s-1}, \ldots, v_s)$ of length s from v_0 to v_s. If I contains two distinct paths $(v_0, \ldots, v_l)$, $(v'_0, \ldots, v'_{l'})$ with $v_0 = v'_0$, $v_l = v'_{l'}$ and $l, l' \leqslant k$, let a be minimal such that $v_a \neq v'_a$, let $b \geqslant a$ be minimal such that $v_b \in \{v'_a, \ldots, v'_{l'}\}$ and let $d \geqslant a$ be minimal such that $v'_d = v_b$. Then $(v_{a-1}, \ldots, v_b, v'_{d-1}, \ldots, v'_a)$ is a cycle in I of length $2s \leqslant 2k$.

(f) $\Rightarrow$ (a) holds by Theorem 6.5(b): If each $(\alpha, \beta) \in \mathscr{W}_k^I$ satisfies $\alpha = \beta$, then Eq. (14) shows that $\Psi_I(\epsilon, z)$ as defined in Eq. (13) is constant in ϵ. ◻

Remark 9.2. The equivalence (b) $\Leftrightarrow$ (g) is a noncommutative analogue to [14, Prop. 2.5(ii)].

Remark 9.3. In [15, Th. 2.7], the condition of Th. 9.1(f) is visualised in another way: a closed walk $P = (c_1, r_1, \ldots, c_s, r_s) \sim (\alpha, \beta)$ in $\mathbb{N} \times \mathbb{N}$ is considered as the polygonal closed curve γ in $\mathbb{C}$ with sides parallel to the coordinate axes whose successive vertices are $r_1 + \mathrm{i}c_1$, $r_1 + \mathrm{i}c_2$, $r_2 + \mathrm{i}c_2$, $\ldots$, $r_{s-1} + \mathrm{i}c_s$, $r_s + \mathrm{i}c_s$, $r_s + \mathrm{i}c_1$ and again $r_1 + \mathrm{i}c_1$. Then $\alpha = \beta$ if and only if the index with respect to γ of every point not on γ is zero, if and only if γ can be shrunk to a point inside of the set of its points.

Remark 9.4. One cannot drop the assumption that x has real matrix coefficients in Th. 9.1(e). Consider a 2×2 matrix x. Then $\operatorname{tr} x^*x = \sum |x_q|^2$ and $\det x^*x = |x_{00}x_{11} - x_{01}x_{10}|^2$. This shows that if $\Re(\overline{x_{00}x_{11}} x_{01}x_{10}) = 0$, e.g. $x = \begin{pmatrix} 1 & 1 \\ 1 & \mathrm{i} \end{pmatrix}$, then the singular values of x do not depend on the real sign of the matrix coefficients of x, whereas $(\mathrm{col}\,0, \mathrm{row}\,0, \mathrm{col}\,1, \mathrm{row}\,1)$ is a cycle of length 4.

Remark 9.5. Theorem 9.1(h) $\Rightarrow$ (a) is the isometric counterpart to [9, Th. 3.1], which shows in particular that I is an unconditional basic sequence in S^{2k} if the number of walks in I between two given vertices of length k and with no edge repeated has a uniform bound. The following combinatorial problem arises naturally: if I satisfies this latter condition, is it so that I is the union of a finite number of sets I_j such that there is at most one path of length at most k in I_j between two given vertices? In the simplest case, $k = 2$, William Banks, Ilijas Farah, Asma Harcharras and Dominique Lecomte [2] have deduced from [24] that it is not so.

10 Metric unconditional approximation property for S^p_I

Let R, C be two copies of $\mathbb{N}$. It is well known that, apart from S^2, no S^p has an unconditional basis or just a local unconditional structure (see [23, § 4].) S^1 and S^∞ cannot even be embedded in a space with unconditional basis. If $1 < p < \infty$, then S^p has the unconditional finite dimensional decomposition

$$\bigoplus_{n\in\mathbb{N}} S^p_{\{(r,c):r\leqslant n,c=n\}} \oplus S^p_{\{(r,c):r=n+1,c\leqslant n\}}$$

because the triangular projection associated to the idempotent Schur multiplier $(\chi_{r\leqslant c})$ is bounded on S^p.

Definition 10.1. Let X be a separable Banach space and $\mathbb{S} = \mathbb{T}$ (vs. $\mathbb{S} = \{-1, 1\}$.)

- A sequence (T_k) of operators on X is an *approximating sequence* if each T_k has finite rank and $\|T_k x - x\| \to 0$ for every $x \in X$. An approximating sequence of commuting projections is a *finite-dimensional decomposition*.
- ([18].) The *difference sequence* (ΔT_k) of (T_k) is given by $\Delta T_1 = T_1$ and $\Delta T_k = T_k - T_{k-1}$ for $k \geqslant 2$. X has the *unconditional approximation property* (*uap*) if there is an approximating sequence (T_k) such that for some constant D

$$\Big\|\sum_{k=1}^{n} \epsilon_k \Delta T_k\Big\| \leqslant D \quad \text{for all } n \text{ and } \epsilon_k \in \mathbb{S}.$$

  The *complex (vs. real) unconditional constant* of (T_k) is the least such constant D.
- ([5, § 3], [7, § 8].) X has the *complex (vs. real) metric unconditional approximation property* (*muap*) if, for every $\delta > 0$, X has an approximating sequence with complex (vs. real) unconditional constant $1 + \delta$. By [5, Th. 3.8] and [7, Lemma 8.1], this is the case if and only if there is an approximating sequence (T_k) such that

$$\sup_{\epsilon\in\mathbb{S}} \|T_k + \epsilon(\mathrm{Id} - T_k)\| \longrightarrow 1. \tag{17}$$

X has (*muap*) if and only if, for every given $\delta > 0$, X is isometric to a 1-complemented subspace of a space with a $(1+\delta)$-unconditional finite-dimensional decomposition [6, Cor. IV.4]. If X has (*muap*), then, for any given $\delta > 0$, X is isometric to a subspace of a space with a $(1 + \delta)$-unconditional basis.

Example 10.2. The simplest example is the subspace in S^p of operators with an upper triangular matrix. In fact, if $I \subseteq R \times C$ is such that all columns $I \cap R \times \{c\}$ (vs. all rows $I \cap \{r\} \times C$) are finite, then S^p_I admits a 1-unconditional finite-dimensional decomposition in the corresponding finitely supported idempotent Schur multipliers $\chi_{I\cap R\times\{c\}}$ (vs. $\chi_{I\cap\{r\}\times C}$.)

Our results on complete 1-unconditional basic sequences yield the following theorem.

Theorem 10.3. *Let $1 \leqslant p \leqslant \infty$. Let $R_r \subseteq R$, $r \in \mathbb{N}$, be pairwise disjoint and finite. Let $C_c \subseteq C$, $c \in \mathbb{N}$, be pairwise disjoint and finite. Let $J \subseteq \mathbb{N} \times \mathbb{N}$ and $I = \bigcup_{(r,c)\in J} R_r \times C_c$. Then the sequence of Schur multipliers $(\chi_{R_r\times C_c})_{(r,c)\in J}$ forms a complex 1-unconditional finite-dimensional decomposition for S^p_I if and only if J is a forest or p is an even integer and J contains no cycle of length $4, 6, \dots, p$.*

We may always suppose that approximating sequences on spaces S^p_I are associated to Schur multipliers. More precisely, we have

Proposition 10.4. *Let $1 \leqslant p \leqslant \infty$ and $I \subseteq R \times C$. Let (T_n) be an approximating sequence on S^p_I. Then there is a sequence of Schur multipliers (φ_n) such that (M_{φ_n}) is an approximating sequence on S^p_I and such that if (T_n) satisfies (17), then so does (M_{φ_n}).*

Proof. Let $\delta_n > 0$ be such that $\delta_n \to 0$. As T_n has finite rank, there is a finite $R_n \times C_n \subseteq R \times C$ such that the projection $P_{R_n \times C_n}$ of S^p onto $\mathrm{S}^p_{R_n \times C_n}$ defined by the Schur multiplier $\chi_{C_n} \otimes \chi_{R_n}$ satisfies $\|P_{R_n \times C_n} T_n - T_n\| < \delta_n$. Let φ_n be the Schur multiplier associated to $[T_n]_{R_n \times C_n}$. With the notation of Eq. (5),

$$M_{\varphi_n}(x) - x = \int_{\mathbb{T}^R} \mathrm{d}\eta \int_{\mathbb{T}^C} \mathrm{d}\zeta\, \mathrm{M}_{\zeta^* \otimes \eta^*}(P_{R_n \times C_n} T_n - \mathrm{Id})(\mathrm{M}_{\zeta \otimes \eta} x).$$

As $P_{R_n \times C_n} T_n$ tends to the identity uniformly on compact sets, this shows that M_{φ_n} is an approximating sequence. As

$$\mathrm{M}_{\varphi_n} + \epsilon(\mathrm{Id} - \mathrm{M}_{\varphi_n}) = [P_{R_n \times C_n} T_n + \epsilon(\mathrm{Id} - P_{R_n \times C_n} T_n)]\ ,$$

the norm of this operator is at most $\|T_n + \epsilon(\mathrm{Id} - T_n)\| + 2\delta_n$. ☐

This proposition shows together with Prop. 2.1 the following results.

Corollary 10.5. *Let $1 \leqslant p \leqslant \infty$ and $I \subseteq R \times C$.*

- *If S^p_I has (muap), then some sequence of Schur multipliers realises it.*
- *Let $J \subseteq I$. If S^p_I has (muap), then so does S^p_J.*
- *If S^∞_I has (muap), then so does S^p_I.*

Let us define the following asymptotic properties.

Definition 10.6. Let $1 \leqslant p \leqslant \infty$, $I \subseteq R \times C$ and $\mathbb{S} = \mathbb{T}$ (vs. $\mathbb{S} = \{-1, 1\}$.)

- S^p_I is *asymptotically unconditional* if for every $x \in \mathrm{S}^p_I$ and for every bounded sequence (y_n) in S^p_I such that each matrix coefficient of y_n tends to 0

$$\max_{\epsilon \in \mathbb{S}} \|x + \epsilon y_n\|_p - \min_{\epsilon \in \mathbb{S}} \|x + \epsilon y_n\|_p \longrightarrow 0.$$

- I enjoys the property $(\mathscr{U})$ of *block unconditionality* in S^p if for each $\delta > 0$ and finite $F \subseteq I$, there is a finite $G \subseteq I$ such that

$$\forall\, x \in B_{\mathrm{S}^p_F}\ \forall\, y \in B_{\mathrm{S}^p_{I \setminus G}} \quad \max_{\epsilon \in \mathbb{S}} \|x + \epsilon y\|_p - \min_{\epsilon \in \mathbb{S}} \|x + \epsilon y\|_p < \delta.$$

The arguments of [14, § 6.2] show mutatis mutandis

Theorem 10.7. *Let $1 \leqslant p \leqslant \infty$, $I \subseteq R \times C$ and $\mathbb{S} = \mathbb{T}$ (vs. $\mathbb{S} = \{-1, 1\}$.) Consider the following properties.*

(*a*) *S^p_I is asymptotically unconditional.*

(*b*) *I enjoys $(\mathscr{U})$ in S^p.*

(*c*) *S^p_I has (muap).*

Then (c) $\Rightarrow$ (a) $\Leftrightarrow$ (b). If $1 < p < \infty$, then (b) $\Leftrightarrow$ (c). If $p = 1$, S^1_I has (muap) if and only if S^1_I has (uap) and I enjoys $(\mathscr{U})$ in S^1.

The case $p = \infty$ is extreme in the sense that the following properties are equivalent for S^∞_I: to be a dual space, to be reflexive, to have a finite cotype, not to contain c_0, because they are equivalent for I not to contain any sequence (r_n, c_n) with (r_n) and (c_n) injective, that is for I to be contained in the union of a finite set of lines and a finite set of columns, so that S^∞_I is isomorphic to ℓ^2_I.

Let us now introduce the asymptotic property on I that reflects the combinatorics imposed by (*muap*).

Definition 10.8. Let $I \subseteq R \times C$ and $k \geqslant 1$.

- I enjoys property $\mathscr{J}_k$ if for every path $P = (c_0, r_0, \dots, c_j, r_j)$ of odd length $2j+1 \leqslant k$ in I there is a finite set $R' \times C'$ such that P cannot be completed with edges in $I \setminus R' \times C'$ to a cycle of length $2s \in \{4j+2, \dots, 2k\}$.

- The *asymptotic distance* $d_\infty(r, c)$ of $r \in R$ and $c \in C$ in I is the supremum, over all finite rectangle sets $R' \times C'$, of the distance from r to c in $I \setminus R' \times C'$.

The asymptotic distance takes its values in $\{1, 3, 5, \dots, \infty\}$. Note that $\mathscr{J}_1$ is true and that $\mathscr{J}_k \Rightarrow \mathscr{J}_{k-1}$. This implication is strict: let R, C be two copies of $\mathbb{N}$ and, given $j \geqslant 1$, consider the union I_j of all paths $(\operatorname{col} 0, \operatorname{row} nj+1, \operatorname{col} nj+1, \dots, \operatorname{row} nj+j, \operatorname{col} nj+j, \operatorname{row} 0)$ of length $2j+1$. Then I_j contains no cycle of length $2s \in \{4, \dots, 4j\}$ and therefore enjoys $\mathscr{J}_{2j}$, but fails $\mathscr{J}_{2j+1}$; $I_j \cup \{(\operatorname{row} 0, \operatorname{col} 0)\}$ contains no cycle of length $2s \in \{4, \dots, 2j\}$ and thus enjoys $\mathscr{J}_j$, but fails $\mathscr{J}_{j+1}$. In particular, the properties $\mathscr{J}_k$, $k \geqslant 2$, are not stable under union with a singleton.

Let us now explicit the relationship between $\mathscr{J}_k$ and d_∞.

Proposition 10.9. *Let $I \subseteq R \times C$ and $k \geqslant 1$.*

(*a*) *I enjoys $\mathscr{J}_k$ if and only if any two vertices $r \in R$ and $c \in C$ at distance $2j+1 \leqslant k$ satisfy $d_\infty(r, c) \geqslant 2k - 2j + 1$.*

(*b*) *If $d_\infty(r, c) \geqslant 2k+1$ for all $(r, c) \in R \times C$, then I enjoys $\mathscr{J}_k$.*

(*c*) *If $d_\infty(r, c) \leqslant k$ for some $(r, c) \in R \times C$, then I fails $\mathscr{J}_k$.*

(*d*) *I enjoys $\mathscr{J}_k$ for every k if and only if $d_\infty(r, c) = \infty$ for every $(r, c) \in R \times C$.*

Proof. (*a*) is but a reformulation of the definition of $\mathscr{J}_k$ and implies (*b*).

(*d*) is a consequence of (*b*) and (*c*).

(*c*). If $d_\infty(r, c) \leqslant k$, then there is $0 \leqslant j \leqslant (k-1)/2$ such that there are infinitely many paths of length $2j+1$ from c to r: there is a path $(c, r_1, c_1, \dots, r_j, c_j, r)$ that can be completed with edges outside any given finite set to a cycle of length $4j+2 \leqslant 2k$. ☐

Theorem 10.10. *Let $I \subseteq R \times C$ and $1 \leqslant p \leqslant \infty$. If p is an even integer, then S^p_I has complex or real (muap) if and only if I enjoys $\mathscr{J}_{p/2}$. If $p = \infty$ or if p is not an even integer, then S^p_I has real (muap) only if I enjoys $\mathscr{J}_k$ for every k.*

Proof. Suppose that I enjoys $(\mathscr{U})$ in S^p and fails $\mathscr{J}_k$. Then, for some $s \leqslant k$, I contains a sequence of cycles $(c_0, r_0, \dots, c_{j-1}, r_{j-1}, c^n_j, r^n_j, \dots, c^n_{s-1}, r^n_{s-1})$ with the property that $\|x-y\|_p \leqslant (1+1/n)\|x+y\|_p$ for all x with support in $\{(r_0, c_0), (r_0, c_1), \dots, (r_{j-2}, c_{j-1}), (r_{j-1}, c_{j-1})\}$ and all y with support in $\{(r_{j-1}, c^n_j), (r^n_j, c^n_j), \dots, (r^n_{s-1}, c^n_{s-1}), (r^n_{s-1}, c_0)\}$. With the notation of Section 7, this amounts to stating that the multiplier on $I = \{(i, i), (i, i+1)\} \subseteq \mathbb{Z}/s\mathbb{Z} \times \mathbb{Z}/s\mathbb{Z}$ given by $\epsilon_{rc} = 1$ if $r, c \in \{0, \dots, j-1\}$ and $\epsilon_{rc} = -1$ otherwise actually is an isometry on S^p_I. As $\overline{\epsilon_{00}}\epsilon_{01} \dots \overline{\epsilon_{s-1\,s-1}}\epsilon_{s-1\,0} = (-1)^{2s-2j+1} = -1$, this implies by Prop. 7.1(*e*) that $p/2 \in \{1, 2, \dots, s-1\}$.

Suppose that I enjoys $\mathscr{J}_k$. We claim that for every finite $F \subseteq I$ there is a finite $G \subseteq I$ such that every closed walk $P \sim (\alpha, \beta)$ of length $2k$ in I satisfies $\sum_{q \in I \setminus G} \beta_q - \alpha_q = 0$. This signifies that given a closed walk $(v_0, \dots, v_{2k-1})$ and $0 = a_0 < b_0 < \dots < a_m < b_m < a_{m+1} = 2k$ such that $v_{a_i}, \dots, v_{b_i - 1} \in I \setminus G$ and $v_{b_i}, \dots, v_{a_{i+1}-1} \in F$,

$$\bigl\{i \in \{0, \dots, m\} : a_i, b_i \text{ even}\bigr\} = \bigl\{i \in \{0, \dots, m\} : a_i, b_i \text{ odd}\bigr\}.$$

Suppose that this is not true: then there is an $s \leqslant k$, there are $0 = a_0 < b_0 < \dots < a_m < b_m < 2s$ and there are cycles $(v^n_{a_0}, \dots, v^n_{b_0-1}, v_{b_0}, \dots, v_{a_1-1}, \dots, v^n_{a_m}, \dots, v^n_{b_m-1}, v_{b_m}, \dots, v_{2s-1})$ such that the $(v^n_i)_{n \geqslant 0}$ are injective sequences of vertices and $b_i - a_i$ is even for at least one index i: let us suppose so for $i = 0$. If $b_0 - a_0 \geqslant s-1$, consider the path $P = (v_{b_0}, \dots, v_{a_0-1}, v^0_{a_0}, \dots, v^0_{b_m-1}, v_{b_m}, \dots, v_{2s-1})$ of odd length $2s-1-(b_0-a_0)$; if $b_0 - a_0 \leqslant s-1$, consider the path $P = (v_{2s-1}, v^0_{a_0}, \dots, v^0_{b_0-1}, v_{b_0})$ of odd length $b_0 - a_0 + 1$. Then P can be completed with vertices outside any given finite set to a cycle of length at most $2s$ because $(v_{2s-1}, v^n_{a_0}, \dots, v^n_{b_0-1}, v_{b_0})$ is a path of length $b_0 - a_0 + 1$ in I for every n. This proves that I fails $\mathscr{J}_s$.

The claim shows that I enjoys $(\mathscr{U})$ in S^p for $p = 2k$. In fact, if $\tilde{\epsilon} \in \mathbb{T}^{F\cup(I\setminus G)}$ is defined by $\tilde{\epsilon}_q = 1$ for $q \in F$ and $\tilde{\epsilon}_q = \epsilon \in \mathbb{T}$ for $q \in I \setminus G$, then, with the notation of Th. 6.5,

$$\Phi_{F\cup(I\setminus G)}(\tilde{\epsilon}, a) = \sum_{(\alpha,\beta)\in\mathscr{W}_k^{F\cup(I\setminus G)}} n_{\alpha\beta}\epsilon^{\sum_{q\in I\setminus G}\beta_q-\alpha_q}\overline{a}^\alpha a^\beta$$

does not depend on ϵ, so that $\|x + \epsilon y\|_{2k} = \|x + y\|_{2k}$ if $x \in \mathrm{S}^{2k}_F$ and $y \in \mathrm{S}^{2k}_{I\setminus G}$, and S^{2k}_I has complex $(muap)$ by Th. 10.7$(b) \Rightarrow (c)$. $\square$

Remark 10.11. This theorem is a noncommutative analogue to [14, Th. 7.5].

11 Examples

One of Varopoulos' motivations for the study of the projective tensor product $\ell_\infty \overset{\wedge}{\otimes} \ell_\infty$ are lacunary sets in a locally compact abelian group.

Let Γ be a discrete abelian group and $\Lambda \subseteq \Gamma$. Let us say that Λ is *n-independent* if every element of Γ admits at most one representation as the sum of n terms in Λ, up to a permutation. For example, the geometric sequence $\{j^k\}_{k\geqslant 0}$ with $j \in \{2, 3, \dots\}$ is n-independent in $\mathbb{Z}$ if and only if $j \geqslant n$ [14, § 3]. If Λ is n-independent for all n, then Λ is *independent*. Let

$$\mathrm{Z}_n = \{\zeta \in \mathbb{Z}^\Lambda : \sum_{\gamma\in\Lambda} \zeta_\gamma = 0 \text{ and } \sum_{\gamma\in\Lambda} |\zeta_\gamma| \leqslant 2n]$$

and $\mathrm{Z} = \bigcup \mathrm{Z}_n$. Then Λ is n-independent if and only if, for every $\zeta \in \mathrm{Z}_n$,

$$\sum_{\gamma\in\Lambda} \zeta_\gamma\gamma = 0 \;\Rightarrow\; \zeta = 0$$

and Λ is independent if and only if this holds for every $\zeta \in \mathrm{Z}$.

Let us say that Λ is *n-independent modulo 2* if in every representation of an element of Γ as the sum of n terms in Λ, each element of Λ appears the same number of times modulo 2. In other words, for every $\zeta \in \mathrm{Z}_n$,

$$\sum_{\gamma\in\Lambda} \zeta_\gamma\gamma = 0 \quad\Longrightarrow\quad \forall\,\gamma \in \Lambda \quad \zeta_\gamma = 0 \pmod 2\;;$$

Λ is *independent modulo 2* if this holds for every $\zeta \in \mathrm{Z}$. If Γ contains no element of order 2, then one may always suppose that at least one coefficient ζ_γ of a nontrivial relation $\sum \zeta_\gamma\gamma = 0$ is odd, so that these two latter notions "modulo 2" coincide with the two former ones.

Let $G = \hat{\Gamma}$, so that Γ is the group of characters on G. Then the computation presented in [14, Prop. 2.5(ii)] for the case $\Gamma = \mathbb{Z}$ shows that Λ is a complex (vs. real) 1-unconditional basic sequence in $\mathrm{L}^p(G)$ with $p \in 2\mathbb{N}^*$ if and only if Λ is $p/2$-independent (vs. modulo 2). Furthermore Λ is a complex (vs. real) 1-unconditional basic sequence in $\mathrm{L}^p(G)$ with $p \in (0, \infty] \setminus 2\mathbb{N}^*$ if and only if Λ is independent (vs. modulo 2). If Γ contains no element of order 2, then a real 1-unconditional basic sequence in $\mathrm{L}^p(G)$ is also complex 1-unconditional. All these results hold also for the complete counterparts to 1-unconditional basic sequences.

Results on lacunary sets in a discrete abelian group transfer to lacunary matrices in the following way, as in [30, Th. 4.2].

Proposition 11.1. *Let Γ be a discrete abelian group and R, C be countable subsets of Γ. To every $\Lambda \subseteq R + C$ associate $I_\Lambda = \{(r, c) \in R \times C : r + c \in \Lambda\}$. Let $G = \hat{\Gamma}$.*

- (a) *If Λ is a complex 1-unconditional basic sequence in $\mathrm{L}^4(G)$, then I_Λ is a 1-unconditional basic sequence in S^4.*
- (b) *Suppose that each element of Γ admits at most one representation as the sum of an element of R with an element of C. Then every $I \subseteq R\times C$ has the form $I = I_\Lambda$ with $\Lambda = \{r + c : (r, c) \in I\}$. If Λ is a real 1-unconditional basic sequence in $\mathrm{L}^p(G)$, then I_Λ is a 1-unconditional basic sequence in S^p.*

(c) *Let $p = 2k$ be a positive even integer. Suppose that $R \cap C = \emptyset$ and $R \cup C$ is k-independent modulo 2. I_Λ is a 1-unconditional basic sequence in S^p if and only if Λ is a real 1-unconditional basic sequence in $\mathrm{L}^p(G)$.*

Proof. (a). Let $P = (c, r, c', r')$ be a closed walk in I_Λ. Then $r + c$, $r' + c'$, $r + c'$ and $r' + c$ are in Λ while $(r + c) + (r' + c') = (r + c') + (r' + c)$: if Λ is 2-independent, then $r + c \in \{r + c', r' + c\}$, so that $c = c'$ or $r = r'$ and P is not a cycle.

(b). For each $\gamma \in \Lambda$, let $q_\gamma = (r_\gamma, c_\gamma)$ be the unique element of I such that $r_\gamma + c_\gamma = \gamma$. If Λ is a real 1-unconditional basic sequence in $\mathrm{L}^p(G)$, then it is also a complete real 1-unconditional basic sequence in $\mathrm{L}^p(G)$. Let $\varphi \in \{-1, 1\}^{I_\Lambda}$, so that $\varphi_{q_\gamma} \in \{-1, 1\}$ for all $\gamma \in \Lambda$. Then, as in Eq. (6),

$$\Big\|\sum_{q\in I_\Lambda} a_q \mathrm{e}_q\Big\|_{\mathrm{S}^p_{I_\Lambda}(\mathrm{S}^p)} = \Big\|\sum_{(r,c)\in I_\Lambda} r(g)c(g)a_{rc}\mathrm{e}_{rc}\Big\|_{\mathrm{S}^p_{I_\Lambda}(\mathrm{S}^p)}$$
$$= \Big\|\sum_{\gamma\in\Lambda} a_{q_\gamma}\mathrm{e}_{q_\gamma}\gamma(g)\Big\|_{\mathrm{S}^p_{I_\Lambda}(\mathrm{S}^p)} = \Big\|\sum_{\gamma\in\Lambda} a_{q_\gamma}\mathrm{e}_{q_\gamma}\gamma\Big\|_{\mathrm{L}^p_\Lambda(G,\mathrm{S}^p(\mathrm{S}^p))}$$

so that as in Eq. (2), by complete real 1-unconditionality of Λ in $\mathrm{L}^p(G)$,

$$\Big\|\sum_{q\in I_\Lambda} \varphi_q a_q \mathrm{e}_q\Big\|_{\mathrm{S}^p_{I_\Lambda}(\mathrm{S}^p)} = \Big\|\sum_{\gamma\in\Lambda} \varphi_{q_\gamma} a_{q_\gamma}\mathrm{e}_{q_\gamma}\gamma\Big\|_{\mathrm{L}^p_\Lambda(G,\mathrm{S}^p(\mathrm{S}^p))} = \Big\|\sum_{q\in I_\Lambda} a_q \mathrm{e}_q\Big\|_{\mathrm{S}^p_{I_\Lambda}(\mathrm{S}^p)}.$$

(c). Each element of Γ admits at most one representation as the sum of an element of R with an element of C, so that (b) yields sufficiency. Suppose that Λ is not a real 1-unconditional basic sequence in $\mathrm{L}^p(G)$ and let $\zeta \in \mathrm{Z}_k$ such that $\sum_{\gamma\in\Lambda}\zeta_\gamma\gamma = 0$ and $J = \{(r, c) \in I_\Lambda : \zeta_{r+c} \neq 0 \pmod 2\}$ is nonempty; J has at most $2k$ elements. Let $P = (v_1, \ldots, v_j)$ be a path in J of maximal length. Then $\zeta_{v_{j-1}+v_j}$ is odd and $\sum\{\zeta_{v_j+v} : v_j + v \in \Lambda\}$ is even because it is the coefficient of v_j in the relation $\sum_{\gamma\in\Lambda}\zeta_\gamma\gamma = 0$ and $R \cup C$ is k-independent modulo 2. There is therefore v_{j+1} distinct from v_{j-1} such that $\zeta_{v_j+v_{j+1}}$ is odd. As j is maximal and $R \cap C = \emptyset$, $v_{j+1} = v_{j+1-2i}$ for some $2 \leqslant i \leqslant k$, so that $(v_{j+1-2i}, \ldots, v_j)$ is a cycle of length $2i$ in J: I_Λ is not a 1-unconditional basic sequence in S^p. ☐

Let R and C be any countable sets. Consider $G = \{-1, 1\}^C \times \{-1, 1\}^R$. If we denote by $\big((\epsilon_c)_{c\in C}, (\epsilon'_r)_{r\in R}\big)$ a generic point in G, then the set of Rademacher functions $\{\epsilon_c\}_{c\in C} \cup \{\epsilon'_r\}_{r\in R}$ is a real 1-unconditional basic sequence in $\mathrm{C}(G)$, so that it is independent modulo 2 in $\hat{G}$. Similarly, the set of Steinhaus functions $\{z_c\}_{c\in C} \cup \{z'_r\}_{r\in R}$ is independent in the dual of $\mathbb{T}^C \times \mathbb{T}^R$. This yields:

Corollary 11.2. *Let $I \subseteq R \times C$ and $p \in (0, \infty]$. The following are equivalent:*

- *I is a 1-unconditional basic sequence in S^p.*
- *$\{\epsilon_c\epsilon'_r : (r, c) \in I\}$ is a real 1-unconditional basic sequence in $\mathrm{L}^p(G)$.*
- *$\{z_c z'_r : (r, c) \in I\}$ is a 1-unconditional basic sequence in $\mathrm{L}^p(\mathbb{T}^C \times \mathbb{T}^R)$.*

Remark 11.3. The isomorphic counterpart is also true: I is a completely unconditional basic sequence in S^p (i.e., a complete $\sigma(p)$ set) if and only if $\{\epsilon_c\epsilon'_r : (r, c) \in I\}$ is a completely unconditional basic sequence in $\mathrm{L}^p(G)$ (a $\Lambda(p)_{\mathrm{cb}}$ *set* in $\hat{G}$, see [8] and [21, § 8.1],) if and only if $\{z_c z'_r : (r, c) \in I\}$ is a completely unconditional basic sequence in $\mathrm{L}^p(\mathbb{T}^C \times \mathbb{T}^R)$. This follows e.g. from the proof of Prop. 11.1(b) and the iterated noncommutative Khinchin inequality [21, Eq. (8.4.11)].

Harcharras [8] used Peller's discovery [19] of the link between Fourier and Hankel Schur multipliers to produce unconditional basic sequences in S^p that are unions of antidiagonals in $\mathbb{N} \times \mathbb{N}$. We have in our context the rather disappointing

Proposition 11.4. *Let $\Lambda \subseteq \mathbb{N} \subseteq \mathbb{Z}$ and $I = \{(r, c) \in \mathbb{N} \times \mathbb{N} : r + c \in \Lambda\}$.*

(a) *I is a 1-unconditional basic sequence in S^4 if and only if $\{z^\lambda\}_{\lambda\in\Lambda}$ is a 1-unconditional basic sequence in $\mathrm{L}^4(\mathbb{T})$.*

(*b*) *If* Λ *contains three elements* $\lambda<\mu<\nu$ *such that* $\lambda+\mu\geqslant\nu$, *then* I *is not a 1-unconditional basic sequence in* S^p *if* $p\in(0,\infty]\setminus\{2,4\}$.

(*c*) *If* $\Lambda=\{\lambda_k\}$ *with* $\lambda_{k+1}>2\lambda_k$ *for all* k, *then* I *is a 1-unconditional basic sequence in* S^p *for every* p.

Proof. (*a*). Sufficiency follows from Prop. 11.1(*a*) with $R=C=\mathbb{N}$. Conversely, if Λ contains a solution to $\lambda+\mu=\lambda'+\mu'$ with $\lambda<\lambda'\leqslant\mu'<\mu$, then I contains the cycle $(\operatorname{col}0,\operatorname{row}\lambda,\operatorname{col}\lambda'-\lambda,\operatorname{row}\mu')$.

(*b*). Consider the cycle $(\operatorname{col}0,\operatorname{row}\lambda,\operatorname{col}\nu-\lambda,\operatorname{row}\mu-\nu+\lambda,\operatorname{col}\nu-\mu,\operatorname{row}\mu)$.

(*c*). In fact, I is a forest. Let $P=(c_1,r_1,\dots,c_k,r_k)$ be a closed walk in I. We may suppose without loss of generality that r_1+c_2 is a maximal element of $\{r_1+c_1,r_1+c_2,\dots,r_k+c_k,r_k+c_1\}$. Then $r_1+c_1\leqslant r_1+c_2$ and $r_2+c_2\leqslant r_1+c_2$. One of these inequalities must be an equality and P is not a cycle: for otherwise $2(r_1+c_1)<r_1+c_2$ and $2(r_2+c_2)<r_1+c_2$ because $r_1+c_1,r_1+c_2,r_2+c_2\in\Lambda$, so that $2(r_1+c_1+r_2+c_2)<2(r_1+c_2)$ and $c_1+r_2<0$. ☐

Remark 11.5. Further computations yield the following result. If $\{z^\lambda\}_{\lambda\in\Lambda}$ is a 1-unconditional basic sequence in $\mathrm{L}^6(\mathbb{T})$ and if $\{\lambda<\mu<\nu\}\subseteq\Lambda\Rightarrow\lambda+\mu<\nu$, then I is a 1-unconditional basic sequence in S^6; the converse does not hold.

Let us now give an overview of the known extremal bipartite graphs without cycle of length $4,6,\dots,2k$ and their size. Look up [3, Def. I.3.1] for the definition of a Steiner system and [29, Def. 1.3.1] for the definition of a generalised polygon. An elementary example is given in the introduction with (1).

Proposition 11.6. *Let* $2\leqslant n\leqslant m$, $I\subseteq R\times C$ *with* $\#C=n$ *and* $\#R=m$, *and* $e=\#I$.

(*a*) *If* I *is a 1-unconditional basic sequence in* S^4, *then*

$$n\geqslant 1+\Big(\frac{e}{m}-1\Big)+\Big(\frac{e}{m}-1\Big)\Big(\frac{e}{n}-1\Big),$$

that is $e^2-me-mn(n-1)\leqslant 0$. *Equality holds if and only if* I *is the incidence graph of a Steiner system* $\mathrm{S}(2,e/m;n)$ *on* n *points and* m *blocks.*

(*b*) *If* I *is a 1-unconditional basic sequence in* S^6, *then*

$$n\geqslant 1+\Big(\frac{e}{m}-1\Big)+\Big(\frac{e}{m}-1\Big)\Big(\frac{e}{n}-1\Big)+\Big(\frac{e}{m}-1\Big)^2\Big(\frac{e}{n}-1\Big),$$

that is $e^3-(m+n)e^2+2mne-m^2n^2\leqslant 0$. *Equality holds if and only if* I *is the incidence graph of the* quadrangle *(the cycle of length 8) or of a generalised quadrangle with* n *points and* m *lines.*

(*c*) *If* I *is a 1-unconditional basic sequence in* S^{2k} *with* $k\geqslant 1$ *an integer, then*

$$n\geqslant\sum_{i=0}^{k}\Big(\frac{e}{m}-1\Big)^{\lceil\frac{i}{2}\rceil}\Big(\frac{e}{n}-1\Big)^{\lfloor\frac{i}{2}\rfloor}. \tag{18}$$

Equality holds if I *is the incidence graph of the* $(k+1)$-gon *(the cycle of length* $2k+2$*) or of a generalised* $(k+1)$*-gon with* n *points and* m *lines.*

Proof. By Theorem 9.1(*b*) $\Rightarrow$ (*g*), I is a 1-unconditional basic sequence in S^{2k}, with $k\geqslant 1$ an integer, if and only if I is a graph of girth $2k+2$ in the sense of [10]. Therefore (*a*) and (*b*) are shown in [16, Prop. 4, Th. 8, Rem. 10]. Inequality (18) is [10, Eq. (1)] and the sufficient condition for equality follows from [29, Lemma 1.5.4]. ☐

Consult [3, Tables A1.1, A5.1] for examples of Steiner systems and [29, Table 2.1] for examples of generalised polygons. In both cases, the corresponding incidence graph is biregular: every vertex in R has same degree $s+1$ and every vertex in C has same degree $t+1$. Arbitrarily large generalised $(k+1)$-gons exist only if $2k \in \{4, 6, 10, 14\}$ [29, Lemma 1.7.1]; for $2k \in \{6, 10, 14\}$, it follows from [29, Lemma 1.5.4] that

$$n = (s+1)\frac{(st)^{(k+1)/2} - 1}{st - 1}, \quad m = (t+1)\frac{(st)^{(k+1)/2} - 1}{st - 1}.$$

Remark 11.7. Let $I \subseteq R \times C$ with $\# C = \# R = n$. Inequality (18) shows that if I is a 1-unconditional basic sequence in S^{2k}, then $\# I \leqslant n^{1+1/k} + (s-1)n/s$. If $p \notin \{4, 6, 10\}$, the existence of 1-unconditional basic sequences in S^{2k} such that $\# I \succcurlyeq n^{1+1/k}$ is in fact an important open problem in graph theory: extremal graphs cannot correspond to generalised polygons and necessarily have less structure.

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