Lattice-ordered groups generated by an ordered group and regular systems of ideals

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Abstract

Unbounded entailment relations, introduced by Paul Lorenzen (1951), are a slight variant of a notion which plays a fundamental rôle in logic (see Scott 1974) and in algebra (see Lombardi and Quitté 2015). We call *systems of ideals* their single-conclusion counterpart. If they preserve the order of a commutative ordered monoid G and are equivariant w.r.t. its law, we call them *equivariant systems of ideals for* G: they describe all morphisms from G to meet-semilattice-ordered monoids generated by (the image of) G. Taking an article by Lorenzen (1953) as a starting point, we also describe all morphisms from a commutative ordered group G to lattice-ordered groups generated by G through unbounded entailment relations that preserve its order, are equivariant, and satisfy a regularity property invented by Lorenzen (1950); we call them *regular entailment relations*. In particular, the free lattice-ordered group generated by G is described through the finest regular entailment relation for G, and we provide an explicit description for it; it is order-reflecting if and only if the morphism is injective, so that the Lorenzen-Clifford-Dieudonné theorem fits into our framework. Lorenzen's research in algebra starts as an inquiry into the system of Dedekind ideals for the divisibility group of an integral domain R, and specifically into Wolfgang Krull's "Fundamentalsatz" that R may be represented as an intersection of valuation rings if and only if R is integrally closed: his constructive substitute for this representation is the *regularisation* of the system of Dedekind ideals, i.e. the lattice-ordered group generated by it when one proceeds as if its elements are comparable.

Keywords: Ordered monoid; system of ideals; equivariant system of ideals; morphism from an ordered monoid to a meet-semilattice-ordered monoid; ordered group; unbounded entailment relation; regular entailment relation; regular system of ideals; morphism from an ordered group to a lattice-ordered group; Lorenzen-Clifford-Dieudonné theorem; Fundamentalsatz for integral domains; Grothendieck ℓ-group; cancellativity.

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Introduction

In this article, all monoids and groups are supposed to be commutative, and orders are tacitly partial.

The idea of generating a semilattice and a distributive lattice by a logic-free and set-theory-free formal system, called respectively "system of ideals" and "unbounded entailment relation" in this article, dates back to Lorenzen (1951, §2) and is motivated there as capturing how ideal theory provides formal gcds and lcms, i.e. formal meets and joins, for elements of an integral domain. Multiplicative ideal theory gives rise to "equivariant" counterparts to these formal systems.

After studying Lorenzen 1953, we have isolated a new axiom that we call "regularity". In this article, our aim is to give a precise account of Lorenzen's results through "regular" entailment relations. Our main theorem, Theorem 3.4, shows that by means of this axiom, an equivariant entailment relation generates an ℓ-group.

Lorenzen (1950) introduces a construction that embodies the right to compute in an equivariant system of ideals as if it was linearly ordered; we formulate it as "regularisation" in Definition 4.1. Theorem 4.9 states that this gives rise to an ℓ-group, the "Lorenzen group" associated with the equivariant system of ideals. The literature on ℓ-groups seems not to have taken notice of these results.

In Lorenzen's work, this approach supersedes another, based on a procedure for forcing the cancellativity of an equivariant system of ideals, ideated by Prüfer (1932) and generalised to the setting of ordered monoids in Lorenzen's Ph.D. thesis (1939). In Section 5, we also provide an account for that.

The key step in our presentation is to show that a regular entailment relation defines by restriction a cancellative equivariant system of ideals; in both approaches, the sought-after ℓ-group is constructed as the Grothendieck ℓ-group of a cancellative monoid of ideals (Theorem 3.3).

The Fundamentalsatz for integral domains

The motivating example for Lorenzen's analysis of the concept of ideal is Wolfgang Krull's "Fundamentalsatz", which states that an integral domain is an intersection of valuation rings if and only if it is integrally closed. As Krull (1935, page 111) himself emphasises, "Its main defect, that one must not overlook, lies in that it is a purely existential theorem", resulting from a well-ordering argument. In a letter to Heinrich Scholz[1] dated 18th April 1953, Krull writes: "At working with the uncountable, in particular with the well-ordering theorem, I

[1] Scholz-Archiv, Universitäts- und Landesbibliothek Münster, http://www.uni-muenster.de/IVV5WS/ScholzWiki/doku.php?id=scans:blogs:ko-05-0647, accessed 21st September 2016, published in Neuwirth 2018, § V.

always had the feeling that one uses fictions there that need to be replaced some day by more reasonable concepts. But I was not getting upset over it, because I was convinced that at a careful application of the common 'fictions' nothing false comes out, and because I was firmly counting on the man who would some day put all in order. Lorenzen has now found according to my conviction the right way [...]".

Lorenzen shows that the well-ordering argument in Krull's proof may be replaced by the performance of computations as if the monoid of Dedekind ideals was linearly ordered (see Comment 4.3), that integral closedness guarantees that such computations do not add new relations of divisibility to the integral domain, and that this performance, formulated as regularisation, generates a lattice-ordered group. Theorem 4.9 is in fact an abstract version of the following theorem (see Theorem 4.22).

Theorem. *The divisibility group of an integral domain embeds into an ℓ-group that contains the system of Dedekind ideals if and only if the integral domain is integrally closed.*

Outline of the article

Section 1 deals with meet-semilattices as generated by systems of ideals, discusses equivariant systems of ideals for an ordered monoid and the meet-monoid they generate (Theorem 1.10). Section 2 deals with distributive lattices as generated by unbounded entailment relations and discusses regular entailment relations. Section 3 introduces the Grothendieck ℓ-group of a meet-monoid as a means for proving Theorem 3.4. Section 4 investigates regularisation: applied to the finest equivariant system of ideals, it leads to the finest regular entailment relation and to the ℓ-group freely generated by an ordered group (Sections 1.5, 4.3, and 4.4); applied to the system of Dedekind ideals for the divisibility group of an integral domain, it captures the concept of integral dependence and leads to Lorenzen's theory of divisibility (Sections 1.6, 1.8, 4.5, and 4.6). Section 5 reminds us of an important theorem by Prüfer which has led to the historically first approach to the Lorenzen group associated with an equivariant system of ideals.

This article is written in Errett Bishop's style of constructive mathematics (Bishop 1967; Bridges and Richman 1987; Mines, Richman, and Ruitenburg 1988; Lombardi and Quitté 2015): all theorems can be viewed as providing an algorithm that constructs the conclusion from the hypotheses.

1 Meet-semilattice-ordered monoids and equivariant systems of ideals

1.1 Meet-semilattices and systems of ideals

Let us define a *meet-semilattice* as a purely equational algebraic structure with just one law $\wedge$ that is idempotent, commutative, and associative. We are leaving out the axiom of meet-semilattices providing a greatest element because it does not suit monoid theory: meets are only supposed to exist for *nonempty* finitely enumerated sets.

Let $\mathrm{P}_{\mathrm{fe}}^{*}(G)$ be the set of nonempty finitely enumerated subsets of an arbitrary set G. For a meet-semilattice S, let us denote by $A \rhd b$ the relation defined between the sets $\mathrm{P}_{\mathrm{fe}}^{*}(S)$ and S in the following way (see Lorenzen 1951, Satz 1):[2]

$$A \rhd b \quad \overset{\text{def}}{\iff} \quad \bigwedge A \leqslant_S b \quad \overset{\text{def}}{\iff} \quad b \wedge \bigwedge A =_S \bigwedge A.$$

This relation is reflexive, monotone (a property also called “thinning” and “weakening”), and transitive (a property also called “cut” because it “cuts” c) in the following sense, expressed without the law $\wedge$:

S0	$a \rhd a$	(reflexivity);
S1	if $A \rhd b$, then $A, A' \rhd b$	(monotonicity);
S2	if $A \rhd c$ and $A, c \rhd b$, then $A \rhd b$	(transitivity).

Note that in the context of relations, we shall make the following abuses of notation for finitely enumerated sets: we write a for the singleton consisting of a, and A, A' for the union of the sets A and A'. These three properties correspond respectively to the “tautologic assertions”, the “immediate deductions”, and to an elementary form of the “syllogisms” of the systems of axioms introduced by Paul Hertz (1923, § 1), so that the following definition may be attributed to him;[3] see also Gerhard Gentzen (1933, § 2), who has coined the vocables “thinning” and “cut”. This definition is introduced as description of a meet-semilattice (see Theorem 1.4) in Lorenzen (1951, § 2).

Definition 1.1. A *system of ideals* for a set G is a reflexive, monotone, and transitive relation $\rhd$ between $\mathrm{P}_{\mathrm{fe}}^{*}(G)$ and G.

[2] The sign $\rhd$ has been introduced with this meaning and with the terminology “single-conclusion entailment relation” by Rinaldi, Schuster, and Wessel (2017).

[3] Jean-Yves Béziau (2006, § 6) discusses the relationship of systems of ideals with Alfred Tarski’s consequence operation, which may be compared to the relationship of our Definition 1.7 of an equivariant system of ideals with the set-theoretic star-operation: see Item (2) of Remarks 1.8.

Comment 1.2. By our terminology, we emphasise the feedback of algebra to logic while being faithful to Lorenzen. In a letter to Krull[4] dated 13 March 1944, he writes: "the insight that a system of ideals is intrinsically nothing more than a supersemilattice, and a valuation nothing more than a linear order [see Section 1.6], strikes me as the most essential result of my effort". ◇

Remark 1.3. If instead of nonempty subsets, we had considered nonempty multisets, we would have had to add a contraction rule, and if we had considered nonempty lists, we would have had to add also a permutation rule. ◇

Note the following banal generalisation of cut, using monotonicity: if $A \rhd c$ and $A', c \rhd b$, with A' possibly empty, then $A, A' \rhd b$.

1.2 Fundamental theorem of systems of ideals

A fundamental theorem holds for a system of ideals for a given set G: it states that the relation generates a meet-semilattice S whose order reflects the relation. This is the single-conclusion analogue of the better known Theorem 2.2.

Theorem 1.4 (fundamental theorem of systems of ideals, see Lorenzen 1951, Satz 3)**.**[5] *Let G be a set and $\rhd$ a system of ideals for G. Let us consider the meet-semilattice S defined by generators and relations in the following way: the generators are the elements of G and the relations are the*

$$\bigwedge A \leqslant_S b \text{ whenever } A \rhd b.$$

Then, for all (A, b) in $\mathrm{P}^*_{\mathrm{fe}}(G) \times G$, we have the reflection of entailment

$$\text{if } \bigwedge A \leqslant_S b, \text{ then } A \rhd b.$$

In fact, S can be defined as the ordered set obtained by descending to the quotient of $(\mathrm{P}^*_{\mathrm{fe}}(G), \leqslant_\rhd)$ by $=_\rhd$, where $\leqslant_\rhd$ is the meet-semilattice preorder defined by

$$(*) \qquad A \leqslant_\rhd B \overset{\text{def}}{\iff} A \rhd b \text{ for all } b \in B.$$

Proof. Let $A, B \in \mathrm{P}^*_{\mathrm{fe}}(G)$: one has $\bigwedge A \leqslant_S \bigwedge B$ if and only if $\bigwedge A \leqslant_S b$ for all $b \in B$, i.e. $A \leqslant_\rhd B$. The meet-semilattice S may therefore be generated in two steps.

[4] Philosophisches Archiv, Universität Konstanz, PL 1-1-131, published in Neuwirth 2018, § M.

[5] Our statement is the natural counterpart to Lorenzen's when using basic notions of universal algebra, and follows readily from his sketch of proof.

1. Let us check that $\leqslant_{\triangleright}$ is a preorder on $\mathrm{P}^*_{\mathrm{fe}}(G)$ that is compatible with the idempotent, commutative, and associative law of set union. Reflexivity of $\leqslant_{\triangleright}$ follows from Properties *S0* and *S1*. Transitivity of $\leqslant_{\triangleright}$ follows from Property *S1* and a repeated application of Property *S2*: if $A \triangleright c$ for every $c \in C$ and $C \triangleright b$, then one may cut successively the $c \in C$ and obtain $A \triangleright b$. Compatibility means that if $A \leqslant_{\triangleright} B$ and $A' \leqslant_{\triangleright} B'$, then $A, A' \leqslant_{\triangleright} B, B'$: this follows from Property *S1*.

2. We may therefore define S as the quotient of $(\mathrm{P}^*_{\mathrm{fe}}(G), \leqslant_{\triangleright})$ by $=_{\triangleright}$, with law $\curlywedge_S$ obtained by descending the law of set union to the quotient. □

Note that the preorder $a \triangleright b$ on G makes its quotient a subobject of S in the category of ordered sets.

Remark 1.5. The relation $a \triangleright b$ is a priori just a preorder relation for G, not an order relation. Let us denote the element a viewed in the ordered set $\overline{G}$ associated to this preorder by $\overline{a}$, and let $\overline{A} = \{\, \overline{a} \mid a \in A \,\}$ for a subset A of G. In Theorem 1.4, we construct a meet-semilattice S endowed with an order $\leqslant_S$ that, loosely said, coincides with $\triangleright$ on $\mathrm{P}^*_{\mathrm{fe}}(G) \times G$; for the sake of rigour, we should have written above $\bigwedge \overline{A} \leqslant_S \overline{b}$ rather than $\bigwedge A \leqslant_S b$ in order to deal with the fact that the equality of S is coarser than the equality of G. In particular, it is $\overline{G}$ rather than G which can be identified with a subset of S. ◇

Definition 1.6. The system of ideals $\triangleright_2$ is *coarser* than the system of ideals $\triangleright_1$ if $A \triangleright_1 y$ implies $A \triangleright_2 y$. One says also that $\triangleright_1$ is *finer* than $\triangleright_2$.

This terminology has the following explanation: to say that the relation $\triangleright_2$ is coarser than the relation $\triangleright_1$ is to say this for the associated preorders, i.e. that $A \leqslant_{\triangleright_1} B$ implies $A \leqslant_{\triangleright_2} B$, and this corresponds to the usual meaning of "coarser than" for preorders, since $A =_{\triangleright_1} B$ implies accordingly $A =_{\triangleright_2} B$, i.e. the equivalence relation $=_{\triangleright_2}$ is coarser than $=_{\triangleright_1}$.

1.3 Equivariant systems of ideals

Now suppose that $(G, \leqslant_G)$ is an ordered monoid,[6] $(M, \leqslant_M)$ a meet-semilattice-ordered monoid,[7] a *meet-monoid* for short, and $\varphi\colon G \to M$ a morphism of ordered monoids. The relation

$$a_1, \ldots, a_k \triangleright b \quad \overset{\text{def}}{\iff} \quad \varphi(a_1) \curlywedge \cdots \curlywedge \varphi(a_k) \leqslant_M \varphi(b)$$

[6] I.e. a monoid $(G, +, 0)$ endowed with a (partial) order relation $\leqslant_G$ compatible with addition: $x \leqslant_G y \implies x + z \leqslant_G y + z$. We shall systematically omit the epithet "partial".

[7] I.e. a monoid endowed with a meet-semilattice law $\curlywedge$ inducing $\leqslant_M$ and compatible with addition: the equality $x + (y \curlywedge z) = (x + y) \curlywedge (x + z)$ holds.

defines a system of ideals for G that satisfies furthermore the following properties:

S3 if $a \leqslant_G b$, then $a \rhd b$ (preservation of order);

S4 if $A \rhd b$, then $x + A \rhd x + b \quad (x \in G)$ (equivariance).

Definition 1.7. An *equivariant system of ideals* for an ordered monoid G is a system of ideals $\rhd$ for G satisfying Properties *S3* and *S4*.

We propose to introduce equivariant systems of ideals in a purely logical form, i.e. as relations that require only a naive set theory for finitely enumerated sets: this definition has been extracted from Lorenzen 1939, Definition 1 (compare Jaffard 1960, I, § 3, 1). One may also give them the form of predicates on $\mathrm{P}^*_{\mathrm{fe}}(G)$: see Coquand, Lombardi, and Neuwirth 2018, § 3. The traditional form of a meet-monoid for equivariant systems of ideals may be recovered by Theorem 1.10 below.

Remarks 1.8. 1. We find that it is more natural to state a direct implication rather than an equivalence in Property *S3*; we deviate here from Lorenzen and Paul Jaffard 1960, page 16. The reverse implication expresses the supplementary property that the equivariant system of ideals is order-reflecting.

2. Lorenzen (1939), following at first Richard Dedekind (1897) and Heinz Prüfer (1932, § 2) in subordinating algebra to set theory, is describing a (finite) "r-system" of ideals through a set-theoretic map

$$\mathrm{P}^*_{\mathrm{fe}}(G) \longrightarrow \mathrm{P}(G), \quad A \longmapsto \{\, x \in G \mid A \rhd x \,\} \overset{\mathrm{def}}{=} A_r$$

(here $\mathrm{P}(G)$ stands for the set of all subsets of G, and r is just a variable name for distinguishing different systems) that satisfies the following properties:

I1 $A_r \supseteq A$;

I2 $A_r \supseteq B \implies A_r \supseteq B_r$;

I3 $\{a\}_r = \{\, x \in G \mid a \leqslant_G x \,\}$ (preservation and reflection of order);

I4 $(x + A)_r = x + A_r$ (equivariance).

This map has been called $'$-operation by Krull (1935, Nr. 43) and is called star-operation today. Let us note that the containment $A_r \supseteq B_r$ corresponds to the inequality $A \leqslant_{\rhd} B$ for the preorder associated with the system of ideals $\rhd$ by the definition $(*)$ above. As previously indicated, in contradistinction to Lorenzen and Jaffard, we find it more natural to relax the equality in Property *I3* to a containment: if we do so, the reader can prove that the definition of star-operation is equivalent to Definition 1.7; Properties *I1* and *I2* correspond to the definition of a system of ideals,[8] and Properties *I3* (relaxed) and *I4* correspond to Properties *S3* and *S4* in Definition 1.7; compare Lorenzen 1950, pages 504–505.

[8] They can also be read as a finite version of Tarski's consequence operation (see Footnote 3).

3. In the set-theoretic framework of the previous item, the r_2-system is coarser than the r_1-system exactly if $A_{r_2} \supseteq A_{r_1}$ holds for all $A \in \mathrm{P}^*_{\mathrm{fe}}(G)$ (see Lorenzen 1950, page 509, and Jaffard 1960, I, § 3, Proposition 2). ◇

Comment 1.9. Lorenzen unveils the lattice theory hiding behind multiplicative ideal theory step by step, the decisive one being dated back by him to 1940. In a footnote to his definition, Lorenzen (1939, page 536) writes: "If one understood hence by a system of ideals every lattice that contains the principal ideals and satisfies Property [*I4*], then this definition would be only unessentially more comprehensive" (it seems that Lorenzen is lacking the concept of *semi*lattice at this stage of his research). Lorenzen (1950, page 486) emphasises the transparency of this presentation as compared to the set-theoretic ideals: "But if one removes this set-theoretic clothing, then the concept of ideal may be defined quite simply: a system of ideals of a preordered set is nothing other than an embedding into a semilattice." ◇

1.4 The meet-monoid generated by an equivariant system of ideals

The effectiveness of Definition 1.7 is shown by the following straightforward theorem, which boils down to acknowledging that the meet operation of set union on the preordered meet-semilattice $(\mathrm{P}^*_{\mathrm{fe}}(G), \leqslant_\triangleright)$, described in the proof of Theorem 1.4, is compatible with the monoid operation of set addition $A + B$.

Theorem 1.10. *Let $\triangleright$ be an equivariant system of ideals for an ordered monoid G. Let S be the meet-semilattice generated by the system of ideals $\triangleright$. Then there is a (unique) monoid law on S which is compatible with its semilattice structure and such that the natural morphism (of ordered sets) $G \to S$ is a monoid morphism. The resulting meet-monoid S is called the* monoid of ideals *associated with $\triangleright$.*

Proof. We define $A + B = \{\, a + b \mid a \in A,\, b \in B \,\}$ in $\mathrm{P}^*_{\mathrm{fe}}(G)$. We have to check that this law descends to the quotient S. It suffices to show that $B \leqslant_\triangleright C$ implies $A + B \leqslant_\triangleright A + C$. In fact, $B \leqslant_\triangleright C$ implies $x + B \leqslant_\triangleright x + C$ by equivariance, and $A + B \leqslant_\triangleright x + C$ for every $x \in A$ by monotonicity. Finally, let us verify the compatibility of $\curlywedge_S$ with addition: we note that already in $\mathrm{P}^*_{\mathrm{fe}}(G)$, set union is compatible with set addition, i.e. $A + (B, C) = A + B, A + C$. □

1.5 The finest equivariant system of ideals for an ordered monoid

The finest equivariant system of ideals admits the following description.

Proposition 1.11 (Lorenzen 1950, Satz 14). *Let $(G, \leqslant_G)$ be an ordered monoid. The finest equivariant system of ideals for G is defined by*

$$A \rhd_{\mathrm{s}} b \overset{\text{def}}{\iff} a \leqslant_G b \text{ for some } a \in A.$$

Note that $\rhd_{\mathrm{s}}$ is order-reflecting: $a \rhd_{\mathrm{s}} b \iff a \leqslant_G b$. The associated monoid of ideals is the meet-monoid freely generated by $(G, \leqslant_G)$ (in the sense of the left adjoint functor of the forgetful functor).

Proof. Left to the reader. □

1.6 The system of Dedekind ideals

Lorenzen's goal is to unveil the constructive content of Krull's Fundamentalsatz, i.e. to express it without reference to valuations. In order to do so, consider an integral domain R and its divisibility group $G = K^\times/R^\times$ ordered by divisibility, where K is the field of fractions of R. A valuation is a linear preorder $\preccurlyeq$ on G such that $1 \preccurlyeq x$ for $x \in R^*$ and

$$(\dagger) \qquad \min(a_1, a_2) \preccurlyeq a_1 + a_2 \quad \text{if } a_1 + a_2 \neq 0.$$

Property ($\dagger$) implies that $\min(a_1, \dots, a_k) \preccurlyeq x_1a_1+\cdots+x_ka_k$ if $x_1a_1+\cdots+x_ka_k \neq 0$, where $x_1, \dots, x_k \in R$. Let us write $\langle A \rangle_R$ for these linear combinations, where $A = \{a_1, \dots, a_k\}$: we have $\langle A \rangle_R \ni b \implies \min A \preccurlyeq b$. This motivates the following definition and observation.

Definition 1.12. Let R be an integral domain, K its field of fractions, and $G = K^\times/R^\times$ its divisibility group ordered by divisibility. The *system of Dedekind ideals* for G is defined by

$$A \rhd_{\mathrm{d}} b \overset{\text{def}}{\iff} \langle A \rangle_R \ni b,$$

where $\langle A \rangle_R$ is the fractional ideal generated by A over R in K: if $a_1, \dots, a_k$ are the elements of A, then $\langle A \rangle_R = R\,a_1 + \cdots + R\,a_k$.

Proposition 1.13. *The system of Dedekind ideals for the divisibility group G of an integral domain is an equivariant system of ideals for G.*

The above argument shows that a valuation may be defined as a linear preorder that is coarser than the system of Dedekind ideals, so that it gives rise to a homomorphism from the preordered meet-monoid of Dedekind ideals into a linearly preordered group. In a letter to Krull dated 6 June 1944,[9] Lorenzen writes:

[9]Philosophisches Archiv, Universität Konstanz, PL 1-1-133, published in Neuwirth 2018, § R.

“If e.g. I replace the concept of valuation by ‘homomorphism of a semilattice into a linearly preordered set’, then I see therein a conceptual simplification and not a complication. For the introduction of the concept of valuation (e.g. the at first arbitrary triangular inequality [(†)]) is only justified by the subsequent success, whereas the concept of homomorphism bears its justification in itself. I would say that the homomorphism into a linear preorder is the ‘pure concept’ that underlies the concept of valuation.”

1.7 Forcing the positivity of an element

Definition 1.14. Let $\triangleright$ be an equivariant system of ideals for an ordered monoid G and $x \in G$. The system $\triangleright_x$ is the equivariant system of ideals coarser than $\triangleright$ obtained by forcing the property $0 \triangleright x$.

The precise description of $\triangleright_x$ given in the proposition below is the counterpart for a system of ideals to the submonoid generated by adding an element x to a submonoid in an ordered monoid (the “r-extension” $\dot{\mathfrak{g}}(x)_r$ of the submonoid $\dot{\mathfrak{g}}$, Lorenzen 1950, page 516).

Proposition 1.15. *Let $\triangleright$ be an equivariant system of ideals for an ordered monoid G and $x \in G$. We have the equivalence*

$$A \triangleright_x b \quad \Longleftrightarrow \quad \text{there is } p \geqslant 0 \text{ such that } A, A+x, \ldots, A+px \triangleright b.$$

Unlike the case of regular entailment relations (see Lemma 2.9), it is not possible to omit $A+x, \ldots, A+(p-1)x$ and to keep only $A, A+px$ to the left of $\triangleright$: this can be seen in the equivalence (‡) on page 11, which is the application of Proposition 1.15 to Dedekind ideals; see Coquand, Lombardi, and Neuwirth 2018, Examples 8.1 and 8.2.

Proof. Let us denote by $A \triangleright' b$ the right-hand side in the equivalence above. In any meet-monoid, $0 \leqslant x$ implies $\bigwedge(A, A+x, \ldots, A+px) = \bigwedge A$, so that $A \triangleright' b$ implies $A \mathrel{\widetilde{\triangleright}} b$ for any equivariant system of ideals $\widetilde{\triangleright}$ coarser than $\triangleright$ and satisfying $0 \mathrel{\widetilde{\triangleright}} x$.

It remains to prove that $A \triangleright' b$ defines an equivariant system of ideals for G (clearly $0 \triangleright' x$ and $\triangleright'$ is coarser than $\triangleright$). Reflexivity, preservation of order, equivariance, and monotonicity are straightforward. It remains to prove transitivity. Assume that $A \triangleright' c$ and $A, c \triangleright' b$. We have to show that $A \triangleright' b$. E.g. we have

$$(\flat) \qquad A, A+x, A+2x, A+3x \triangleright c,$$
$$(\natural) \qquad A, A+x, A+2x, c, c+x, c+2x \triangleright b.$$

($\flat$) gives by equivariance $A+2x, A+3x, A+4x, A+5x \rhd c+2x$. By a cut with ($\natural$) we may cancel out $c+2x$ and get

$$(\sharp) \qquad A, A+x, A+2x, A+3x, A+4x, A+5x, c, c+x \rhd b.$$

The same argument allows us to cancel successively $c+x$ and c out of ($\sharp$). $\square$

1.8 Forcing an element to be positive w.r.t. the system of Dedekind ideals

Proposition 1.16. *Let R be an integral domain, K its field of fractions and $G = K^\times/R^\times$ its divisibility group. Let $x \in G$. Then the system $(\rhd_\mathrm{d})_x$ obtained from the system of Dedekind ideals $\rhd_\mathrm{d}$ for G by forcing $1 \rhd_\mathrm{d} x$ is the system of Dedekind ideals for the divisibility group of the extension $R[x]$ of R by a representative of x in K.*

Proof. Forcing $1 \rhd_\mathrm{d} x$ for an $x \in G$ amounts to replacing R by $R[x]$ since Proposition 1.15 tells that the resulting equivariant system of ideals satisfies

$$(\ddagger) \qquad A \; (\rhd_\mathrm{d})_x \; b \iff \begin{array}{l}\text{there is } p \geqslant 0 \text{ such that} \\ A, Ax, \dots, Ax^p \rhd_\mathrm{d} b \text{ holds,}\end{array}$$

which means that $\langle A \rangle_{R[x]} \ni b$. $\square$

This is explained in Lorenzen 1953, § 3, and has suggested Proposition 1.15 to us.

2 Lattice-ordered groups and regular entailment relations

2.1 Distributive lattices and entailment relations

Let us define a *distributive lattice* as a purely equational algebraic structure with two laws $\wedge$ and $\vee$ satisfying the axioms of distributive lattices; we are leaving out the two axioms providing a greatest and a least element.

For a distributive lattice L, let us denote by $A \vdash B$ the relation defined on the set $\mathrm{P}^*_{\mathrm{fe}}(L)$ in the following way (see Lorenzen 1951, Satz 5):

$$A \vdash B \overset{\text{def}}{\iff} \bigwedge A \leqslant_L \bigvee B.$$

This relation is reflexive, monotone, and transitive in the following sense, expressed without the laws $\wedge$ and $\vee$:

R0	$a \vdash a$	(reflexivity);
R1	if $A \vdash B$, then $A, A' \vdash B, B'$	(monotonicity);
R2	if $A \vdash B, c$ and $A, c \vdash B$, then $A \vdash B$	(transitivity);

we insist on the fact that A and B must be nonempty.

Note the following banal generalisation of cut, using monotonicity: if $A \vdash B', x$ and $A', x \vdash B$, with A' and B' possibly empty, then $A, A' \vdash B, B'$.

The following definition is a slight variant of a notion whose name has been coined by Dana Scott (1974, page 417). It is introduced as description of a distributive lattice (see Theorem 2.2) in Lorenzen 1951, § 2.

Definition 2.1. Let G be an arbitrary set.

1. A binary relation $\vdash$ on $\mathrm{P}^*_{\mathrm{fe}}(G)$ which is reflexive, monotone, and transitive is called an *unbounded entailment relation*.

2. The unbounded entailment relation $\vdash_2$ is *coarser* than the unbounded entailment relation $\vdash_1$ if $A \vdash_1 B$ implies $A \vdash_2 B$. One says also that $\vdash_1$ is *finer* than $\vdash_2$.

Remark 1.3 applies again verbatim for Definition 2.1.

2.2 Fundamental theorem of unbounded entailment relations

The counterpart to Theorem 1.4 for unbounded entailment relations is Theorem 2.2, a slight variant of the fundamental theorem of entailment relations (Cederquist and Coquand 2000, Theorem 1, obtained independently), which may in fact be traced back to Lorenzen (1951, Satz 7). It states that an unbounded entailment relation for a set G generates a distributive lattice L whose order reflects the relation. The proof is the same as in Cederquist and Coquand 2000 or in Lombardi and Quitté 2015, Theorem XI-5.3.

Theorem 2.2 (fundamental theorem of unbounded entailment relations, see Lorenzen 1951, Satz 7).[10] *Let G be a set and $\vdash$ an unbounded entailment relation on $\mathrm{P}^*_{\mathrm{fe}}(G)$. Let us consider the distributive lattice L defined by generators and*

[10]Footnote 5 applies verbatim. Lorenzen's Satz 7 yields directly that if for every distributive lattice L and every $f\colon G \to L$ with $X \vdash Y \implies \bigwedge f(X) \leqslant_L \bigvee f(Y)$ one has $\bigwedge f(A) \leqslant_L \bigvee f(B)$, then $A \vdash B$. This may be considered as a result of completeness for the semantics of distributive lattices.

relations in the following way: the generators are the elements of G and the relations are the

$$\bigwedge A \leqslant_L \bigvee B \textit{ whenever } A \vdash B.$$

Then, for all A, B in $\mathrm{P}^*_{\mathrm{fe}}(G)$, we have the reflection of entailment

$$\textit{if } \bigwedge A \leqslant_L \bigvee B, \textit{ then } A \vdash B.$$

Remark 1.5 applies again mutatis mutandis.

2.3 Regular entailment relations

Let $(G, \leqslant_G)$ be an ordered monoid, $(H, \leqslant_H)$ a distributive lattice-ordered monoid,[11] and $\varphi\colon G \to H$ a morphism of ordered monoids. The laws $\wedge$ and $\vee$ on H provide a distributive lattice structure, and the relation

$$a_1, \ldots, a_k \vdash b_1, \ldots, b_\ell \quad \overset{\text{def}}{\iff} \quad \varphi(a_1) \wedge \cdots \wedge \varphi(a_k) \leqslant_H \varphi(b_1) \vee \cdots \vee \varphi(b_\ell)$$

defines an unbounded entailment relation for G that satisfies furthermore the following straightforward properties:

R3	if $a \leqslant_G b$, then $a \vdash b$	(preservation of order);
R4	if $A \vdash B$, then $x + A \vdash x + B \quad (x \in G)$	(equivariance).

Now suppose that $(H, \leqslant_H)$ is a lattice-ordered group,[12] an *ℓ-group* for short. Then the following further property holds:

R5	$x + a, y + b \vdash y + a, x + b$	(regularity).

This follows from the observation that if x', a', y', b' are elements of H, then the difference of right-hand side and left-hand side of

$$(\S) \qquad (x' + a') \wedge (y' + b') \leqslant_H (y' + a') \vee (x' + b')$$

is

$$\begin{aligned} \big((y' + a') \vee (x' + b')\big) &+ \big((-x' - a') \vee (-y' - b')\big) \\ &=_H (y' - x') \vee (a' - b') \vee (b' - a') \vee (x' - y') \\ &=_H |y' - x'| \vee |b' - a'|. \end{aligned}$$

We assemble these observations into the following new purely logical definitions (compare Lorenzen 1953, § 1), given for ordered monoids even though we study them only in the case of ordered groups.

[11] I.e. a meet-monoid endowed with a join-semilattice law $\vee$ inducing $\leqslant_H$ that is distributive over $\wedge$ and compatible with addition.

[12] An *ordered group* is a group that is an ordered monoid. If it is meet-semilattice-ordered, then it turns out that it is a *lattice-ordered group* with join defined by $a \vee b = -(-a \wedge -b)$.

Definition 2.3. Let G be an ordered monoid.

1. An *equivariant* entailment relation for G is an unbounded entailment relation $\vdash$ for G satisfying Properties *R3* and *R4*.

2. A *regular entailment relation* for G is an equivariant entailment relation for G satisfying Property *R5*.

3. An equivariant system of ideals for G is *regular* if it is the restriction of a regular entailment relation to $\mathrm{P}_{\mathrm{fe}}^*(G) \times G$.

We prefer the terminology in Item (1) to Lorenzen's vocable "upper system of ideals"; note that a fundamental theorem is also available for this concept, but we shall not need it. A key fact to be established is that a regular entailment relation is determined by its restriction to $\mathrm{P}_{\mathrm{fe}}^*(G) \times G$ (see Item (2) of Corollary 2.12). This allows one to give it the form of a predicate on $\mathrm{P}_{\mathrm{fe}}^*(G)$: see Coquand, Lombardi, and Neuwirth 2018, § 2.

Comment 2.4. Lorenzen discovers the property of regularity in his analysis of the case of noncommutative groups: he isolates Inequality (§), which is trivially verified in a commutative ℓ-group, but not in a noncommutative one. Lorenzen (1950, Satz 13) proves by a well-ordering argument that a (noncommutative) preordered ℓ-group satisfying Inequality (§) is a subdirect product of linearly preordered groups. In the commutative setting, this corresponds to the theorem (in classical mathematics) that any commutative preordered ℓ-group is a subdirect product of linearly preordered commutative groups. $\diamond$

2.4 Regularity as the right to assume elements linearly ordered

Let us now undertake an investigation of regular entailment relations as defined in Definition 2.3.

Lemma 2.5. *Let $\vdash$ be an unbounded entailment relation for an ordered group G. Property R5 may be restated as follows:*

$$\text{if } x_1 + x_2 =_G y_1 + y_2, \text{ then } x_1, x_2 \vdash y_1, y_2.$$

Proof. By Property *R5*, $y_1 + (y_2 - x_2), x_2 \vdash y_1, x_2 + (y_2 - x_2)$, and if $x_1 + x_2 =_G y_1 + y_2$, then $y_1 + (y_2 - x_2) =_G x_1$. $\square$

In the remainder of this section, $\vdash$ is a regular entailment relation for an ordered group G.

Lemma 2.6. *Let $A \in \mathrm{P}_{\mathrm{fe}}^*(G)$ and $x \in G$. In the distributive lattice L generated by $\vdash$ (Theorem 2.2), $\bigwedge A \leqslant_L (\bigwedge A + x) \vee (\bigwedge A - x)$ holds.*

Proof. For every $a, a' \in A$, $a, a' \vdash a + x, a' - x$ holds by Lemma 2.5. Therefore

$$\bigwedge A \leqslant_L \bigwedge_{a,a' \in A} (a + x \vee a' - x) =_L \Big(\bigwedge_{a \in A} a + x\Big) \vee \Big(\bigwedge_{a' \in A} a' - x\Big). \qquad \square$$

Lemma 2.7. *Let $A, B \in \mathrm{P}^*_{\mathrm{fe}}(G)$ and $x \in G$.*

1. *If $A, A + x \vdash B$ and $A, A - x \vdash B$, then $A \vdash B$.*
2. *$A, A + x \vdash B$ holds if and only if $A \vdash B, B - x$.*

Proof. 1. Theorem 2.2 allows us to work in the distributive lattice L generated by $\vdash$. We have $\bigwedge A \wedge (\bigwedge A + x) \leqslant_L \bigvee B$ and $\bigwedge A \wedge (\bigwedge A - x) \leqslant_L \bigvee B$. By Lemma 2.6,

$$\begin{aligned} \bigwedge A &=_L \bigwedge A \wedge \Big(\Big(\bigwedge A + x\Big) \vee \Big(\bigwedge A - x\Big)\Big) \\ &=_L \Big(\bigwedge A \wedge \Big(\bigwedge A + x\Big)\Big) \vee \Big(\bigwedge A \wedge \Big(\bigwedge A - x\Big)\Big) \leqslant_L \bigvee B. \end{aligned}$$

2. Suppose that $A, A + x \vdash B$. Then $A - x, A \vdash B - x$, so that $A, A + x \vdash B, B - x$ and $A, A - x \vdash B, B - x$, and we may apply Item (1). The converse holds because the relation converse to $\vdash$ is a regular entailment relation for $(G, \geqslant_G)$. $\square$

Lemma 2.8. *Let $a, x \in G$ and $0 \leqslant p \leqslant q$. Then $a, a + qx \vdash a + px$.*

Proof. By induction on q. This is trivial if $p = 0$ or $p = q$. Suppose that $a, a + q'x \vdash a + px$ whenever $0 \leqslant p \leqslant q' < q$. Consider first $1 \leqslant p \leqslant q - p$. By hypothesis, $a, a + (q - p)x \vdash a + px$. By regularity, $a, a + qx \vdash a + px, a + (q - p)x$. A cut yields $a, a + qx \vdash a + px$. Make now an induction on p with $q - p \leqslant p < q$. Suppose that $a, a + qx \vdash a + p'x$ for $0 \leqslant p' < p$. As $0 \leqslant 2p - q < p$, we have $a, a + qx \vdash a + (2p - q)x$. By regularity (and contraction), $a + qx, a + (2p - q)x \vdash a + px$. A cut yields $a, a + qx \vdash a + px$. $\square$

Lemma 2.9. *Let $A, B \in \mathrm{P}^*_{\mathrm{fe}}(G)$ and $x \in G$. Let $0 \leqslant p \leqslant q$. If $A, A + px \vdash B$ holds, or merely $A, A + px, A + qx \vdash B$, then so does $A, A + qx \vdash B$.*

Proof. Cut successively the $a + px$ for $a \in A$ in the given entailment with the entailment $a, a + qx \vdash a + px$ holding by Lemma 2.8. $\square$

Let us now give a description of the regular entailment relation obtained by forcing an element x to be positive.

Proposition 2.10. *Let $\vdash$ be a regular entailment relation for an ordered group G. Let us define the relation $\vdash_x$ on $\mathrm{P}^*_{\mathrm{fe}}(G)$ by writing $A \vdash_x B$ if there is $p \geqslant 0$ such that $A, A + px \vdash B$. Then $\vdash_x$ is a regular entailment relation, and it is the finest equivariant entailment relation $\vdash'$ coarser than $\vdash$ such that $0 \vdash' x$.*

Proof. Only transitivity needs an argument. Suppose that $A, A + px \vdash B, c$ and $A, c, A + qx, c + qx \vdash B$. By Lemma 2.9, we may suppose $p = q$; let $y = px = qx$. By equivariance, $A + y, A + 2y \vdash B + y, c + y$. Let us consider $A' = A, A + y, A + 2y$ and prove $A' \vdash B$. By monotonicity, $A' \vdash B, c$ and $A', c, c+y \vdash B$ and $A' \vdash B+y, c+y$. The two last yield by a cut $A', c \vdash B, B+y$, which by Item (2) of Lemma 2.7 yields $A', c, A' - y, c - y \vdash B$. But monotonicity also yields $A', c, A' + y, c + y \vdash B$, so that by Item (1) of Lemma 2.7 follows $A', c \vdash B$. A cut yields $A' \vdash B$. Lemma 2.9 produces $A, A + 2y \vdash B$, and therefore $A \vdash_x B$.

The relation $\vdash_x$ is clearly coarser than $\vdash$ and satisfies $0 \vdash_x x$. Conversely, suppose that $A \vdash_x B$, i.e. $A, A + px \vdash B$ for some $p \geqslant 0$, and consider an equivariant entailment relation $\vdash'$ coarser than $\vdash$ and satisfying $0 \vdash' x$. Then $A, A + px \vdash' B$ and, because $a \vdash' a + x \vdash \cdots \vdash' a + px$ for each $a \in A$, we may cut successively the $a + px$ and obtain $A \vdash' B$. □

Theorem 2.11. *Let $\vdash$ be a regular entailment relation for an ordered group G. If $A \vdash_x B$ and $A \vdash_{-x} B$, then $A \vdash B$.*

Proof. By Proposition 2.10, $A, A+px \vdash B$ and $A, A-qx \vdash B$ for some $p, q \geqslant 0$. By Lemma 2.9, we may suppose $p = q$, and conclude by Item (1) of Lemma 2.7. □

The meaning of Theorem 2.11 is that if one wants to establish an entailment involving certain elements, one can always assume that these elements are linearly ordered. Lombardi and Quitté (2015, Principle XI-2.10) call this the "Principle of covering by quotients for ℓ-groups".

2.5 Consequences of assuming elements linearly ordered: cancellativity

If A and B are linearly ordered nonempty finitely enumerated subsets of G, then

$$\min(A + B) \leqslant_G \min B \implies \min A \leqslant_G 0,$$
$$\min A \leqslant_G \max B \iff \min(A - B) \leqslant_G 0 \iff 0 \leqslant_G \max(B - A).$$

We have therefore the following corollary to Theorem 2.11.

Corollary 2.12. *Let $\vdash$ be a regular entailment relation for $(G, \leqslant_G)$ and $A, B \in \mathrm{P}^*_{\mathrm{fe}}(G)$.*

1. *If $A + B \vdash b$ for every $b \in B$, then $A \vdash 0$.*
2. *$A \vdash B$ holds if and only if $A - B \vdash 0$, if and only if $0 \vdash B - A$.*

Let $\rhd$ be the system of ideals defined as the restriction of $\vdash$ to $\mathrm{P}^*_{\mathrm{fe}}(G) \times G$.

- Item (1) expresses that the meet-monoid associated with $\triangleright$ is cancellative (see Item (3) of Theorem 3.3). In Section 3.1, we provide the construction of its Grothendieck group, which is an ℓ-group, and draw the conclusion that the underlying distributive lattice coincides with the one generated by $\vdash$.
- Item (2) expresses that $\vdash$ is determined by $\triangleright$. Conversely, given a system of ideals $\triangleright$, there are several unbounded entailment relations that reflect $\triangleright$: see Lorenzen 1952, § 3, and Rinaldi, Schuster, and Wessel 2017, § 3.1.

3 Consequences of cancellativity

3.1 The Grothendieck ℓ-group of a meet-semilattice-ordered monoid

Definition 3.1. Let $(M, +, 0)$ be a monoid. The *Grothendieck group of M* is the group freely generated by $(M, +, 0)$ (in the sense of the left adjoint functor of the forgetful functor).

Lemma 3.2. *The Grothendieck group H of M may be obtained by considering the monoid of formal differences $a-b$ for $a, b \in M$, equipped with the addition $(a-b)+(c-d) = (a+c)-(b+d)$ and the neutral element $0-0$, and by taking its quotient by the equality*

$$a - b =_H c - d \overset{\text{def}}{\iff} \exists x \in M \ a + d + x =_M b + c + x.$$

Every equality $a - b =_H c - d$ may be reduced to two elementary ones, i.e. of the form $e - f =_H (e + y) - (f + y)$.

Proof. See Bourbaki 1974, I, § 2.4, but for the last assertion, which follows from transitivity and symmetry of $=_H$:

$$a - b =_H (a + d + x) - (b + d + x) =_H (b + c + x) - (b + d + x) =_H c - d. \quad \square$$

The following easy construction, for which we did not locate a good reference (but compare Cignoli, D'Ottaviano, and Mundici 2000, § 2.4), is particularly significant in the case where the meet-monoid associated with an equivariant system of ideals proves to be cancellative.

Theorem 3.3. *Let $(M, +, 0, \wedge)$ be a meet-monoid. Let H be the Grothendieck group of M with monoid morphism $\varphi\colon M \to H$.*

1. There is a unique meet-monoid structure on H such that φ is a morphism of ordered sets.

2. $(H, +, -, 0, \wedge)$ *is an* ℓ*-group: it is the* ℓ*-group freely generated by* $(M, +, 0, \wedge)$ *(in the sense of the left adjoint functor of the forgetful functor), and called the* Grothendieck ℓ-group *of* M*.*

3. Assume that M *is cancellative, i.e. that* $a + x =_M b + x$ *implies* $a =_M b$*. Then* φ *is an embedding of meet-monoids.*

Proof. (1) • When trying to define $z = (e - f) \wedge (i - j)$ we need to ensure that $f + j + z =_M (e + j) \wedge (i + f)$: so we claim that $z \stackrel{\text{def}}{=} ((e + j) \wedge (i + f)) - (f + j)$ will do. Let us show first that the law $\wedge$ is well-defined on H: by Lemma 3.2, it suffices to show that $z =_H ((e + y) - (f + y)) \wedge (i - j)$, which reduces successively to $((e + j) \wedge (i + f)) - (f + j) =_H ((e + j + y) \wedge (i + f + y)) - (f + j + y)$ and to $((e + j) \wedge (i + f)) + (f + j + y) =_M ((e + j + y) \wedge (i + f + y)) + (f + j)$. Since $\wedge$ is compatible with $+$ in M, both sides are equal to $(e + 2j + f + y) \wedge (i + 2f + j + y)$.

• The map $\varphi\colon M \to H$ preserves $\wedge$: in fact $\varphi(a) \stackrel{\text{def}}{=} a - 0$, and the checking is immediate.

• The law $\wedge$ on H is idempotent, commutative, and associative. This is easy to check and left to the reader.

• The law $\wedge$ is compatible with $+$ on H. This is easy to check and left to the reader.

(2) This construction yields the ℓ-group freely generated by M because it only uses the hypothesis that there is a meet-monoid morphism $\varphi\colon M \to H$.

(3) Cancellativity may be read precisely as the injectivity of φ. The meet-monoid structure is purely equational, so that an injective morphism is always an embedding. □

3.2 The ℓ-group generated by a regular entailment relation

The main result of this article is Theorem 3.4 below: it states that regular entailment relations provide a description of all morphisms from an ordered group G to ℓ-groups generated by (the image of) G.

Theorem 3.4. *Let* $\vdash$ *be a regular entailment relation for an ordered group* G*. Let* H *be the distributive lattice generated by the entailment relation* $\vdash$*. Then there is a (unique) group law on* H *which is compatible with its lattice structure and such that the natural morphism (of ordered sets)* $G \to H$ *is a group morphism. The resulting* ℓ*-group is called the* group of ideals *associated with* $\vdash$*.*

Proof. By Item (1) of Corollary 2.12, the meet-monoid associated with the restriction $\rhd$ of $\vdash$ is cancellative, so that by Item (3) of Theorem 3.3 it embeds into its Grothendieck ℓ-group H. The underlying distributive lattice coincides with the one generated by $\vdash$ by Item (2) of Corollary 2.12. Uniqueness follows from Item (2) of Theorem 3.3. $\square$

Let us state a variant of Theorem 3.4.

Corollary 3.5. *Let $(G, \leqslant_G)$ be an ordered group and $\rhd$ an equivariant system of ideals for G. The following are equivalent:*

1. The equivariant system of ideals $\rhd$ is regular (i.e. it is the restriction of a regular entailment relation $\vdash$).

2. The meet-monoid associated with the equivariant system of ideals $\rhd$ for G (Theorem 1.10) is cancellative.

Proof. (1) $\implies$ (2). The subset $M \subseteq H$ of those elements that may be written $\varphi(x_1) \wedge \cdots \wedge \varphi(x_n)$ for some $x_1, \ldots, x_n \in G$ is the meet-semilattice associated with the equivariant system of ideals $\rhd$ obtained by restricting $\vdash$ to $\mathrm{P}^*_{\mathrm{fe}}(G) \times G$. This subset is stable by addition, so that the restriction of addition to M endows it with the structure of a cancellative meet-monoid. Thus H is necessarily (naturally isomorphic to) the Grothendieck ℓ-group of M.

(2) $\implies$ (1). See the proof of Theorem 3.4. $\square$

Comment 3.6. Theorem 3.4 is new and replaces the second step of the proof of Satz 1 in Lorenzen 1953 (see page 21), which establishes that the distributive lattice H is in fact an ℓ-group by constructing "by hand" a group law without emphasis on the rôle of regularity. This rôle is revealed by our presentation, which allows for more conceptual arguments. $\diamond$

4 The regularisation of an equivariant system of ideals for an ordered group

4.1 Definition

An equivariant system of ideals gives rise to a regular entailment relation if one proceeds as if elements occurring in a computation are comparable. More precisely, this idea gives the following definition.

Definition 4.1 (see Lorenzen 1953, (2.2) and page 23)**.** Let $\rhd$ be an equivariant system of ideals for an ordered group G.

1. For $y_1, \ldots, y_n \in G$, consider the equivariant system of ideals $\rhd_{y_1,\ldots,y_n}$ coarser than $\rhd$ obtained by forcing the properties $0 \rhd y_1, \ldots, 0 \rhd y_n$. The *regularisation* of $\rhd$ is the relation on $\mathrm{P}^*_{\mathrm{fe}}(G)$ defined by

$$A \vdash_{\rhd} B \overset{\text{def}}{\iff} \begin{array}{l}\text{there are } x_1, \ldots, x_m \in G \text{ such that for every}\\ \text{choice of signs } \pm,\ A - B \rhd_{\pm x_1,\ldots,\pm x_m} 0 \text{ holds.}\end{array}$$

2. An element b of G is $\rhd$-*dependent on* A if $A \vdash_\rhd b$.

3. The group G is $\rhd$-*closed* if $\vdash_\rhd$ reflects the order on G, i.e. if the implication $a \vdash_\rhd b \implies a \leqslant_G b$ holds for all $a, b \in G$.

The terminology of Items (2) and (3) comes from integral domains. Regularisation is an early occurrence of dynamical algebra (see Coste, Lombardi, and Roy 2001): we shall see that $\rhd$-closedness is the dynamical counterpart to being embedded into a product of linearly preordered groups. Let us go through a simple example that illustrates a relevant feature of this construction (compare Proposition 4.11).

Example 4.2. Let us apply a case-by-case reasoning in order to prove that in a linearly ordered group, if $n_1 a_1 + \cdots + n_k a_k \leqslant 0$ for some integers $n_i \geqslant 0$ not all zero, then $a_j \leqslant 0$ for some j. If $a_j \leqslant 0$ for some j, everything is all right. If $0 \leqslant a_j$ for all j, take i such that $n_i \geqslant 1$: then $a_i \leqslant n_i a_i \leqslant n_1 a_1 + \cdots + n_k a_k \leqslant 0$. The conclusion holds in each case. Similarly, assume that $n_1 a_1 + \cdots + n_k a_k \rhd 0$ with $n_i \geqslant 0$ not all zero. We have $a_j \rhd_{-a_j} 0$ for each j. By monotonicity, $a_1, \ldots, a_k \rhd_{\epsilon_1 a_1, \ldots, \epsilon_k a_k} 0$ holds if at least one ϵ_j is equal to -1. If we force $0 \rhd a_j$ for all j, take i such that $n_i \geqslant 1$: then $a_i \leqslant_\rhd n_i a_i \leqslant_\rhd n_1 a_1 + \cdots + n_k a_k \leqslant_\rhd 0$. This proves that $a_1, \ldots, a_k \rhd_{+a_1, \ldots, +a_k} 0$. We conclude that $a_1, \ldots, a_k \vdash_\rhd 0$. $\diamond$

Comment 4.3. Lorenzen (1950, pages 488–489) describes the basic idea of regularisation. He says that (the single-conclusion counterpart to) it is his answer to the following question for a given equivariant system of ideals $\rhd$ with associated monoid of ideals H_r ("H" for "Halbverband", semilattice, r a variable name for distinguishing different monoids):

> How can the distinguished brauchbar [equivariant] system of ideals of a preordered group G be constructed—assumed that it actually exists?
>
> We thus assume that the preorder of G is representable as conjunction of allowable linear preorders of G. But we do not want to use the knowledge of these linear preorders, because we precisely want to determine the allowable linear preorders by the aid of the equivariant system of ideals.

As customary in the literature, we do not translate the epithet "brauchbar" (which means "usable") introduced by Krull. Let us say that a linear preorder $\preccurlyeq$ on G is $\rhd$-*allowable* if $A \rhd b \implies \min A \preccurlyeq b$; Lorenzen notes that it defines a "linear" equivariant system of ideals $\rhd'$ coarser than $\rhd$: $\preccurlyeq$ extends in a unique way to the linear preorder on the monoid of ideals given by $A \leqslant_{\rhd'} B \overset{\text{def}}{\iff} \min A \preccurlyeq \min B$. Lorenzen's question is therefore about the equivariant system of ideals $\rhd_{\mathrm{a}}$

defined by

$$A \rhd_{\mathrm{a}} b \overset{\text{def}}{\iff} \text{for every } \rhd\text{-allowable linear preorder } \preccurlyeq, \\ \min A \preccurlyeq b \text{ holds,}$$

and his answer is that $A \rhd_{\mathrm{a}} b \iff A \vdash_{\rhd} b$ (Lorenzen 1950, Satz 24). More precisely, it is straightforward that every linear preorder coarser than $\rhd$ is also coarser than $\vdash_{\rhd}$, so that $A \vdash_{\rhd} b \implies A \rhd_{\mathrm{a}} b$. Conversely, consider A and b such that $A \vdash_{\rhd} b$ does not hold: a well-ordering argument grants a coarsest equivariant system of ideals $\rhd'$ coarser than $\rhd$ such that $A \vdash_{\rhd'} b$ still does not hold; it cannot be other than linear, for if both $0 \rhd' x$ and $x \rhd' 0$ did not hold for some $x \in G$, then both $\rhd'_x$ and $\rhd'_{-x}$ would be coarser than $\rhd'$, so that $A \vdash_{\rhd'_x} b$ and $A \vdash_{\rhd'_{-x}} b$ would hold, i.e. $A \vdash_{\rhd'} b$.

In the literature, "endlich arithmetisch brauchbar" is the property of cancellativity of a monoid of ideals, to be found in several places of this article: Item (1) of Corollary 2.12 (w.r.t. regular entailment relations); Item (3) of Theorem 3.3 (w.r.t. the Grothendieck ℓ-group); Corollary 3.5 (equivalence with regularity); Corollary 4.25 (Macaulay's theorem); Lemma 5.1 (Prüfer's property Γ); Theorem 5.5 (Prüfer's theorem). In our presentation, cancellativity of the regularisation is a key feature.

In the case of integral domains, the allowable linear preorders are exactly the valuations, and the equivariant system of ideals $\rhd_{\mathrm{a}}$ is designed so as to capture the characterisation of integral dependence of an element b on the ideal generated by a finitely enumerated set A through valuations (the letter "a" has been chosen by Prüfer 1932 for "algebraically representable"). ◇

Note that Sections 4.3 and 4.5 below do not resort to the fundamental theorem of unbounded entailment relations, Theorem 2.2, i.e. the reasoning takes place on the level of the entailment relations and not in the generated distributive lattice.

4.2 The regularisation is the regular entailment relation generated by an equivariant system of ideals

Lorenzen (1953, § 2) starts with an equivariant system of ideals $\rhd$ for an ordered group and uses the heuristics of Item (2) of Corollary 2.12 to define the regularisation $\vdash_{\rhd}$ as in Definition 4.1. Then he applies the fundamental theorem for unbounded entailment relations, Theorem 2.2, and obtains a distributive lattice. This article wishes to assess the following remarkable theorem (which holds also for noncommutative groups).

Satz 1 (Lorenzen 1953)**.** *Let $\rhd$ be an equivariant system of ideals for an ordered group G. Its regularisation $\vdash_{\rhd}$ is a regular entailment relation and the action* *of G on the distributive lattice H generated by $\vdash_{\rhd}$ may be extended to a group law for H.*

Theorem 4.6 below strengthens the first step of the proof of Satz 1, in which the entailment relation $\vdash_{\rhd}$ is constructed and shown to be regular as in Proposition 4.4. In our analysis of Lorenzen's proof, we separate this step from its second step, the explicit construction of a group law for the regularisation. Our presentation makes regularity (Property *R5*) the lever for sending G homomorphically into an ℓ-group in Theorem 3.4.

Proposition 4.4. *Let $\rhd$ be an equivariant system of ideals for an ordered group G. Its regularisation $\vdash_{\rhd}$ is a regular entailment relation for G.*

Proof. The regularisation is clearly reflexive and monotone, and satisfies Properties *R3* and *R4*.

Let us prove that the regularisation is transitive. Suppose that $A, 0 \vdash_{\rhd} B$ and $A \vdash_{\rhd} 0, B$: there are $x_1, \ldots, x_m, y_1, \ldots, y_n$ such that for every choice of signs $\pm$, $A - B, -B \rhd_{\pm x_1,\ldots,\pm x_m} 0$ and $A, A - B \rhd_{\pm y_1,\ldots,\pm y_n} 0$ hold.

Let $A = \{a_1, \ldots, a_k\}$. If $a_i \rhd 0$ for some i, then $A \leqslant_{\rhd} A, 0$ and $A - B \leqslant_{\rhd} A - B, -B$. Thus

$$A - B \leqslant_{\rhd_{-a_i,\pm x_1,\ldots,\pm x_m}} A - B, -B \leqslant_{\rhd_{-a_i,\pm x_1,\ldots,\pm x_m}} 0 \quad \text{for } i = 1, \ldots, k.$$

If $0 \rhd a_1$, …, $0 \rhd a_k$, then $0 \leqslant_{\rhd} A, 0$ and $-B \leqslant_{\rhd} A - B, -B$. Thus

$$\begin{aligned} -B &\leqslant_{\rhd_{a_1,\ldots,a_k,\pm x_1,\ldots,\pm x_m}} 0 \\ A - B &\leqslant_{\rhd_{a_1,\ldots,a_k,\pm x_1,\ldots,\pm x_m}} A. \end{aligned}$$

As $A, A - B \rhd_{\pm y_1,\ldots,\pm y_n} 0$, we have

$$A - B \leqslant_{\rhd_{a_1,\ldots,a_k,\pm x_1,\ldots,\pm x_m,\pm y_1,\ldots,\pm y_n}} 0.$$

All together, we conclude that

$$A - B \rhd_{\pm a_1,\ldots,\pm a_k,\pm x_1,\ldots,\pm x_m,\pm y_1,\ldots,\pm y_n} 0.$$

Let us prove that the regularisation is regular, i.e. that $x+a, y+b \vdash_{\rhd} x+b, y+a$ holds for all $a, b, x, y \in G$: it suffices to note that

$$\begin{aligned} &\text{if } a - b \rhd 0, \text{ then } a - b, x - y, y - x, b - a \rhd 0; \\ &\text{if } b - a \rhd 0, \text{ then } a - b, x - y, y - x, b - a \rhd 0. \end{aligned}$$

□

The following lemma justifies the terminology of Definition 4.1. One may formulate it as follows: "regularisation leaves a regular entailment relation unchanged".

Lemma 4.5. *Let G be an ordered group and $\vdash$ a regular entailment relation for G. Let $\rhd_{\vdash}$ be the equivariant system of ideals given as the restriction of $\vdash$ to $\mathrm{P}^*_{\mathrm{fe}}(G) \times G$. Then $\vdash$ coincides with the regularisation of $\rhd_{\vdash}$.*

Proof. This is a consequence of Theorem 2.11 and Item (2) of Corollary 2.12. ☐

Theorem 4.6. *Let $\rhd$ be an equivariant system of ideals for an ordered group G. The regularisation $A \vdash_{\rhd} B$ given in Definition 4.1 is the finest regular entailment relation for G whose restriction to $\mathrm{P}^*_{\mathrm{fe}}(G) \times G$ is coarser than $\rhd$.*

Proof. Proposition 4.4 tells that $\vdash_{\rhd}$ is a regular entailment relation, and it is clear from its definition that its restriction to $\mathrm{P}^*_{\mathrm{fe}}(G) \times G$ is coarser than $\rhd$. Now let $\vdash$ be a regular entailment relation whose restriction $\rhd_{\vdash}$ to $\mathrm{P}^*_{\mathrm{fe}}(G) \times G$ is coarser than $\rhd$. Then the same holds for their regularisation, i.e., by Lemma 4.5, $\vdash$ is coarser than $\vdash_{\rhd}$. ☐

These results give rise to the following construction and theorem, that one can find in Lorenzen 1953, § 2 and page 23.

Definition 4.7. Let $\rhd$ be an equivariant system of ideals for an ordered group G. The *Lorenzen group* associated with $\rhd$ is the ℓ-group provided by Theorems 3.4 and 4.6.

Comment 4.8. Lorenzen (1939, § 4) and Jaffard (1960, II, § 2, 2) follow the Prüfer approach (see Definition 5.7) for defining the Lorenzen group associated with an equivariant system of ideals. The present approach leading to Definition 4.7 is inspired by Lorenzen (1953, § 2). The two definitions are equivalent according to Proposition 5.8. ◇

Theorem 4.9. *Let $\rhd$ be an equivariant system of ideals for an ordered group G. If G is $\rhd$-closed, then G embeds into the Lorenzen group associated with $\rhd$.*

4.3 The regularisation of the finest equivariant system of ideals

We shall now give a precise description of the regularisation $\vdash_{\rhd_{\mathrm{s}}}$ of the finest equivariant system of ideals introduced in Proposition 1.11.

Lemma 4.10. *Let $(G, \leqslant_G)$ be an ordered group and $\vdash$ a regular entailment relation for G. Let $a_1, \dots, a_k \in G$. If*

$$(\|) \qquad n_1 a_1 + \cdots + n_k a_k \leqslant_G 0 \quad \textit{for some integers } n_i \geqslant 0 \textit{ not all zero,}$$

then $a_1, \dots, a_k \vdash 0$.

Proof. This follows from the argument of Example 4.2 because of Theorem 2.11. ☐

Let us write $A^{(n)}$ for the $n$fold sum of the set A with itself: $A^{(n)} = A + \cdots + A$ (n times).

Proposition 4.11. *Let $(G, \leqslant_G)$ be an ordered group. T.f.a.e. for $A \in \mathrm{P}^*_{\mathrm{fe}}(G)$.*

1. $A \vdash_{\rhd_{\mathrm{s}}} 0$.

2. There is an integer $n \geqslant 1$ such that $A^{(n)} \rhd_{\mathrm{s}} 0$, i.e. there are $a_1, \ldots, a_k \in A$ such that (‖) holds.

Proof. Let us denote Property (‖) by $\varrho(a_1, \ldots, a_k)$.

(1) $\Longrightarrow$ (2). Proposition 1.15 shows that if $A \rhd_{\epsilon_1 x_1, \ldots, \epsilon_m x_m} 0$, then $A, A + \epsilon_m x_m, \ldots, A + p\epsilon_m x_m \rhd_{\epsilon_1 x_1, \ldots, \epsilon_{m-1} x_{m-1}} 0$ for some integer p. One may therefore proceed by induction on m. Firstly, it is clear that $a_1, \ldots, a_k \rhd_{\mathrm{s}} 0$ implies that $\varrho(a_1, \ldots, a_k)$ holds. Secondly, suppose that for some integers p and q,

$$\varrho(a_1, \ldots, a_k, a_1 + x_m, \ldots, a_k + x_m, \ldots, a_1 + px_m, \ldots, a_k + px_m) \text{ and}$$
$$\varrho(a_1, \ldots, a_k, a_1 - x_m, \ldots, a_k - x_m, \ldots, a_1 - qx_m, \ldots, a_k - qx_m) \text{ hold.}$$

Let us show that $\varrho(a_1, \ldots, a_k)$ holds. The hypothesis implies that there are integers $n_i, n \geqslant 0$, at least one n_i nonzero, such that $n_1 a_1 + \cdots + n_k a_k + nx_m \leqslant_G 0$, and integers $n'_i, n' \geqslant 0$, at least one n'_i nonzero, such that $n'_1 a_1 + \cdots + n'_k a_k - n' x_m \leqslant_G 0$. If $n = 0$ or if $n' = 0$, then we are done; otherwise, $(n' n_1 + n n'_1) a_1 + \cdots + (n' n_k + n n'_k) a_k \leqslant_G 0$ with at least one $n' n_i + n n'_i$ nonzero.

(2) $\Longrightarrow$ (1). This follows from Lemma 4.10. ☐

Corollary 4.12. *Let G be an ordered group. The regularisation of the finest equivariant system of ideals for G is the finest regular entailment relation for G.*

Proof. This follows from Lemma 4.10 and Proposition 4.11 because of Item (2) of Corollary 2.12. ☐

Corollary 4.13. *An ordered group $(G, \leqslant_G)$ is $\rhd_{\mathrm{s}}$-closed if and only if*

$$(\circ) \qquad 0 \leqslant_G na \text{ implies } 0 \leqslant_G a \quad (a \in G,\ n > 1).$$

Corollary 4.14. *Let $(G, \leqslant_G)$ be an ordered group. T.f.a.e.*

1. $A \vdash_{\rhd_{\mathrm{s}}} B$.

2. There is an integer $n \geqslant 1$ such that for some elements $a^{(n)} \in A^{(n)}$ and $b^{(n)} \in B^{(n)}$, $a^{(n)} \leqslant_G b^{(n)}$ holds.

Proof. Suppose that there are $a_1, \ldots, a_k, b_1, \ldots, b_\ell \in G$ such that $a^{(n)} = n_1 a_1 + \cdots + n_k a_k \leqslant_G b^{(n)} = m_1 b_1 + \cdots + m_\ell b_\ell$ with integers $n_i, m_j \geqslant 0$ such that $n_1 + \cdots + n_k = m_1 + \cdots + m_\ell = n$. By the Riesz refining lemma (see e.g. Lombardi and Quitté 2015, Theorem XI-2.11), there are integers $p_{ij} \geqslant 0$ such that $n_i = \sum_{j=1}^{\ell} p_{ij}$ for each i and $m_j = \sum_{i=1}^{k} p_{ij}$ for each j, so that Item (2) may be written

$$\textstyle\sum_{i=1}^{k} \sum_{j=1}^{\ell} p_{ij}(a_i - b_j) \leqslant_G 0 \text{ for some integers } p_{ij} \geqslant 0 \text{ not all zero.}$$

The equivalence follows by Proposition 4.11. □

4.4 The ℓ-group freely generated by an ordered group

As an application, we provide the following description for the ℓ-group freely generated by an ordered group.

Theorem 4.15. *For every ordered group G we can construct an ℓ-group H with a morphism $\varphi\colon G \to H$ such that $0 \leqslant_H \varphi(a)$ holds if and only if $0 \leqslant_G na$ for some $n \geqslant 1$. More precisely, H is the ℓ-group freely generated by G (in the sense of the left adjoint functor of the forgetful functor) and can be constructed as the Lorenzen group associated with the finest equivariant system of ideals, characterised by: $\varphi(a_1) \wedge \cdots \wedge \varphi(a_k) \leqslant_H \varphi(b_1) \vee \cdots \vee \varphi(b_\ell)$ holds if and only if there are integers $n_1, \ldots, n_k, m_1, \ldots, m_\ell \geqslant 0$ with $n_1 + \cdots + n_k = m_1 + \cdots + m_\ell \geqslant 1$ such that $n_1 a_1 + \cdots + n_k a_k \leqslant_G m_1 b_1 + \cdots + m_\ell b_\ell$.*

Theorem 4.15 is in fact a reformulation of the following proposition, enriched with an account of Corollaries 4.12 to 4.14.

Proposition 4.16. *Let $(G, \leqslant_G)$ be an ordered group. The Lorenzen group associated with the finest equivariant system of ideals for G is the ℓ-group freely generated by $(G, \leqslant_G)$ (in the sense of the left adjoint functor of the forgetful functor).*

Proof. The finest monoid of ideals for G is the meet-monoid M freely generated by G, and its Grothendieck ℓ-group H is the ℓ-group freely generated by M: therefore H is the ℓ-group freely generated by G as a monoid, and therefore also as a group. □

Theorem 4.15 may be seen as a generalisation of the following corollary, the constructive core of the classical Lorenzen-Clifford-Dieudonné theorem.

Corollary 4.17 (Lorenzen-Clifford-Dieudonné, see Lorenzen 1939, Satz 14 for the s-system of ideals ; Clifford 1940, Theorem 1; Dieudonné 1941, Section 1)**.** *The ordered group $(G, \leqslant_G)$ is embeddable into an ℓ-group if and only if*

$$(^\circ) \qquad 0 \leqslant_G na \text{ implies } 0 \leqslant_G a \quad (a \in G,\ n > 1).$$

Proof. The condition is clearly necessary. Theorem 4.15 shows that it yields the injectivity of the morphism $\varphi\colon G \to H$ as well as the fact that $\varphi(x) \leqslant_H \varphi(y)$ implies $x \leqslant_G y$. □

Comments 4.18. 1. In each of the three references given in Corollary 4.17, the authors invoke a maximality argument for showing that G embeds in fact into a direct product of linearly ordered groups. The goal of Lorenzen (1950, § 4; 1953) is to avoid the necessarily nonconstructive reference to linear orders in conceiving embeddings into an ℓ-group, and this endeavour culminates in Corollary 3.5.

2. The reader will recognise Condition ($^\circ$) of $\rhd_{\mathrm{s}}$-closedness of Corollary 4.13 in the condition of embeddability stated here. In fact, in his Ph.D. thesis (1939), Lorenzen proves Corollary 4.17 as a side-product of his enterprise of generalising the concepts of multiplicative ideal theory to the framework of preordered groups. He is following the Prüfer approach presented in Section 5, in which $\rhd_{\mathrm{s}}$-closedness is being introduced according to Definition 5.3 and the equivalence with Condition ($^\circ$) is easy to check (see Lorenzen 1939, page 358, or Jaffard 1960, I, § 4, Théorème 2). ◇

4.5 The regularisation of the system of Dedekind ideals

Let us resume Section 1.8 with a crucial lemma.

Lemma 4.19. *One has $A \vdash_{\rhd_{\mathrm{d}}} 1$ if and only if $\langle A\rangle_{R[A]} \ni 1$.*

Proof. Suppose that $A \vdash_{\rhd_{\mathrm{d}}} 1$, i.e. by Proposition 1.16 that there are elements $x_1, \dots, x_n \in G$ such that $\langle A\rangle_{R[x_1^{\pm 1},\dots,x_n^{\pm 1}]} \ni 1$. It suffices to prove the following fact and to use it in an induction argument: suppose that $\langle A\rangle_{R[A,x]} \ni 1$ and $\langle A\rangle_{R[A,x^{-1}]} \ni 1$; then $\langle A\rangle_{R[A]} \ni 1$. In fact, the hypothesis means that $\langle A, Ax, \dots, Ax^p\rangle_{R[A]} \ni 1$ and $\langle A, Ax^{-1}, \dots, Ax^{-p}\rangle_{R[A]} \ni 1$ for some $p \geqslant 0$, which implies that

$$\forall k \in [\![-p..p]\!] \quad \left\langle Ax^{-p}, \dots, Ax^{-1}, A, Ax, \dots, Ax^p\right\rangle_{R[A]} \ni x^k,$$

i.e. that there is a matrix M with coefficients in $\langle A\rangle_{R[A]}$ such that $M(x^k)_{-p}^p = (x^k)_{-p}^p$, i.e. $(1-M)(x^k)_{-p}^p = 0$. Let us now apply the determinant trick: multiplying $1-M$ by the matrix of its cofactors and expanding it yields that $\langle A\rangle_{R[A]} \ni 1$.

Conversely, let $a_1, \dots, a_k$ be the elements of A. For each i, $a_i a_i^{-1} = 1$, so that $\langle A\rangle_{R[a_i^{-1}]} \ni 1$ and $A\ (\rhd_{\mathrm{d}})_{a_1^{\pm 1},\dots,a_k^{\pm 1}}\ 1$ for every choice of signs with at least one negative sign: the only missing choice of signs consists in the hypothesis $\langle A\rangle_{R[A]} \ni 1$. □

An element $b \in K$ is said to be *integral over the ideal* $\langle A \rangle_R$ when an integral dependence relation $b^p = \sum_{k=1}^{p} c_k b^{p-k}$ with $c_k \in {\langle A \rangle_R}^k$ holds for some $p \geqslant 1$. If $A = \{1\}$, then this reduces to the same integral dependence relation with $c_k \in R$, i.e. to b being integral over R.

Note that if A contains nonintegral elements, i.e. elements not in R, then ${\langle A \rangle_R}^2$ may or may not be contained in $\langle A \rangle_R$: consider respectively e.g. the ideal $\langle 1, \frac{u}{t} \rangle$ in $k[T,U]/(T^3 - U^2) = k[t,u]$ and ideals in a Prüfer domain.

Theorem 4.20 (Lorenzen 1953, Satz 2)**.** *Let R be an integral domain and $\triangleright_{\mathrm{d}}$ its system of Dedekind ideals.*

1. One has $A \vdash_{\triangleright_{\mathrm{d}}} b$—i.e. the element b is $\triangleright_{\mathrm{d}}$-dependent on A; there are $x_1, \ldots, x_n$ such that $\langle A \rangle_{R[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]} \ni b$ for every choice of signs—if and only if b is integral over the ideal $\langle A \rangle_R$.

2. One has $A \vdash_{\triangleright_{\mathrm{d}}} B$—that is, there are $x_1, \ldots, x_n$ such that $\langle AB^{-1} \rangle_{R[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]} \ni 1$ for every choice of signs—if and only if $\sum_{k=1}^{p} {\langle AB^{-1} \rangle_R}^k \ni 1$ for some $p \geqslant 1$, i.e. there is an equality $\sum_{k=1}^{p} f_k = 1$ with each f_k a homogeneous polynomial of degree k in the elements of AB^{-1} with coefficients in R.

3. The divisibility group G is $\triangleright_{\mathrm{d}}$-closed, i.e. the equivalence

$$a \vdash_{\triangleright_{\mathrm{d}}} b \iff a \text{ divides } b$$

holds, if and only if R is integrally closed.

Proof. (1–2) This follows from the previous lemma because

$$\begin{aligned} A \vdash_{\triangleright_{\mathrm{d}}} b &\iff Ab^{-1} \vdash_{\triangleright_{\mathrm{d}}} 1, \\ \textstyle\sum_{k=1}^{p} c_k b^{p-k} = b^p \text{ with } c_k \in {\langle A \rangle_R}^k &\iff \textstyle\sum_{k=1}^{p} {\langle Ab^{-1} \rangle_R}^k \ni 1, \\ \langle A \rangle_{R[A]} \ni 1 &\iff \exists p \geqslant 1 \ \textstyle\sum_{k=1}^{p} {\langle A \rangle_R}^k \ni 1. \end{aligned}$$

(3) $\triangleright_{\mathrm{d}}$-closedness is equivalent to $1 \vdash_{\triangleright_{\mathrm{d}}} b \implies R \ni b$; by Item (1), $1 \vdash_{\triangleright_{\mathrm{d}}} b$ holds if and only if b is integral over R. □

4.6 The Lorenzen divisor group of an integral domain

In this section, we note consequences of Theorems 3.4 and 4.6 for Lorenzen's theory of divisibility presented in Section 4.5.

Definition 4.21. Let R be an integral domain. The *Lorenzen divisor group* $\mathrm{Lor}(R)$ of R is the Lorenzen group associated by Definition 4.7 with the system of Dedekind ideals $\triangleright_{\mathrm{d}}$ for the divisibility group of R.

The following version of Theorems 4.6 and 4.9 takes into account the informations provided by Theorem 4.20; Item (1) emphasises the fact that a regular entailment relation is characterised by its restriction to $\mathrm{P}_{\mathrm{fe}}^{*}(G) \times G$ (Item (2) of Corollary 2.12).

Theorem 4.22. *Let R be an integral domain with field of fractions K and divisibility group $G = K^{\times}/R^{\times}$. The entailment relation $\vdash_{\rhd_{\mathrm{d}}}$ generates the Lorenzen divisor group $\mathrm{Lor}(R)$ together with a morphism of ordered groups $\varphi\colon G \to \mathrm{Lor}(R)$ that satisfies the following properties.*

1. *The "ideal Lorenzen gcd" of $a_1, \ldots, a_k \in K^*$ is characterised by*

$$\varphi(a_1) \wedge \cdots \wedge \varphi(a_k) \leqslant \varphi(b) \iff b \text{ is integral over the ideal } \langle a_1, \ldots, a_k \rangle_R.$$

2. *The morphism φ is an embedding if and only if R is integrally closed.*

Item (1) lends itself to an extensional formulation in terms of the integral closure $\mathrm{Icl}_K(a_1, \ldots, a_k)$ of ideals $\langle a_1 \ldots, a_k \rangle_R$ in the field of fractions K. If $a_1, \ldots, a_k \in R^*$, i.e. if one considers integral finitely generated ideals, it seems more appropriate to find a formulation in terms of the integral closure $\mathrm{Icl}(a_1, \ldots, a_k)$ in the integral domain. This works because the elements $a_1, \ldots, a_k, b \in K^*$ in a relation $a_1, \ldots, a_k \vdash_{\rhd_{\mathrm{d}}} b$ may be translated by an x into R^*. This yields the following theorem, in which we use the conventional additive notation for divisor groups of an integral domain. It takes into account the construction of the Lorenzen group as the Grothendieck ℓ-group of the meet-monoid associated with the regularisation of the system of Dedekind ideals in the proof of Theorem 3.4, i.e. as formal differences $\bigwedge \varphi(A) - \bigwedge \varphi(B)$; we take advantage of the fact that $\bigwedge \varphi(A) - \bigwedge \varphi(B) = \bigwedge \varphi(xA) - \bigwedge \varphi(xB)$ for every x, so that it suffices to use integral ideals in this construction.

Theorem 4.23. *Let R be an integral domain. The Lorenzen divisor group* $\mathrm{Lor}(R)$ *can be realised extensionally in the following way.*

- *A* basic divisor *is realised as the integral closure* $\mathrm{Icl}(a_1 \ldots, a_k)$ *of an ordinary, i.e. integral finitely generated ideal $\langle a_1 \ldots, a_k \rangle_R$ with $a_1, \ldots, a_k \in R^*$.*
- *The neutral element of the group, i.e. the divisor* 0, *is realised as* $\mathrm{Icl}(1)$.
- *The meet of two basic divisors is realised as*

$$\mathrm{Icl}(a_1, \ldots, a_k) \wedge \mathrm{Icl}(b_1, \ldots, b_\ell) = \mathrm{Icl}(a_1, \ldots, a_k, b_1, \ldots, b_\ell).$$

- *The sum of two basic divisors is realised as*

$$\mathrm{Icl}(a_1, \ldots, a_k) + \mathrm{Icl}(b_1, \ldots, b_\ell) = \mathrm{Icl}(a_1 b_1, \ldots \ldots, a_k b_\ell).$$

- *The order relation between basic divisors is realised as*

$$\mathrm{Icl}(a_1,\dots,a_k) \leqslant \mathrm{Icl}(b_1,\dots,b_\ell) \iff \mathrm{Icl}(a_1,\dots,a_k) \supseteq \mathrm{Icl}(b_1,\dots,b_\ell).$$

In particular, $\mathrm{Icl}(a) \leqslant \mathrm{Icl}(b)$ *holds if and only if* b *is integral over* $\langle a\rangle_R$*.*

- *Every divisor is realised as the formal difference of two basic divisors.*

Remarks 4.24. 1. This theorem holds without condition of integral closedness, but beware of the following fact: if some $b \in K^* \setminus R^*$ is integral over R, then $\mathrm{Icl}_K(1) \ni b$ and $0 \leqslant \varphi(b)$; however $\mathrm{Icl}(1) = R$ and $\varphi(b)$ is realised as a nonbasic divisor. An example for this is $R = \mathbb{Q}[t^2,t^3]$, $b = \frac{t^3}{t^2}$, $\varphi(b) = \varphi(t^3) - \varphi(t^2)$, $\mathrm{Icl}(t^3) = \langle t^3,t^4\rangle_R$, $\mathrm{Icl}(t^2) = \langle t^2,t^3\rangle_R$.

2. If every positive divisor is basic, then one can show the domain to be Prüfer.

3. When R is a Prüfer domain, the Lorenzen divisor group $\mathrm{Lor}(R)$ coincides with the usual divisor group, the group of finitely generated fractional ideals defined by Dedekind and Kronecker. In fact, all finitely generated ideals are integrally closed in a Prüfer domain, so that $\mathrm{Icl}(a_1,\dots,a_k) = \langle a_1,\dots,a_k\rangle_R$.

4. The integral domain $R = \mathbb{Q}[t,u]$ is a gcd domain of dimension $\geqslant 2$, so that its divisibility group G is an ℓ-group. The domain R is not Prüfer and the Lorenzen divisor group is much greater than G: e.g. the ideal gcd of t^3 and u^3 in $\mathrm{Lor}(R)$ corresponds to the integrally closed ideal $\langle t^3,t^2u,tu^2,u^3\rangle$, whereas their gcd in R^* is 1, corresponding to the ideal $\langle 1\rangle$. In this case, we see that G is a proper quotient of $\mathrm{Lor}(R)$. ◇

The following corollary concentrates upon the cancellation property holding in ℓ-groups. Note that the integral closure of an integral finitely generated ideal in an integrally closed integral domain is equal to its integral closure in the field of fractions.

Corollary 4.25 (see Macaulay 1916, pages 108–109)**.** *Let* R *be an integrally closed integral domain. When* $\mathfrak{a}$ *is a finitely generated integral ideal* $\langle a_1,\dots,a_k\rangle_R$ *with* $a_1,\dots,a_k \in R^*$*, we let* $\overline{\mathfrak{a}} = \mathrm{Icl}(a_1,\dots,a_k)$ *be the integral closure of* $\mathfrak{a}$*. Then, if* $\mathfrak{a}$*,* $\mathfrak{b}$*, and* $\mathfrak{c}$ *are nonzero finitely generated integral ideals, we have the cancellation property*

$$\overline{\mathfrak{a}\,\mathfrak{b}} \supseteq \overline{\mathfrak{a}\,\mathfrak{c}} \implies \overline{\mathfrak{b}} \supseteq \overline{\mathfrak{c}}.$$

This corollary is a key result for “containment in the wider sense” as considered by Leopold Kronecker (1883) (see Penchèvre (preprint), pages 36–37). H. S. Macaulay (1916) gives a proof based on the multivariate resultant. We may also deduce it as a consequence of Prüfer's Theorem 5.5 (see Item (2) of Remarks 5.9, compare Prüfer 1932, § 6, Krull 1935, Nr. 46).

5 Equivariant systems of ideals and Prüfer's theorem

In this section, we account for another way to obtain the Lorenzen group associated with an equivariant system of ideals for an ordered group (Definition 4.7). This way has historical precedence, as it dates back to Lorenzen's Ph.D. thesis (1939), that builds on earlier work by Prüfer (1932). In the case of the system of Dedekind ideals, this approach provides another way of understanding the Lorenzen divisor group of an integral domain.

5.1 Prüfer's properties Γ and Δ

Let us now express cancellativity of the meet-monoid as a property of the equivariant system of ideals itself (a.k.a. "endlich arithmetisch brauchbar", "e.a.b.", see Comment 4.3), as in Prüfer 1932, § 3.

Lemma 5.1 (Prüfer's Property Γ of cancellativity)**.** *Let $\rhd$ be an equivariant system of ideals for an ordered group G. The associated meet-monoid M is cancellative, i.e. $\bigwedge(A+X) =_M \bigwedge(B+X)$ implies $\bigwedge A =_M \bigwedge B$, if and only if the following property holds:*

$$(\text{※}) \qquad A + X \leqslant_\rhd b + X \implies A \rhd b.$$

This holds if and only if $A + X \leqslant_\rhd X \implies A \rhd 0$.

Proof. The second implication, a particular case of the first one, implies the first one by equivariance. Let us work with the first implication. Cancellativity means that if $A+X \leqslant_\rhd B+X$, then $A \leqslant_\rhd B$. Property (※) is necessary: take $B = \{b\}$. Let us show that it is sufficient. Assume $A + X \leqslant_\rhd B + X$ and let $b \in B$. As $B \rhd b$, we have $B + X \leqslant_\rhd b + X$, whence $A + X \leqslant_\rhd b + X$. So $A \rhd b$. Since this holds for each $b \in B$, we get $A \leqslant_\rhd B$. □

Remark 5.2. The original version of Prüfer's Property Γ states, for a set-theoretical star-operation $A \mapsto A_r$ on nonempty finitely enumerated subsets of G as considered in Item (2) of Remarks 1.8, the cancellation property $(A + X)_r \supseteq (B + X)_r \implies A_r \supseteq B_r$. ⋄

Prüfer's Theorem 5.5 will reveal the significance of the following definition. We shall check in Proposition 5.8 that it agrees with Definition 4.1.

Definition 5.3 (Prüfer's Property Δ of integral closedness)**.** Let $\rhd$ be an equivariant system of ideals for an ordered group G. The group G is $\rhd$*-closed* if $X \leqslant_\rhd b + X \implies 0 \leqslant_G b$.

Remark 5.4. The original version of Prüfer's Property Δ states the cancellation property $X_r \supseteq b + X_r \implies 0 \leqslant_G b$. $\diamond$

5.2 Forcing cancellativity: Prüfer's theorem

When the monoid M in Theorem 1.10 is not cancellative, it is possible to adjust the equivariant system of ideals in order to straighten the situation. A priori, it suffices to consider the Grothendieck ℓ-group of M (Theorem 3.3). But we have to see that this corresponds to an equivariant system of ideals for G, and to provide a description for it. The following theorem is a reformulation of Prüfer's theorem (Prüfer 1932, § 6). We follow the proofs in Jaffard 1960, pages 42–43. In fact, the language of systems of ideals simplifies the proofs. Jaffard's statement corresponds to Items (1) and (4), and Items (2) and (3) have been added by us.

Theorem 5.5 (Prüfer's theorem)**.** *Let $\rhd$ be an equivariant system of ideals for an ordered group G. We define the relation $\rhd_{\mathrm{a}}$ between $\mathrm{P}_{\mathrm{fe}}^*(G)$ and G by*

$$A \rhd_{\mathrm{a}} b \overset{\text{def}}{\iff} \exists X \in \mathrm{P}_{\mathrm{fe}}^*(G) \;\; A + X \leqslant_{\rhd} b + X.$$

1. *The relation $\rhd_{\mathrm{a}}$ is an equivariant system of ideals for G, and the associated meet-monoid M_{a} (Theorem 1.10) is cancellative.*
2. *The meet-monoid M_{a} embeds into its Grothendieck ℓ-group H_{a}.*
3. *The system $\rhd_{\mathrm{a}}$ is the finest equivariant system of ideals $\rhd'$ coarser than $\rhd$ such that M_{a} is cancellative, i.e. forcing*

$$A + X \leqslant_{\rhd'} b + X \quad\implies\quad A \rhd' b.$$

4. *The implication $a \rhd_{\mathrm{a}} b \implies a \leqslant_G b$ holds if (and only if) G is $\rhd$-closed (Definition 5.3); in this case, G embeds into H_{a}.*

Proof. Note that if $A+X \leqslant_{\rhd} b+X$, then $A+X+Y \leqslant_{\rhd} b+X+Y$ for all Y (see the proof of Theorem 1.10 on page 8). This makes the definition of $\rhd_{\mathrm{a}}$ very easy to use. In the proof below, we have two preorder relations on $\mathrm{P}_{\mathrm{fe}}^*(G)$ ($\leqslant_{\rhd}$ and $\leqslant_{\mathrm{a}}$), and we shall proceed as if they were order relations (i.e. we shall descend to the quotients).

(1) • *Reflexivity and preservation of order* (of the relation $\rhd_{\mathrm{a}}$). Setting $X = \{0\}$ in the definition of $\rhd_{\mathrm{a}}$ shows that $a \leqslant_G b$ implies $a \rhd_{\mathrm{a}} b$.

• *Monotonicity.* It suffices to note that the elements $(A, A') + X$ and $A + X, A' + X$ of $\mathrm{P}_{\mathrm{fe}}^*(G)$ are the same: therefore, if $A + X \leqslant_{\rhd} b + X$, then $(A, A') + X \leqslant_{\rhd} b + X$.

• *Transitivity.* Assume $A \rhd_{\mathrm{a}} c$ and $A, c \rhd_{\mathrm{a}} b$: we have an X such that $A + X \leqslant_{\rhd} c + X$ and a Y such that $(A, c) + Y \leqslant_{\rhd} b + Y$; these inequalities imply respectively $A + X + Y \leqslant_{\rhd} c + X + Y$ and $A + X + Y, c + X + Y \leqslant_{\rhd} b + X + Y$; we deduce $A + X + Y \leqslant_{\rhd} b + X + Y$, so that $A \rhd_{\mathrm{a}} b$.

• *Equivariance.* If $A \rhd_{\mathrm{a}} b$, we have an X such that $A + X \leqslant_{\rhd} b + X$, so that, since $\leqslant_{\rhd}$ is equivariant, $x + A + X \leqslant_{\rhd} x + b + X$. This yields $x + A \rhd_{\mathrm{a}} x + b$.

• *Cancellativity* (of the meet-monoid M_{a}). Let us denote by $\leqslant_{\mathrm{a}}$ the order relation associated to $\rhd_{\mathrm{a}}$. By Lemma 5.1, it suffices to suppose that $A + X \leqslant_{\mathrm{a}} X$ and to deduce that $A \rhd_{\mathrm{a}} 0$. But the hypothesis means that $A + X \rhd_{\mathrm{a}} x$ for each $x \in X$, i.e. that for each $x \in X$ there is a Y_x such that $A + X + Y_x \leqslant_{\rhd} x + Y_x$. Let $Y = \sum_{x \in X} Y_x$: we have $A + X + Y \leqslant_{\rhd} x + Y$. As $x \in X$ is arbitrary, $A + X + Y \leqslant_{\rhd} X + Y$: this yields $A \rhd_{\mathrm{a}} 0$ as desired.

(2) Follows from Item (1) by Theorem 3.3.

(3) This is immediate from the definition of $\rhd_{\mathrm{a}}$: it has been defined in a minimal way as coarser than $\rhd$ and forcing the cancellativity of the monoid M_{a} as characterised in Lemma 5.1.

(4) If $a \rhd_{\mathrm{a}} b$, then we have an X such that $a + X \leqslant_{\rhd} b + X$, so that by a translation $X \leqslant_{\rhd} (b - a) + X$. The hypothesis on G yields $0 \leqslant_G b - a$. By a translation, we get $a \leqslant_G b$. □

Comment 5.6. This is the approach proposed in Lorenzen 1939, § 4. Lorenzen abandoned it in favour of Definition 4.1 for the purpose of generalising his theory to noncommutative groups. See also Comment 4.3 and Comments 4.18. ◇

Definition 5.7 (see Lorenzen 1939, page 546, or Jaffard 1960, II, § 2, 2)**.** Let $\rhd$ be an equivariant system of ideals for an ordered group G. The ℓ-group in Item (2) of Theorem 5.5 is the *Lorenzen group* associated with $\rhd$.

Proposition 5.8 (Lorenzen 1950, Satz 27)**.** *The definition of $A \rhd_{\mathrm{a}} 0$ in Theorem 5.5 agrees with Definition 4.1 of $A \vdash_{\rhd} 0$. So Definition 5.3 of $\rhd$-closedness agrees with that of Definition 4.1, and Definition 5.7 of the Lorenzen group agrees with that of Definition 4.7.*

Proof. This proposition expresses that, given an equivariant system of ideals $\rhd$ for an ordered group G and an $A \in \mathrm{P}^*_{\mathrm{fe}}(G)$, we have $A \vdash_{\rhd} 0$ (Definition 4.1) if and only if $A + X \leqslant_{\rhd} X$ for some $X \in \mathrm{P}^*_{\mathrm{fe}}(G)$. First, $A + Y \leqslant_{\rhd_x} Y$ and $A + Z \leqslant_{\rhd_{-x}} Z$ imply $A + X \leqslant_{\rhd} X$ for some X. In fact, we have p and q such that

$$\begin{aligned} &A + Y, A + Y + x, \ldots, A + Y + px \leqslant_{\rhd} Y \text{ and} \\ &A + Z, A + Z - x, \ldots, A + Z - qx \leqslant_{\rhd} Z \text{ hold,} \end{aligned}$$

which yield that for $z \in Z$, $j \leqslant q$, $y \in Y$, and $k \leqslant p$,

$$\begin{aligned} &A + Y + z - jx, \ldots, A + Y + z + (p - j)x \leqslant_{\rhd} Y + z - jx \text{ and} \\ &A + y + Z + kx, \ldots, A + y + Z + (k - q)x \leqslant_{\rhd} y + Z + kx \text{ hold,} \end{aligned}$$

so that $A + X \leqslant_{\rhd} X$ for $X = Y + Z + \{-qx, \ldots, px\}$.

In the other direction, assume that $A + X \rhd x_i$ for each x_i in $X = \{x_1, \ldots, x_m\}$. Let $x_{i,j} = x_i - x_j$ ($i < j \in [\![1..m]\!]$) and let us prove that $A \rhd_{\pm x_{1,2}, \pm x_{1,3}, \ldots, \pm x_{m-1,m}} 0$. In fact, for any system of constraints $(\epsilon_{1,2} x_{1,2}, \epsilon_{1,3} x_{1,3}, \ldots, \epsilon_{m-1,m} x_{m-1,m})$ with $\epsilon_{i,j} = \pm 1$, the elements x_i are linearly ordered in the associated meet-monoid M_ϵ. E.g. $x_1 \leqslant_{M_\epsilon} x_2 \leqslant_{M_\epsilon} \cdots \leqslant_{M_\epsilon} x_m$ holds, in which case

$$\bigwedge(A + x_1, \ldots, A + x_m) =_{M_\epsilon} \bigwedge(A + x_1) \leqslant_{M_\epsilon} x_1$$

holds, which yields $\bigwedge A \leqslant_{M_\epsilon} 0$ by a translation. □

Remarks 5.9. 1. Informally, the content of this proposition may be expressed as follows. By starting from $\rhd$ and by adding new pairs (A, b) such that $A \rhd' b$, on the one side Prüfer forces the cancellativity of the meet-monoid M_{a}, and on the other side Lorenzen forces $\rhd$ to become the restriction of an entailment relation (which is still an equivariant system of ideals, as follows trivially from Lorenzen's definition). In fact, each approach realises both aims, but each one realises its own aim in a minimal way. So they give the same result.

2. Theorem 5.5 allows one to recover the results of Theorem 4.20 and of Theorem 4.22 in the Prüfer approach. In particular, one may check that $A\,(\rhd_{\mathrm{d}})_{\mathrm{a}}\, b$ holds if and only if b is integral over the fractional ideal $\langle A \rangle_R$ (by applying the determinant trick, see Prüfer 1932, § 6). One may also check that the hypothesis in Item (4) of Theorem 5.5 holds if and only if R is integrally closed. In this case, the elements $\geqslant 1$ of the meet-monoid M_{a} in Item (2) of Theorem 5.5 can be identified with the integrally closed ideals generated by nonempty finitely enumerated subsets A of R^*; therefore Item (1) of Theorem 5.5 yields the cancellation property stated in Corollary 4.25. ◇

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References

Jean-Yves Béziau. Les axiomes de Tarski. In *La philosophie en Pologne: 1918-1939*, edited by Roger Pouivet and Manuel Rebuschi, 135–149. Analyse et philosophie, J. Vrin, Paris, 2006. Actes du colloque tenu à Nancy du 21 au 22 novembre 2003.

Errett Bishop. *Foundations of constructive analysis.* McGraw-Hill, New York, 1967.

Nicolas Bourbaki. *Elements of mathematics. Algebra I: Chapters 1-3.* Hermann, Paris, 1974.

Douglas Bridges and Fred Richman. *Varieties of constructive mathematics.* London mathematical society lecture note series, 97, Cambridge university press, Cambridge, 1987.

Jan Cederquist and Thierry Coquand. Entailment relations and distributive lattices. In *Logic Colloquium '98: proceedings of the annual European summer meeting of the Association for symbolic logic, held in Prague, Czech Republic, August 9–15, 1998*, edited by Samuel R. Buss, Petr Hájek, and Pavel Pudlák, 127–139. Lecture notes in logic, 13, Association for symbolic logic, Urbana, 2000.

Roberto L. O. Cignoli, Itala M. L. D'Ottaviano, and Daniele Mundici. *Algebraic foundations of many-valued reasoning.* Trends in Logic: Studia Logica Library, 7, Kluwer, Dordrecht, 2000.

A. H. Clifford. Partially ordered abelian groups. *Ann. of Math. (2)*, 41, 465–473, 1940. doi:10.2307/1968728.

Thierry Coquand, Henri Lombardi, and Stefan Neuwirth. Regular entailment relations, 2018. Preprint.

Michel Coste, Henri Lombardi, and Marie-Françoise Roy. Dynamical method in algebra: effective Nullstellensätze. *Ann. Pure Appl. Logic*, 111(3), 203–256, 2001. doi:10.1016/S0168-0072(01)00026-4.

R. Dedekind. Über Zerlegungen von Zahlen durch ihre grössten gemeinsamen Theiler. In *Fest-Schrift der Herzoglichen Technischen Hochschule Carolo-Wilhelmina: dargeboten den naturwissenschaftlichen Theilnehmern an der 69. Versammlung deutscher Naturforscher und Ärzte*, edited by Heinr. Beckurts, 1–40. Friedrich Vieweg und Sohn, Braunschweig, 1897. `https://nbn-resolving.org/urn:nbn:de:gbv:084-09102209048`. Also in *Werke*, zweiter Band, 103–147.

Jean Dieudonné. Sur la théorie de la divisibilité. *Bull. Soc. Math. France*, 69, 133–144, 1941. `http://eudml.org/doc/86745`.

Gerhard Gentzen. Über die Existenz unabhängiger Axiomensysteme zu unendlichen Satzsystemen. *Math. Ann.*, 107(1), 329–350, 1933. `http://eudml.org/doc/159597`. Translation: *On the existence of independent axiom systems for infinite sentence systems*, in M. E. Szabo (ed.), *The collected papers of Gerhard Gentzen*, 29–52. Studies in logic and the foundations of mathematics, North-Holland, Amsterdam, 1969.

Paul Hertz. Über Axiomensysteme für beliebige Satzsysteme. II, Sätze höheren Grades. *Math. Ann.*, 89(1–2), 76–102, 1923. http://eudml.org/doc/158993.

Paul Jaffard. *Les systèmes d'idéaux.* Travaux et recherches mathématiques, IV, Dunod, Paris, 1960.

L. Kronecker. Zur Theorie der Formen höherer Stufen. *Königl. Preuss. Akad. Wiss. Berlin Sitzungsber.*, 957–960, 1883. http://bibliothek.bbaw.de/bibliothek-digital/digitalequellen/schriften/anzeige?band=10-sitz/1883-2. Also in *Werke*, zweiter Band, 419–424.

Wolfgang Krull. *Idealtheorie.* Ergebnisse der Mathematik und ihrer Grenzgebiete, 4(3), Springer, Berlin, 1935.

Henri Lombardi and Claude Quitté. *Commutative algebra: constructive methods. Finite projective modules.* Algebra and applications, 20, Springer, Dordrecht, 2015. Translated from the French (Calvage & Mounet, Paris, 2011, revised and extended by the authors) by Tania K. Roblot.

Paul Lorenzen. Abstrakte Begründung der multiplikativen Idealtheorie. *Math. Z.*, 45, 533–553, 1939. http://eudml.org/doc/168865.

Paul Lorenzen. Über halbgeordnete Gruppen. *Math. Z.*, 52, 483–526, 1950. http://eudml.org/doc/169131.

Paul Lorenzen. Algebraische und logistische Untersuchungen über freie Verbände. *J. Symbolic Logic*, 16, 81–106, 1951. http://www.jstor.org/stable/2266681. Translation by Stefan Neuwirth: *Algebraic and logistic investigations on free lattices*, http://arxiv.org/abs/1710.08138.

Paul Lorenzen. Teilbarkeitstheorie in Bereichen. *Math. Z.*, 55, 269–275, 1952. http://eudml.org/doc/169251.

Paul Lorenzen. Die Erweiterung halbgeordneter Gruppen zu Verbandsgruppen. *Math. Z.*, 58, 15–24, 1953. http://eudml.org/doc/169331.

F. S. Macaulay. *The algebraic theory of modular systems.* Cambridge tracts in mathematics and mathematical physics, 19, Cambridge university press, Cambridge, 1916.

Ray Mines, Fred Richman, and Wim Ruitenburg. *A course in constructive algebra.* Universitext, Springer, New York, 1988.

Stefan Neuwirth. Lorenzen's reshaping of Krull's Fundamentalsatz for integral domains (1938–1953), 2018. Preprint.

Erwan Penchèvre. La théorie arithmétique des grandeurs algébriques de Kronecker (1882), preprint. http://arxiv.org/abs/1801.04327.

Heinz Prüfer. Untersuchungen über Teilbarkeitseigenschaften in Körpern. *J. Reine Angew. Math.*, 168, 1–36, 1932. http://eudml.org/doc/149823.

Davide Rinaldi, Peter Schuster, and Daniel Wessel. Eliminating disjunctions by disjunction elimination. *Bull. Symb. Log.*, 23(2), 181–200, 2017. http://www.jstor.org/stable/44259447.

Dana Scott. Completeness and axiomatizability in many-valued logic. In *Proceedings of the Tarski symposium: held at the university of California, Berkeley, June 23–30, 1971*, edited by Leon Henkin, John Addison, C. C. Chang, William Craig, Dana Scott, and Robert Vaught, 411–435. Proceedings of symposia in pure mathematics, XXV, American mathematical society, Providence, 1974.