Lorenzen's proof of consistency for elementary number theory

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Abstract

We present a manuscript of Paul Lorenzen that provides a proof of consistency for elementary number theory as an application of the construction of the free countably complete pseudocomplemented semilattice over a preordered set. This manuscript lies in the Oskar-Becker-Nachlass at the Philosophisches Archiv of Universität Konstanz, file OB 5-3b-5. It has probably been written between March and May 1944. We also compare this proof to Gentzen's and Novikov's, and provide a translation of the manuscript.

Keywords: Paul Lorenzen, consistency of elementary number theory, free countably complete pseudocomplemented semilattice, inductive definition, ω-rule.

We present a manuscript of Paul Lorenzen that arguably dates back to 1944. A translation is included as an appendix with the kind permission of Lorenzen's daughter, Jutta Reinhardt.

It provides a constructive proof of consistency for elementary number theory by showing that it is a part of a trivially consistent cut-free calculus. The proof resorts only to the inductive definition of formulas and theorems.

More precisely, Lorenzen proves the admissibility of cut by double induction, on the cut formula and on the complexity of the derivations, without using any ordinal assignment, contrary to the presentation of cut elimination in most standard texts on proof theory.

Prior to that, he proposes to define a countably complete pseudocomplemented semilattice as a deductive calculus, and shows how to present it for constructing the free countably complete pseudocomplemented semilattice over a given preordered set.

In fact, he arrives at the understanding that the existence of this free kind of lattice captures the formal content of consistency, the more so as he has come to know that the existence of another kind of lattice captures the formal content of ideal theory. In this way, lattice theory provides a bridge between algebra and logic: by the concept of preorder, the divisibility of elements in a ring becomes commensurate with the material implication of numerical propositions; the lattice operations give rise to the ideal elements in algebra and to the compound propositions in logic.

The manuscript has not been published because it has been superseded by Lorenzen's 'Algebraische und logistische Untersuchungen über freie Verbände' that appeared in 1951 in *The journal of symbolic logic*. These 'Algebraic and logistic investigations on free lattices' have immediately been recognised as a landmark in the history of infinitary proof theory, but their approach and method of proof have not been incorporated into the corpus of proof theory.

1. The beginnings. In 1938, Paul Lorenzen defends his Ph.D. thesis under the supervision of Helmut Hasse at Göttingen, an 'Abstract foundation of the multiplicative ideal theory', i.e. a foundation of divisibility theory upon the theory of cancellative monoids. He is in a process of becoming more and more aware that lattice theory is the right framework for his research. Lorenzen (*1939a*, footnote on p. 536) thinks of understanding a system of ideals as a lattice, with a reference to *Köthe 1937*; in the definition of a semilattice-ordered monoid on p. 544, he credits Dedekind's two seminal articles of 1897 and 1900 for developing the concept of lattice. On 6 July 1938 he reports to Hasse: 'Momentarily, I am at making a lattice-theoretic excerpt for Köthe.'[1] He also reviews several articles on this subject for the Zentralblatt, *Birkhoff 1938* to start with, then *Klein 1939* and *George 1939* which both introduce semilattices, *Whitman 1941* which studies free lattices. He also knows about the representation theorem for boolean algebras in *Stone 1936* and he discusses the axioms for the arithmetic of real numbers in *Tarski 1937* with Heinrich Scholz.[2]

In 1939, he becomes assistant to Wolfgang Krull at Bonn. During World War II, he serves first as a soldier and then, from 1942 on, as a teacher at the naval college Wesermünde. He devotes his 'off-duty evenings all alone on my own'[3] to mathematics with the goal of habilitating. On 25 April 1944, he writes

[1] ‚Augenblicklich bin ich dabei, für Köthe einen verbandstheoretischen Exzerpt zu machen' (Helmut-Hasse-Nachlass, Niedersächsische Staats- und Universitätsbibliothek Göttingen, Cod. Ms. H. Hasse 1:1022).

[2] See the collection of documents grouped together by Scholz under the title '*Paul Lorenzen*: Gruppentheoretische Charakterisierung der reellen Zahlen [Group theoretic characterisation of the real numbers]' and deposited at the Bibliothek des Fachbereichs Mathematik und Informatik of the Westfälische Wilhelms-Universität Münster, as well as several letters filed in its Scholz-Archiv, the earliest dated 7 April 1944, `http://www.uni-muenster.de/IVV5WS/ScholzWiki/doku.php?id=scans:blogs:hb-01-1040`, accessed 14 March 2017.

[3] ‚ganz allein auf mich gestellt – [...] die dienstfreien Abende' (carbon copy of a letter to

to his advisor that '[...] it became clear to me—about 4 years ago—that an ideal system is nothing but a semilattice.'[4]

He will later recall a talk by Gerhard Gentzen on the consistency of elementary number theory in 1937 or 1938 as a trigger for his discovery that the reformulation of ideal theory in lattice-theoretic terms reveals that his 'algebraic works [...] were concerned with a problem that had *formally* the same structure as the problem of freedom from contradiction of the classical calculus of logic';[5] compare also his letter to Eckart Menzler-Trott (*2001*, p. 260).

In his letter of 13 March 1944 he announces: 'Subsequently to an algebraic investigation of orthocomplemented semilattices, I am now trying to get out the connection of these questions with the freedom from contradiction of classical logic. [...] actually I am much more interested into the algebraic side of proof theory than into the purely logical.'[6] The concept of 'orthocomplementation'[7] (see p. 15 for the definition) must have been motivated by logical negation from the beginning. On the one hand, such lattices correspond to the calculus of sequents considered by Gentzen (*1936*, section IV), who shows that a given derivation can be transformed into a derivation 'in which the connectives $\vee$, $\exists$ and $\supset$ no longer occur' and provides a proof of consistency for this calculus (see section 3 below). On the other hand, note that Lorenzen reviewed *Ogasawara 1939* for the Zentralblatt.

2. The 1944 manuscript. The result of this investigation can be found in 'Ein halbordnungstheoretischer Widerspruchsfreiheitsbeweis'.[8]

We believe that this manuscript is the one that he assertedly sends to Wilhelm Ackermann, Gentzen, Hans Hermes and Heinrich Scholz between March and May 1944, and for which he gets a dissuasive answer from Gentzen, dated 12 September 1944: 'I have looked through your attempt at a consistency proof,

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Krull, 13 March 1944, Paul-Lorenzen-Nachlass, Philosophisches Archiv, Universität Konstanz, PL 1-1-131).

[4] ‚mir vor etwa 4 Jahren – [...] klar wurde, daß ein Idealsystem nichts anderes als ein Halbverband ist' (carbon copy of a letter to Krull, PL 1-1-132).

[5] ‚[...] meine algebraischen Arbeiten [...] mit einem Problem beschäftigt waren, das *formal* die gleiche Struktur hatte wie das Problem der Widerspruchsfreiheit des klassischen Logikkalküls" (letter to Carl Friedrich Gethmann (*1991*, p. 76)).

[6] ‚Im Anschluß an eine algebraische Untersuchung über orthokomplementäre Halbverbände versuche ich jetzt, den Zusammenhang dieser Fragen mit der Widerspruchsfreiheit der klassischen Logik herauszubekommen. [...] ich selber eigentlich viel mehr an der algebraischen Seite der Beweistheorie interessiert bin als an der rein logischen' (PL 1-1-131).

[7] The terminology might be adapted from *Stone 1936*, where it has a Hilbert space background; today one says 'pseudocomplementation'.

[8] 'A proof of freedom from contradiction within the theory of partial order', Oskar-Becker-Nachlass, Philosophisches Archiv, Universität Konstanz, OB 5-3b-5, `https://archive.org/details/lorenzen-ein_halbordnungstheoretischer_widerspruchsfreiheitsbeweis`. The file OB 5-3b consists of documents related to Lorenzen, the oldest being the 1944 manuscript and the youngest a letter from 1951. Lorenzen and Becker are both at Bonn from 1945 to 1956 and have been in close contact since at least 1947: see the letter to Gethmann (*1991*, p. 77).

not in detail, for which I lack the time. However I say this much: The consistency of number theory cannot be proven so simply.'[9]

Our identification of the manuscript is made on the basis of the following dating: Lorenzen mentions such a manuscript and its recipients in his letters to Scholz dated 13 May 1944 and 2 June 1944,[10] and in a postcard to Hasse dated 25 July 1945;[11] a letter by Ackermann dated 11 November 1946 states that he lost a manuscript by Lorenzen 'at the partial destruction of his flat by bombs'.[12] Our identification is also consistent with the content of Lorenzen's letter to Menzler-Trott mentioned above. On the other hand, we have not found any hint at another manuscript by Lorenzen for which it could have been mistaken.[13] The generalisation of his proof of consistency to ramified type theory is first mentioned in a letter from Scholz to Bernays dated 11 December 1945:[14] it corresponds to the manuscript 'Die Widerspruchsfreiheit der klassischen Logik mit verzweigter Typentheorie' and is the future part II of his 1951 article.

This manuscript renews the relationship between logic and lattice theory: whereas boolean algebras were originally conceived for modeling the classical calculus of propositions, and Heyting algebras for modeling the intuitionistic one, here logic comes at the rescue of lattice theory for studying countably complete pseudocomplemented semilattices. They are described as deductive calculuses[15] on their own, without any reference to a larger formal framework:[16] this conception dates back to the 'systems of sentences' of Hertz (*1922*, *1923*). The rules of the calculus construct the free countably complete pseudocomplemented semilattice over a given preordered set by taking as axioms the inequalities in the set, by defining inductively formal meets and formal negations, and by introducing inequalities between the formal elements. The introduction rule for formal countable meets, stating that

$$\text{if } c \leqslant a_1, c \leqslant a_2, \ldots, \text{ then } c \leqslant \bigwedge M, \text{ where } M = \{a_1, a_2, \ldots\}$$

(rule c on p. 17), stands out: it has an infinity of premisses, so that it is an 'ω-rule' in today's terminology. Lorenzen's boldness is most probably due to his

[9]The letter is reproduced in *Menzler-Trott 2001*, p. 372, and translated in *Menzler-Trott 2007*.

[10]http://www.uni-muenster.de/IVV5WS/ScholzWiki/doku.php?id=scans:blogs:hb-01-1036 and PL 1-1-138.

[11]Cod. Ms. H. Hasse 1:1022.

[12]„So ist auch ein Manuskript, das Sie mir seiner Zeit zuschickten, bei der teilweisen Zerstörung meiner Wohnung durch Bomben verschwunden“ (PL 1-1-125).

[13]The 'unpublished' manuscript 'Ein finiter Logikkalkül' mentioned by Lorenzen (*1948*, p. 20) may be dated to 1947 even if we have not spotted a copy of it: the review given there shows that it corresponds to a thread of research described in a letter to Bernays dated 21 February 1947 (Hs 975:2950).

[14]ETH-Bibliothek, Hochschularchiv, Hs 975:4111.

[15]We prefer this plural with Curry (*1958*).

[16]In contradistinction to the 'consequence relation' of Tarski (*1930*) which presupposes set theory.

training in algebra, where such a rule is very natural, so that when he arrives at a clear constructive understanding of ideal theory, he has also got a clear constructive understanding of the ω-rule.

Sundholm (*1983*) and Feferman (*1986*) provide a historical account of such rules. Hilbert (*1931b*, *a*) states a restricted ω-rule, in the sense that its premisses must be decidable (i.e. numerical), with the motivation of, respectively, proving the completeness of arithmetic and the law of excluded middle. Lorenzen makes no reference to these articles but, as he states in the 1945 manuscript 'Die Widerspruchsfreiheit der klassischen Logik mit verzweigter Typentheorie', he shares Hilbert's appreciation "that its applications occur to the effect of a 'finitary deduction'".[17]

Lorenzen's introduction of the unrestricted ω-rule happens in a setting that can be sketched as follows. Elementary number theory is an informal theory whose goal is to survey numerical truths. It may be formalised into a 'mechanical' (recursive) calculus in which the rule of complete induction plays a central rôle. The statement of this rule is complex from a logical point of view because of the presence of a free variable, of a universal quantifier, or of an implication. Gödel's theorem expresses that in its presence, numerical truth is out of scope of the calculus.

The ω-rule appears as an analysis of this complexity: the rule of complete induction is the derivation of $(\mathfrak{x})\,\mathfrak{A}(\mathfrak{x})$ from $\mathfrak{A}(1)$ and $\mathfrak{A}(n) \to \mathfrak{A}(n+1)$ for every number n; a repeated application of *modus ponens* (cuts) yields $\mathfrak{A}(2), \mathfrak{A}(3), \ldots, \mathfrak{A}(n), \mathfrak{A}(n+1), \ldots$; therefore this rule is a particular case of the ω-rule that derives $(\mathfrak{x})\,\mathfrak{A}(\mathfrak{x})$ from $\mathfrak{A}(1), \mathfrak{A}(2), \ldots$. Conversely, the only expected effects of the ω-rule correspond to the rule of complete induction and to the rule of introduction of the universal quantifier. This rule has a very simple structure: its premisses are stated without further need of free variables and quantifiers; however, there are infinitely many.

In a letter to Bernays dated 2 April 1931, Gödel points out that such a rule presupposes a framework in which this infinity of premisses may be asserted: "the very complicated and problematical concept 'finitary proof' is assumed [...] without having been made mathematically precise" (see *Feferman, Dawson, Goldfarb, Parsons* et al. *2003*, p. 97). This framework is thus an informal one; and, as the proof of consistency rests on its reliability, this framework is to be the intuitionistic one, as Herbrand (*1931*, 'groupe D', p. 5) and Novikoff (*1943*, p. 231) state, i.e. the constructive one (*Lorenzen 1951*, p. 82). In this sense, a calculus including the ω-rule is of a different nature than a mechanical calculus. In fact, neither the Hilbert program nor Lorenzen's proof of consistency take place in a mechanical formal system.

The proof that the calculus thus defined is a countably complete pseudocom-

[17] „daß ihre Anwendung im Sinne des ‚finiten Schließens' geschieht" ('The consistency of classical logic with ramified type theory'; a version of this manuscript can be found in Niedersächsische Staats- und Universitätsbibliothek Göttingen, Cod. Ms. G. Köthe M 10).

plemented semilattice illustrates, as Lorenzen realises a posteriori,[18] that the strategy of Gentzen's dissertation (*1934, IV, § 3*) for proving the consistency of elementary number theory without complete induction may be maintained for proving the consistency of all of elementary number theory: the introduction rules (rules a to f on p. 17) introduce inequalities for formal elements of increasing complexity, i.e. no inequality can result from a detour; then the corresponding elimination rules (rules γ to ε on p. 17) are shown to hold by an induction on the complexity of the introduced inequality (in Lorenzen's later terminology, one would say that these rules are shown to be 'admissible' and can be considered as resulting from an 'inversion principle'); at last transitivity of the preorder, i.e. the cut rule (rule β on p. 17: if $a \leqslant b$ and $b \leqslant c$, then $a \leqslant c$), is established by proving a stronger rule through an induction on the construction of the cut element b nested with inductions on the complexity of the derivation of the rule's premisses.

The inductions used here are the ones accurately described by Jacques Herbrand (*1930*, pp. 4–5) after having been emphasised by David Hilbert (*1928*, p. 76): the first proceeds along the construction of formulas starting from prime formulas through rules, and has no special name (it will be called "formula induction" in *Lorenzen 1951*); the second proceeds along the construction of theorems starting from prime theorems through deduction rules, and is called 'premiss induction'.[19]

In other words, Lorenzen starts with a preordered set P, constructs the free countably complete pseudocomplemented semilattice K over P and emphasises conservativity, i.e. that no more inequalities come to hold among elements of P viewed as a subset of K than the ones that have been holding before.[20]

Then the consistency of elementary number theory with complete induction is established in § 3 by constructing the free countably complete pseudocomplemented semilattice over its 'prime formulas', i.e. the numerical formulas, viewed as a set preordered by material implication.

Note the presence of rule g on pp. 17 and 25, a contraction rule. This should be put in relation

- with the rôle of contraction, especially for steps 13. 5 1–13. 5 3 in *Gentzen*'s proofs of consistency (*1936*, *1974*);
- with the calculus of P. S. Novikoff (*1943*, lemma 6), in which contraction may be proved.

3. Comparison with Gentzen's proof of consistency. There are common points and differences with respect to the strategy developed by Gentzen

[18]This is how we interpret the beginning of the second paragraph p. 13: 'Without knowledge of [. . .]'.

[19]See *Lorenzen 1939b* for his interest in the foundation of inductive definitions.

[20]This is exactly the approach of Skolem (*1921*, § 2) for constructing the free lattice over a preordered set, in the course of studying the decision problem for lattices.

for proving the consistency of elementary number theory with complete induction. In his first proof, submitted in August 1935, withdrawn and finally published posthumously by Bernays in *1974* (after its translation by Szabo (*1969*)), Gentzen defines a concept of reduction procedure for a sequent and shows that such a procedure may be specified for every derivable sequent but not for the contradictory sequent $\rightarrow 1 = 2$. Let us emphasise two aspects of this concept.

- If the succedent of the sequent has the form $\forall x\, F(x)$, the next step of the reduction procedure consists in replacing it by $F(n)$, where n is a number to be chosen freely.

- A reduction procedure is defined as the specification of a sequence of steps for all possible free choices, with the requirement that the reduction terminates for every such choice.

In a letter to Bernays of 4 November 1935,[21] Gentzen visualises a reduction procedure as a tree whose every branch terminates.

The proof that a reduction procedure may be specified for every derivable sequent is by theorem induction. For this, a lemma is needed, claiming that if reduction procedurees are known for two sequents $\Gamma \rightarrow D$ and $D, \Delta \rightarrow C$, then a reduction procedure may be specified for their cut sequent $\Gamma, \Delta \rightarrow C$. The proof goes by induction on the construction of the cut formula D and traces the claim back to the same claim with the same cut formula, but with the sequent $D, \Delta \rightarrow C$ replaced by a sequent $D, \Delta^* \rightarrow C^*$ resulting from it after one or more reduction steps and the cut sequent replaced by $\Gamma, \Delta^* \rightarrow C^*$. By definition of the reduction procedure, this tracing back must terminate eventually.

This last kind of argument may be considered as an infinite descent in the reduction procedure. In his letter to Bernays, Gentzen seems to indicate that this infinite descent justifies an induction on the reduction procedure. As analysed by William W. Tait (*2015*), this would be an instance of the Bar theorem. But in his following letter, dated 11 December 1935,[22] he writes that '[his] proof is not satisfactory' and announces another proof, to be submitted in February 1936: in it, he defines the concept of reduction procedure for a derivation (and not for a sequent), associates inductively an ordinal to every derivation, and shows that a reduction procedure may be specified for every derivation by an induction on the ordinal.

Let us compare this strategy with Lorenzen's.

- The free choice is subsumed in a deduction rule, an ω-rule as described above (rules c and j on page 25).[23]

[21] Hs 975:1652, translated by von Plato (*2017*, pp. 241–244).
[22] Hs 975:1653, translated by von Plato (*2017*, page 244).
[23] Compare Bernays' suggestion in his letter to Gentzen of 9 May 1938, Hs 975:1661, translated by von Plato (*2017*, pp. 254–255).

- Elementary number theory is constructed as the cut-free derivations starting from the numerical formulas, so that it is trivially consistent, and the cut rule (rule k on p. 27) is shown to be admissible: if derivations are known for two sequents $\mathfrak{A} \to \mathfrak{B}$ and $\mathfrak{B} \to \mathfrak{C}$, then a cut-free derivation may be specified for their cut sequent $\mathfrak{A} \to \mathfrak{C}$ by a formula induction on the cut formula $\mathfrak{B}$ nested with several instances of a theorem induction.

In this way, Lorenzen's strategy may be used to realise the endeavour expressed by Tait (*2015*): 'the gap in Gentzen's argument is filled, not by the Bar Theorem, but by taking as the basic notion that of a [cut-free] deduction tree in the first place rather than that of a reduction tree'. His 1944 proof can thus be seen as a formal improvement on Gentzen's 1935 argument, which is all the more remarkable given Gentzen's reaction to Lorenzen's proof.

4. Comparison with Novikov's proof of consistency. P. S. Novikoff (*1939*, *1943*) introduces a intuitionistic calculus that contains an ω-rule (rule 6 on p. 233). In § 4, he defines the concept of 'regular formula' that expresses that the formula has a cut-free proof, and shows in § 8 that it is an explanation of classical truth. In fact, he proves essentially that cut ('the rule of inference') is admissible. This proof does not use any induction on the cut formula, contrary to Gentzen's and Lorenzen's proofs (see *Coquand 1995*). He obtains thus a 'point of view' from which the consistency of elementary arithmetic is proved.

5. Mathematical comments. On p. 19, the premiss induction that establishes rule γ is given the form of a *reductio ad absurdum*, but the reasoning may easliy be modified into a direct form.

The calculus N presented on p. 25 is in fact common to intuitionistic and classical arithmetic: recall that 'the connectives $\vee$, $\exists$ and $\supset$ no longer occur'.

On p. 25, the introduction of free variables seems useless in the presence of an ω-rule. Rule j and the corresponding elimination rule p may be omitted from the calculus at the affordable price of giving complete induction the less elegant form $\mathfrak{A}(1) \,\&\, (\mathfrak{x})\,\overline{\mathfrak{A}(\mathfrak{x}) \,\&\, \overline{\mathfrak{A}(\mathfrak{x}')}} \to (\mathfrak{x})\,\mathfrak{A}(\mathfrak{x})$.

6. Conclusion. Proof theory continues to focus on measures of complexity by ordinal numbers. The fact that Lorenzen does not resort to ordinals in his proof of consistency should be considered as a feature of his approach.

Lorenzen's article is remarkable for its metamathematical standpoint. A mathematical object is presented as a construction described by rules. A claim on the object is established by an induction that expresses the very meaning of the construction.

The relations between these objects, of the form of an inequality or of an implication, also admit such a presentation: it has the feature that the construction of a relation proceeds as accumulatively ('without detour', i.e. cut) as the construction of the formulas appearing in the relation. It is only in a second place that the corresponding elimination rules and the cut rule are shown to be admissible.

In number theory and for the free countably complete pseudocomplemented semilattice, the construction of a relation uses an ω-rule that is stronger than the rule of complete induction but requires infinitely many premisses, so that a relation corresponds to a well-founded tree.

Lorenzen's standpoint holds equally well for a logical calculus and for a lattice: 'logical calculuses are semilattices or lattices'.[24] The consistency of the logical calculus of elementary number theory is recognised as a consequence of the following fact: the way a preordered set embeds into the free countably complete pseudocomplemented semilattice it generates is conservative.

The philosophical significance of Lorenzen's approach to logic is addressed by Matthias Wille (*2013*, *2016*).

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[24] ‚Die Tatsache, daß die logischen Kalküle Halbverbände oder Verbände sind' (Lorenzen *1951*, p. 89).

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[P. LORENZEN]

Ein halbordnungstheoretischer Widerspruchsfreiheitsbeweis.

Die Dissertation von G. Gentzen enthält einen Wf-beweis der reinen Zahlentheorie ohne vollständige Induktion, der auf dem folgenden Grundgedanken beruht: jede herleitbare Sequenz muß sich auch ohne Umwege herleiten lassen, sodaß während der Herleitung nur die Verknüpfungen eingeführt werden, die unbedingt notwendig sind, nämlich diejenigen, die in der Sequenz selbst enthalten sind. In dem Wf-beweis der Zahlentheorie mit vollständiger Induktion tritt dieser Grundgedanke gegenüber anderen zurück. Ich möchte jedoch im folgenden zeigen, daß er allein genügt, auch diese Wf. zu erhalten.

Ohne Kenntnis der Dissertation von Gentzen bin ich auf diese Möglichkeit auf Grund einer halbordnungstheoretischen Frage gekommen. Diese lautete: wie läßt sich eine halbgeordnete Menge in einen orthokomplementären vollständigen Halbverband einbetten? Im allgemeinen sind mehrere solche Einbettungen möglich – unter den möglichen Einbettungen ist aber eine ausgezeichnet, nämlich die, welche sich in jede andere homomorph abbilden läßt. Die Existenz dieser ausgezeichneten Einbettung wird in § 2 bewiesen.

Um hieraus in § 3 den gesuchten Wf-beweis zu erhalten, ist nur noch eine Übersetzung des halbordnungstheoretischen Beweises in die logistische Sprache notwendig. Denn der Kalkül, den wir betrachten und auf den sich die üblichen Kalküle zurückführen lassen, ist in der ausgezeichneten Einbettung der halbgeordneten Menge der zahlentheoretischen Primformeln enthalten.

§ 1. Eine Menge M heißt halbgeordnet, wenn in M eine zweistellige Relation $\leqslant$ definiert ist, sodaß für die Elemente $a, b, \ldots$ von M gilt:

$$a \leqslant a$$
$$a \leqslant b,\ b \leqslant c \quad \Rightarrow \quad a \leqslant c.$$

Gilt $a \leqslant b$ und $b \leqslant a$, so schreiben wir $a \equiv b$.

Gilt $a \leqslant x$ für jedes $x \in$ M, so schreiben wir $a \leqslant$. Ebenso schreiben wir $\leqslant a$, wenn $x \leqslant a$ für jedes x gilt. ($\leqslant$ bedeutet also, daß $x \leqslant y$ für jedes $x, y \in$ M gilt.)

Eine halbgeordnete Menge M heißt Halbverband, wenn es zu jedem $a, b \in$ M ein $c \in$ M gibt, sodaß für jedes $x \in$ M gilt

$$x \leqslant a,\ x \leqslant b \quad \Longleftrightarrow \quad x \leqslant c.$$

c heißt die *Konjunktion* von a und b: $c \equiv a \wedge b$

Ein Halbverband M heißt orthokomplementär, wenn es zu jedem $a \in$ M ein $b \in$ M gibt, so daß für jedes $x \in$ M gilt

$$a \wedge x \leqslant \quad \Longleftrightarrow \quad x \leqslant b.$$

[P. LORENZEN]

A proof of freedom from contradiction within the theory of partial order.

The dissertation of G. Gentzen contains a proof of freedom from contradiction of elementary number theory without complete induction that relies on the following basic thought: every derivable sequent must also be derivable without detour, so that during the derivation only those connectives are being introduced that are absolutely necessary, i.e. those that are contained in the sequent itself. In the proof of freedom from contradiction of number theory with complete induction, this basic thought steps back with regard to others. I wish however to show in the following that it alone suffices to obtain also this freedom from contradiction.

Without knowledge of the dissertation of Gentzen I have arrived at this possibility on the basis of a semilattice-theoretic question. This question is: how may a partially ordered set be embedded into an orthocomplemented complete semilattice? In general several such embeddings are possible – but among the possible embeddings one is distinguished, i.e. the one which may be mapped homomorphically into every other. The existence of this distinguished embedding will be proved in § 2.

In order to obtain from this in § 3 the sought-after proof of freedom from contradiction, now just a translation of the semilattice-theoretic proof into the logistic language is necessary. For the calculus that we consider and to which the usual calculuses may be reduced is contained in the distinguished embedding of the partially ordered set of the number-theoretic prime formulas.

§ 1. A set M is called **partially ordered** if a binary relation $\leqslant$ is defined in M so that for the elements $a, b, \ldots$ of M holds:

$$a \leqslant a$$
$$a \leqslant b,\ b \leqslant c \quad \Rightarrow \quad a \leqslant c.$$

If $a \leqslant b$ and $b \leqslant a$ holds, then write we $a \equiv b$.

If $a \leqslant x$ holds for every $x \in$ M, then we write $a \leqslant$. We write as well $\leqslant a$ if $x \leqslant a$ holds for every x. ($\leqslant$ means thus that $x \leqslant y$ holds for every $x, y \in$ M.)

A partially ordered set M is called **semilattice** if to every $a, b \in$ M there is a $c \in$ M so that for every $x \in$ M holds

$$x \leqslant a,\ x \leqslant b \quad \Longleftrightarrow \quad x \leqslant c.$$

c is called the *conjunction* of a and b: $c \equiv a \wedge b$.

b heißt das *Orthokomplement* von a: $b \equiv \bar{a}$.

Ein Halbverband M heißt ω-vollständig, wenn es zu jeder abzählbaren Folge $M = a_1, a_2, \ldots$ in M ein $c \in$ M gibt, so daß für jedes $x \in$ M gilt:

$$(\text{für jedes } n\text{: } x \leqslant a_n) \quad \Longleftrightarrow \quad x \leqslant c.$$

c heißt die *Konjunktion der Elemente von M*: $c \equiv \bigwedge_n a_n \equiv \bigwedge_M$.

Sind M und M′ halbgeordnete Mengen, so heißt M ein Teil von M′, wenn M Untermenge von M′ ist und für jedes $a, b \in$ M genau dann $a \leqslant b$ in M′ gilt, wenn $a \leqslant b$ in M gilt.

Sind M und M′ halbgeordnete Mengen, so verstehen wir unter einer Abbildung von M in M′ eine Zuordnung, die jedem $a \in$ M ein $a' \in$ M′ zuordnet, so daß gilt

$$a \equiv b \quad \Rightarrow \quad a' \equiv b'.$$

Sind M und M′ orthokomplementäre ω-vollständige Halbverbände, so verstehen wir unter einem Homomorphismus von M in M′ eine Abbildung $\to$ von M in M′, so daß für jedes $a, b \in$ M und $a', b' \in$ M′ mit $a \to a'$ und $b \to b'$ gilt:

$$a \wedge b \to a' \wedge b'$$
$$\bar{a} \to \overline{a'}.$$

Ferner soll für jede Folge $M = a_1, a_2, \ldots$ in M und $M' = a'_1, a'_2, \ldots$ in M′ mit $a_n \to a'_n$ gelten:

$$\bigwedge_M \to \bigwedge_{M'}.$$

Wir wollen jetzt beweisen, daß es zu jeder halbgeordneten Menge P einen orthokomplementären ω-vollständigen Halbverband K gibt, so daß

1) P ein Teil von K ist,

2) K in jeden orthokomplementären ω-vollständigen Halbverband, der P als Teil enthält, homomorph abbildbar ist.

Wäre K′ ein weiterer orthokomplementärer ω-vollständiger Halbverband, der die Bedingungen 1) und 2) erfüllt, so gäbe es eine Zuordnung, durch die K in K′ und K′ in K homomorph abgebildet würde, d. h. K und K′ wären isomorph. K ist also durch die Bedingungen 1) und 2) bis auf Isomorphie eindeutig bestimmt. Wir nennen K den ausgezeichneten orthokomplementären ω-vollständigen Halbverband über P.

§ 2. Satz: *Über jeder halbgeordneten Menge gibt es den ausgezeichneten orthokomplementären ω-vollständigen Halbverband.*

Wir konstruieren zu der halbgeordneten Menge P eine Menge K auf folgende Weise:

A semilattice M is called **orthocomplemented** if to every $a \in$ M there is a $b \in$ M so that for every $x \in$ M holds

$$a \wedge x \leqslant \quad \Longleftrightarrow \quad x \leqslant b.$$

b is called the *orthocomplement* of a: $b \equiv \bar{a}$.

A semilattice M is called **ω-complete** if to every countable sequence $M = a_1, a_2, \ldots$ in M there is a $c \in$ M so that for every $x \in$ M holds:

$$(\text{for every } n\colon\ x \leqslant a_n) \quad \Longleftrightarrow \quad x \leqslant c.$$

c is called the *conjunction of the elements of M*: $c \equiv \bigwedge\limits_n a_n \equiv \bigwedge\limits_M$.

If M and M$'$ are partially ordered sets, then M is called **a part** of M$'$ if M is a subset of M$'$ and for every $a, b \in$ M $a \leqslant b$ holds in M$'$ exactly if $a \leqslant b$ holds in M.

If M and M$'$ are partially ordered sets, we understand by a **mapping** of M into M$'$ an assignment that to every $a \in$ M assigns an $a' \in$ M$'$ so that

$$a \equiv b \quad \Rightarrow \quad a' \equiv b'.$$

If M and M$'$ are orthocomplemented ω-complete semilattices, we understand by a **homomorphism** of M into M$'$ a mapping $\to$ of M into M$'$, so that for every $a, b \in$ M and $a', b' \in$ M$'$ with $a \to a'$ and $b \to b'$ holds:

$$a \wedge b \to a' \wedge b'$$
$$\bar{a} \to \overline{a'}.$$

Moreover, for every sequence $M = a_1, a_2, \ldots$ in M and $M' = a'_1, a'_2, \ldots$ in M$'$ with $a_n \to a'_n$ is to hold:

$$\bigwedge_M \to \bigwedge_{M'}.$$

We want to prove now that to every partially ordered set P there is an orthocomplemented ω-complete semilattice K so that

1) P is a part of K,

2) K may be mapped homomorphically into every orthocomplemented ω-complete semilattice that contains P as part.

If K$'$ were a further orthocomplemented ω-complete semilattice that fulfils conditions 1) and 2), then there would be an assignment by which K would be mapped homomorphically into K$'$ and K$'$ into K, i.e. K and K$'$ would be **isomorphic**. K is thus determined uniquely up to isomorphism by conditions 1) and 2). We call K **the distinguished orthocomplemented ω-complete semilattice over** P.

1) K enthalte die Elemente von P. (Diese nennen wir die Primelemente von K.)

2) K enthalte mit endlich vielen Elementen $a_1, a_2, \ldots, a_n$ auch die hieraus gebildete Kombination als Element. (Diese bezeichnen wir durch $a_1 \wedge a_2 \wedge \cdots \wedge a_n$).

3) K enthalte mit jedem Element a auch ein Element $\bar{a}$.

4) K enthalte mit jeder abzählbaren Folge M auch ein Element $\bigwedge\limits_M$.

Jedes Element von K läßt sich also eindeutig als Kombination $a_1 \wedge a_2 \wedge \cdots \wedge a_n$ von Primelementen und Elementen der Form $\bar{a}$ oder $\bigwedge\limits_M$ schreiben.

Wir definieren eine Relation $\leqslant$ in K auf folgende Weise:

1) Für Primelemente p, q gelte $p \leqslant q$ in K, wenn $p \leqslant q$ in P gilt. (Diese Relationen nennen wir die Grundrelationen.)

2) Es soll jede Relation $\leqslant$ in K gelten, die sich aus den Grundrelationen mit Hilfe der folgenden Regeln herleiten läßt:

$$a)\ \frac{c \leqslant a \quad c \leqslant b}{c \leqslant a \wedge b} \qquad d)\ \frac{a \leqslant c}{a \wedge b \leqslant c}$$

$$b)\ \frac{a \wedge c \leqslant}{c \leqslant \bar{a}} \qquad e)\ \frac{a \leqslant b}{a \wedge \bar{b} \leqslant c}$$

$$c)\ \frac{c \leqslant a_1,\ \ldots,\ c \leqslant a_n,\ \ldots}{c \leqslant \bigwedge\limits_M} \qquad f)\ \frac{a_n \wedge b \leqslant c}{\bigwedge\limits_M \wedge b \leqslant c}$$

$$(M = a_1, a_2, \ldots)$$

$$g)\ \frac{a \wedge a \wedge b \leqslant c}{a \wedge b \leqslant c}$$

Wir nennen die Relationen über dem Strich die Prämissen der Relation unter dem Strich.

Wir haben jetzt zunächst zu zeigen, daß K ein orthokomplementärer ω-vollständiger Halbverband bezügl. der Relation $\leqslant$ ist. Dazu müssen wir beweisen

$$\begin{array}{llcll}
\alpha) & a \leqslant a & & & \\
\beta) & a \leqslant b,\ b \leqslant c & \Rightarrow & a \leqslant c & \\
\gamma) & c \leqslant a \wedge b & \Rightarrow & c \leqslant a & \\
\delta) & c \leqslant \bar{a} & \Rightarrow & a \wedge c \leqslant & \\
\varepsilon) & c \leqslant \bigwedge\limits_M & \Rightarrow & c \leqslant a_n & (M = a_1, a_2, \ldots)
\end{array}$$

§ 2. Theorem: *There is over every partially ordered set the distinguished orthocomplemented ω-complete semilattice.*

We construct for the partially ordered set P a set K in the following way:

1) Let K contain the elements of P. (These we call the **prime elements** of K.)

2) Let K contain with finitely many elements $a_1, a_2, \dots, a_n$ also the **combination** formed out of these as element. (These we designate by $a_1 \wedge a_2 \wedge \dots \wedge a_n$.)

3) Let K contain with every element a also an element $\bar{a}$.

4) Let K contain with every countable sequence M also an element $\bigwedge_M$.

Every element of K may thus be written uniquely as combination $a_1 \wedge a_2 \wedge \dots \wedge a_n$ of prime elements and elements of the form $\bar{a}$ or $\bigwedge_M$.

We define a relation $\leqslant$ in K in the following way:

1) For prime elements p, q let $p \leqslant q$ hold in K if $p \leqslant q$ holds in P. (These relations we call the basic relations.)

2) Every relation $\leqslant$ that may be derived from the basic relations by the aid of the following rules is to hold in K:

$$a)\ \frac{c \leqslant a \quad c \leqslant b}{c \leqslant a \wedge b} \qquad\qquad d)\ \frac{a \leqslant c}{a \wedge b \leqslant c}$$

$$b)\ \frac{a \wedge c \leqslant}{c \leqslant \bar{a}} \qquad\qquad e)\ \frac{a \leqslant b}{a \wedge \bar{b} \leqslant c}$$

$$c)\ \frac{c \leqslant a_1 \quad \cdots \quad c \leqslant a_n \quad \cdots}{c \leqslant \bigwedge_M} \qquad f)\ \frac{a_n \wedge b \leqslant c}{\bigwedge_M \wedge b \leqslant c}$$

$$(M = a_1, a_2, \dots)$$

$$g)\ \frac{a \wedge a \wedge b \leqslant c}{a \wedge b \leqslant c}$$

We call the relations above the line the **premisses** of the relation below the line.

We have now to show first that K is an orthocomplemented ω-complete semilattice w.r.t. the relation $\leqslant$. For this we must prove

$$\begin{array}{lrcl}
\alpha) & a \leqslant a & & \\
\beta) & a \leqslant b,\ b \leqslant c & \Rightarrow & a \leqslant c \\
\gamma) & c \leqslant a \wedge b & \Rightarrow & c \leqslant a \\
\delta) & c \leqslant \bar{a} & \Rightarrow & a \wedge c \leqslant \\
\varepsilon) & c \leqslant \bigwedge_M & \Rightarrow & c \leqslant a_n \quad (M = a_1, a_2, \dots)
\end{array}$$

Diese Eigenschaften zusammen mit a), b) und c) drücken nämlich aus, daß K ein orthokomplementärer ω-vollständiger Halbverband ist.

α) gilt für Primelemente. Gilt α) für a und b, so auch für $a \wedge b$ wegen

$$\dfrac{\dfrac{a \leqslant a}{a \wedge b \leqslant a} \quad \dfrac{b \leqslant b}{a \wedge b \leqslant b}}{a \wedge b \leqslant a \wedge b}$$

Gilt α) für jedes $a_n \in M$, so auch für $\bigwedge\limits_M$ wegen

$$\dfrac{\dfrac{a_1 \leqslant a_1}{\bigwedge\limits_M \leqslant a_1} \quad \cdots \quad \dfrac{a_n \leqslant a_n}{\bigwedge\limits_M \leqslant a_n} \quad \cdots}{\bigwedge\limits_M \leqslant \bigwedge\limits_M}$$

Gilt α) für a, so auch für $\bar{a}$, wegen

$$\dfrac{\dfrac{a \leqslant a}{a \wedge \bar{a} \leqslant}}{\bar{a} \leqslant \bar{a}}$$

Dadurch ist α) allgemein bewiesen.

Da β) am schwierigsten zu beweisen ist, nehmen wir zunächst γ).

Um γ) zu beweisen, haben wir zu zeigen, daß, wenn $c \leqslant a \wedge b$ herleitbar ist, dann auch stets $c \leqslant a$ herleitbar sein muß.

Wir führen den Beweis indirekt durch eine *transfinite Induktion.* Es sei $c \leqslant a \wedge b$ herleitbar, aber nicht $c \leqslant a$. Der letzte Schritt der Herleitung von $c \leqslant a \wedge b$ kann dann nicht sein $\dfrac{c \leqslant a \quad c \leqslant b}{c \leqslant a \wedge b}$ ebenfalls nicht $\dfrac{c_1 \leqslant c_2}{c_1 \wedge \overline{c_2} \leqslant a \wedge b}$ $(c = c_1 \wedge \overline{c_2})$ da dann sofort $\dfrac{c_1 \leqslant c_2}{c_1 \wedge \overline{c_2} \leqslant a}$ herleitbar wäre.

Für den letzten Schritt bleiben nur die Möglichkeiten

$$\dfrac{c_1 \leqslant a \wedge b}{c_1 \wedge c_2 \leqslant a \wedge b} \qquad \dfrac{c_1 \wedge c_1 \wedge c_2 \leqslant a \wedge b}{c_1 \wedge c_2 \leqslant a \wedge b} \; (c = c_1 \wedge c_2)$$

$$\dfrac{c_1 \wedge c' \leqslant a \wedge b \quad \cdots \quad c_n \wedge c' \leqslant a \wedge b \quad \cdots}{\bigwedge\limits_M \wedge c' \leqslant a \wedge b} \; \begin{pmatrix} M = a_1, a_2, \ldots \\ c = \bigwedge\limits_M \wedge c' \end{pmatrix}$$

Hier muß jetzt $c_1 \leqslant a$ bezw. $c_1 \wedge c_1 \wedge c_2 \leqslant a$ bezw. für mindestens ein n $c_n \wedge c' \leqslant a$ nicht herleitbar sein, da sonst sofort $c \leqslant a$ herleitbar wäre. In der Herleitung von $c \leqslant a \wedge b$ wäre also schon für eine Prämisse die Behauptung γ) falsch. Gehe ich in der Herleitung von einer Relation zu einer Prämisse über, von dieser wieder zu einer Prämisse usw., so bin ich nach endlich vielen Schritten

These properties together with a), b), and c) express in fact that K is an orthocomplemented ω-complete semilattice.

α) holds for prime elements. If α) holds for a and b, then also for $a \wedge b$ because of

$$\dfrac{\dfrac{a \leqslant a}{a \wedge b \leqslant a} \quad \dfrac{b \leqslant b}{a \wedge b \leqslant b}}{a \wedge b \leqslant a \wedge b}$$

If α) holds for every $a_n \in M$, then also for $\bigwedge\limits_M$ because of

$$\dfrac{\dfrac{a_1 \leqslant a_1}{\bigwedge\limits_M \leqslant a_1} \quad \cdots \quad \dfrac{a_n \leqslant a_n}{\bigwedge\limits_M \leqslant a_n} \quad \cdots}{\bigwedge\limits_M \leqslant \bigwedge\limits_M}$$

If α) holds for a, then also for $\bar{a}$, because of

$$\dfrac{\dfrac{a \leqslant a}{a \wedge \bar{a} \leqslant}}{\bar{a} \leqslant \bar{a}}$$

Hereby α) is proved in general.

As β) is the most difficult to prove, we take first γ).

In order to prove γ), we have to show that if $c \leqslant a \wedge b$ is derivable, then also $c \leqslant a$ must always be derivable.

We lead the proof indirectly by a *transfinite induction.* Let $c \leqslant a \wedge b$ be derivable, but not $c \leqslant a$. Then the last step of the derivation of $c \leqslant a \wedge b$ cannot be $\dfrac{c \leqslant a \quad c \leqslant b}{c \leqslant a \wedge b}$, likewise not $\dfrac{c_1 \leqslant c_2}{c_1 \wedge \overline{c_2} \leqslant a \wedge b}$ $(c = c_1 \wedge \overline{c_2})$, as then $\dfrac{c_1 \leqslant c_2}{c_1 \wedge \overline{c_2} \leqslant a}$ would be derivable at once.

For the last step remain only the possibilities

$$\dfrac{c_1 \leqslant a \wedge b}{c_1 \wedge c_2 \leqslant a \wedge b} \qquad \dfrac{c_1 \wedge c_1 \wedge c_2 \leqslant a \wedge b}{c_1 \wedge c_2 \leqslant a \wedge b} \ (c = c_1 \wedge c_2)$$

$$\dfrac{c_1 \wedge c' \leqslant a \wedge b \quad \cdots \quad c_n \wedge c' \leqslant a \wedge b \quad \cdots}{\bigwedge\limits_M \wedge c' \leqslant a \wedge b} \begin{pmatrix} M = a_1, a_2, \ldots \\ c = \bigwedge\limits_M \wedge c' \end{pmatrix}$$

Here must now $c_1 \leqslant a$ resp. $c_1 \wedge c_1 \wedge c_2 \leqslant a$ resp. for at least one n $c_n \wedge c' \leqslant a$ not be derivable, as otherwise at once $c \leqslant a$ would be derivable. In the derivation of $c \leqslant a \wedge b$ the claim γ) would thus already be false for a premiss. If in the derivation of a relation I go over to a premiss, of this again to a premiss, etc., then I am after finitely many steps at a basic relation. We would thus obtain a bei einer Grundrelation. Wir erhielten also eine Grundrelation, für die die Behauptung γ) falsch wäre. Da dieses aber unmöglich ist, ist damit γ) bewiesen.

Wir nennen die Induktion, die wir hier durchgeführt haben, eine P r ä m i s s e n i n d u k t i o n.

Mit Hilfe von Prämisseninduktionen verläuft der Beweis für δ) und ε) ebenso einfach wie für γ), so daß ich hierauf nicht weiter eingehe.

Es bleibt nur noch β) zu zeigen. Statt dessen beweisen wir die stärkere Behauptung

$$\zeta) \quad a \leqslant b,\ b \wedge b \wedge \cdots \wedge b \wedge c \leqslant d \quad \Rightarrow \quad a \wedge c \leqslant d$$

um hierauf Prämisseninduktionen anwenden zu können.

Es seien zunächst b, c und d Primelemente. Dann gilt ζ) für jede Grundrelation $a \leqslant b$. Wir nehmen als *Induktionsvoraussetzung* an, daß ζ *für jede Prämisse von* $a \leqslant b$ gelte.

Da b ein Primelement ist, kann der letzte Schritt der Herleitung von $a \leqslant b$ nur sein:

$$\frac{a_1 \leqslant b}{a_1 \wedge a_2 \leqslant b} \qquad \frac{a_1 \wedge a_1 \wedge a_2 \leqslant b}{a_1 \wedge a_2 \leqslant b}\ (a = a_1 \wedge a_2)$$

$$\frac{a_1 \leqslant a_2}{a_1 \wedge \overline{a_2} \leqslant b}\ (a = a_1 \wedge \overline{a_2}) \qquad \frac{a_n \wedge a' \leqslant b}{\bigwedge\limits_M \wedge a' \leqslant b} \begin{pmatrix} M = a_1, a_2, \ldots \\ a = \bigwedge\limits_M \wedge a' \end{pmatrix}$$

Nach der Induktionsvoraussetzung ist dann $a_1 \wedge c \leqslant d$ bezw. $a_1 \wedge a_1 \wedge a_2 \wedge c \leqslant d$ bezw. $a_n \wedge a' \wedge c \leqslant d$ herleitbar. In jedem Falle ist sofort $a \wedge c \leqslant d$ herleitbar, ebenso aus $a_1 \leqslant a_2$ wegen

$$\cfrac{\cfrac{a_1 \leqslant a_2}{a_1 \wedge \overline{a_2} \leqslant d}}{a \wedge c \leqslant d}$$

Damit ist ζ) bewiesen für Primelemente b, c und d.

Jetzt sei nur noch b ein Primelement. Dann gilt also ζ) für beliebiges a und Primelemente c, d. Eine *Prämisseninduktion* ergibt jetzt, daß ζ) für jede Relation $b \wedge b_1 \wedge \cdots \wedge b_1 \wedge c \leqslant d$ gilt. Jede Prämisse von $b \wedge b \wedge \cdots \wedge b \wedge c \leqslant d$ hat nämlich wieder die Form $b \wedge \cdots \wedge b \wedge c \leqslant d$. Damit ist ζ) allgemein für *Primelemente* b bewiesen.

Gilt ζ) für Elemente b_1 und b_2, so auch ersichtlich für $b_1 \wedge b_2$. Gilt ζ) für jedes $b_n \in M$, so auch für $b = \bigwedge\limits_M$. (Beweis durch Prämisseninduktion: $\bigwedge\limits_M \wedge \bigwedge\limits_M \wedge \cdots \wedge \bigwedge\limits_M \wedge c \leqslant d$ kann folgende Prämisse haben: $b_n \wedge \bigwedge\limits_M \wedge \cdots \wedge \bigwedge\limits_M \wedge c \leqslant d$. Nach Induktionsvoraussetzung gilt dann $a \leqslant \bigwedge\limits_M$, $b_n \wedge \bigwedge\limits_M \wedge \cdots \wedge \bigwedge\limits_M \wedge c \leqslant d \quad \Rightarrow$ $b_n \wedge a \wedge c \leqslant d$ Da ζ) aber auch für $b = b_n$ vorausgesetzt ist, und wegen

$$a \leqslant \bigwedge\limits_M \quad \Rightarrow \quad a \leqslant b_n$$

basic relation, for which the claim $\gamma)$ would be false. But as this is impossible, $\gamma)$ is thereby proved.

We call the induction that we have undertaken here a **premiss induction**.

By the aid of premiss inductions, the proof for $\delta)$ and $\varepsilon)$ proceeds just as simply as for $\gamma)$, so that I am not going into this any further.

It remains only to show in addition $\beta)$. Instead of this we prove the stronger claim

$$\zeta) \quad a \leqslant b,\ b \wedge b \wedge \cdots \wedge b \wedge c \leqslant d \quad \Rightarrow \quad a \wedge c \leqslant d$$

in order to be able to apply premiss inductions hereupon.

Let first b, c and d be prime elements. Then $\zeta)$ holds for every basic relation $a \leqslant b$. We assume as *induction hypothesis* that ζ holds *for every premiss of* $a \leqslant b$.

As b is a prime element, the last step of the derivation of $a \leqslant b$ can only be:

$$\frac{a_1 \leqslant b}{a_1 \wedge a_2 \leqslant b} \qquad \frac{a_1 \wedge a_1 \wedge a_2 \leqslant b}{a_1 \wedge a_2 \leqslant b} \ (a = a_1 \wedge a_2)$$

$$\frac{a_1 \leqslant a_2}{a_1 \wedge \overline{a_2} \leqslant b} \ (a = a_1 \wedge \overline{a_2}) \qquad \frac{a_n \wedge a' \leqslant b}{\bigwedge_M \wedge a' \leqslant b} \begin{pmatrix} M = a_1, a_2, \ldots \\ a = \bigwedge_M \wedge a' \end{pmatrix}$$

According to the induction hypothesis, then $a_1 \wedge c \leqslant d$ resp. $a_1 \wedge a_1 \wedge a_2 \wedge c \leqslant d$ resp. $a_n \wedge a' \wedge c \leqslant d$ is derivable. In every case $a \wedge c \leqslant d$ is at once derivable, as well from $a_1 \leqslant a_2$ because of

$$\cfrac{\cfrac{a_1 \leqslant a_2}{a_1 \wedge \overline{a_2} \leqslant d}}{a \wedge c \leqslant d}$$

Thereby $\zeta)$ is proved for prime elements b, c and d.

Now let only b still be a prime element. Then $\zeta)$ holds thus for arbitrary a and prime elements c, d. A *premiss induction* results now in $\zeta)$ holding for every relation $b \wedge b_1 \wedge \cdots \wedge b_1 \wedge c \leqslant d$. Every premiss of $b \wedge b \wedge \cdots \wedge b \wedge c \leqslant d$ has in fact again the form $b \wedge \cdots \wedge b \wedge c \leqslant d$. Thereby $\zeta)$ is proved in general for *prime elements* b.

If $\zeta)$ holds for elements b_1 and b_2, then obviously also for $b_1 \wedge b_2$. If $\zeta)$ holds for every $b_n \in M$, then also for $b = \bigwedge_M$. (Proof by premiss induction: $\bigwedge_M \wedge \bigwedge_M \wedge \cdots \wedge \bigwedge_M \wedge c \leqslant d$ can have the following premiss: $b_n \wedge \bigwedge_M \wedge \cdots \wedge \bigwedge_M \wedge c \leqslant d$. According to induction hypothesis holds then $a \leqslant \bigwedge_M,\ b_n \wedge \bigwedge_M \wedge \cdots \wedge \bigwedge_M \wedge c \leqslant d \ \Rightarrow \ b_n \wedge a \wedge c \leqslant d$. But as $\zeta)$ is also assumed for $b = b_n$, and because of

$$a \leqslant \bigwedge_M \quad \Rightarrow \quad a \leqslant b_n,$$

gilt auch $a \leqslant \bigwedge_M,\ b_n \wedge a \wedge c \leqslant d \ \Rightarrow\ a \wedge a \wedge c \leqslant d$. Aus $a \wedge a \wedge c \leqslant d$ ist aber $a \wedge c \leqslant d$ herleitbar. Jede andere Prämisse von $\bigwedge_M \wedge \bigwedge_M \wedge \cdots \wedge \bigwedge_M \wedge c \leqslant d$ ist trivial).

Gilt ζ) für b, so auch für $\bar{b}$. (Beweis durch Prämisseninduktion: $\bar{b} \wedge \bar{b} \wedge \cdots \wedge \bar{b} \wedge c \leqslant d$ kann die folgende Prämisse haben: $\bar{b} \wedge \cdots \wedge \bar{b} \wedge c \leqslant b$. Dann gilt nach Induktionsvoraussetzung

$$a \leqslant \bar{b},\ \bar{b} \wedge \cdots \wedge \bar{b} \wedge c \leqslant b \quad \Rightarrow \quad a \wedge c \leqslant b.$$

Da ζ) auch für b vorausgesetzt ist, gilt auch

$$a \wedge c \leqslant b,\ a \wedge b \leqslant d \quad \Rightarrow \quad a \wedge a \wedge c \leqslant d.$$

Also gilt auch $a \leqslant \bar{b},\ \bar{b} \wedge \cdots \wedge \bar{b} \wedge c \leqslant b \quad \Rightarrow \quad a \wedge c \leqslant d$ wegen $a \leqslant \bar{b} \quad \Rightarrow \quad a \wedge b \leqslant d$ Jede andere Prämisse ist wieder trivial.)

Also ist ζ) allgemein gültig. Damit ist bewiesen, daß K ein orthokomplementärer ω-vollständiger Halbverband ist.

P ist ein Teil von K, da

$$p \leqslant q \text{ in P} \quad \Longleftrightarrow \quad p \leqslant q \text{ in K}$$

gilt. Wir haben uns dazu zu überzeugen, daß keine Relation $p \leqslant q$ in K herleitbar ist, die nicht schon in P gilt. Das ist aber selbstverständlich, da keine der Regeln außer g) überhaupt Relationen $p \leqslant q$ unter dem Strich liefert. Eine Herleitung einer Relation $p \leqslant q$ kann also nur die Regeln d) und g) benutzen. Mit diesen sind aber nur die Grundrelationen herleitbar.

Zum Beweis unseres Satzes bleibt jetzt noch zu zeigen, daß sich K in jeden anderen orthokomplementären ω-vollständigen Halbverband K$'$, der P als Teil enthält, homomorph abbilden läßt. Diese Abbildung definieren wir durch

1) für Primelemente p gilt $p \to p$,

2) ferner soll gelten

$$\begin{aligned} a \to a',\ b \to b' \quad &\Rightarrow \quad a \wedge b \to a' \wedge b' \\ a \to a' \quad &\Rightarrow \quad \bar{a} \to \overline{a'} \\ a_n \to a'_n \quad &\Rightarrow \quad \bigwedge_M \to \bigwedge_{M'} \quad \begin{pmatrix} M = a_1, a_2, \ldots \\ M' = a'_1, a'_2, \ldots \end{pmatrix} \end{aligned}$$

Dadurch wird ersichtlich ein Homomorphismus definiert, denn es gilt für $a \to a'$ und $b \to b'$ stets $a \leqslant b \quad \Rightarrow \quad a' \leqslant b'$.

Jede Herleitung von $a \leqslant b$ beweist nämlich sofort auch $a' \leqslant b'$, da die Herleitungsschritte a) - g) in jedem orthokomplementären ω-vollständigen Halbverband stets richtig sind.

also $a \leqslant \bigwedge_M,\ b_n \wedge a \wedge c \leqslant d \ \Rightarrow\ a \wedge a \wedge c \leqslant d$ holds. But from $a \wedge a \wedge c \leqslant d$ may be derived $a \wedge c \leqslant d$. Every other premiss of $\bigwedge_M \wedge \bigwedge_M \wedge \cdots \wedge \bigwedge_M \wedge c \leqslant d$ is trivial.)

If ζ) holds for b, then also for $\bar{b}$. (Proof by premiss induction: $\bar{b}\wedge\bar{b}\wedge\cdots\wedge\bar{b}\wedge c \leqslant d$ can have the following premiss: $\bar{b} \wedge \cdots \wedge \bar{b} \wedge c \leqslant b$. Then holds according to induction hypothesis

$$a \leqslant \bar{b},\ \bar{b} \wedge \cdots \wedge \bar{b} \wedge c \leqslant b \quad\Rightarrow\quad a \wedge c \leqslant b.$$

As ζ) is also assumed for b, also holds

$$a \wedge c \leqslant b,\ a \wedge b \leqslant d \quad\Rightarrow\quad a \wedge a \wedge c \leqslant d.$$

Thus holds also $a \leqslant \bar{b},\ \bar{b}\wedge\cdots\wedge\bar{b}\wedge c \leqslant b \ \Rightarrow\ a\wedge c \leqslant d$ because of $a \leqslant \bar{b} \ \Rightarrow\ a\wedge b \leqslant d$. Every other premiss is again trivial.)

Thus ζ) is valid in general. Thereby is proved that K is an orthocomplemented ω-complete semilattice.

P is a part of K, as

$$p \leqslant q \text{ in P} \quad\Longleftrightarrow\quad p \leqslant q \text{ in K}$$

holds. We have for this to convince ourselves that no relation $p \leqslant q$ is derivable in K that is not already holding in P. But this goes without saying, as none of the rules except g) actually yield relations $p \leqslant q$ below the line. A derivation of a relation $p \leqslant q$ can thus use only the rules d) and g). But with these only the basic relations are derivable.

For the proof of our theorem it remains now in addition to show that K may be mapped homomorphically into every other orthocomplemented ω-complete semilattice K$'$ that contains P as part. This mapping we define by

1) for prime elements p holds $p \to p$,

2) moreover is to hold

$$\begin{aligned} a \to a',\ b \to b' \quad &\Rightarrow\quad a \wedge b \to a' \wedge b' \\ a \to a' \quad &\Rightarrow\quad \bar{a} \to \overline{a'} \\ a_n \to a'_n \quad &\Rightarrow\quad \bigwedge_M \to \bigwedge_{M'} \quad \begin{pmatrix} M = a_1, a_2, \ldots \\ M' = a'_1, a'_2, \ldots \end{pmatrix} \end{aligned}$$

Hereby obviously a homomorphism is being defined, for with $a \to a'$ and $b \to b'$ always holds $a \leqslant b \Rightarrow a' \leqslant b'$.

Every derivation of $a \leqslant b$ proves in fact at once also $a' \leqslant b'$, as the derivation steps a)–g) are always correct in every orthocomplemented ω-complete semilattice.

§ 3. Um aus dem im § 2 bewiesenen Satz die Widerspruchsfreiheit der reinen Zahlentheorie mit vollständiger Induktion beweisen zu können, benutzen wir die folgende Formalisierung. Als Primformeln nehmen wir die Zeichen für zahlentheoretische Prädikate A(...), B(...), ... mit den Zahlen $1, 1', 1'', \ldots$ als Argumenten, z. B. $1 = 1''$, $1 + 1 = 1'$.

Diese Primformeln $\mathfrak{P}, \mathfrak{Q}, \ldots$ bilden eine halbgeordnete Menge, wenn wir $\mathfrak{P} \to \mathfrak{Q}$ setzen, falls das Prädikat $\mathfrak{P}$ das Prädikat $\mathfrak{Q}$ impliziert. Zu den Grundrelationen $\mathfrak{P} \to \mathfrak{Q}$ nehmen wir auch noch die Relationen der Form $\ \to \mathfrak{P}$, $\mathfrak{P} \to\ $, $\ \to\ $ hinzu, soweit sie inhaltlich richtig sind.

Über dieser halbgeordneten Menge P der Primformeln konstruieren wir jetzt wie in § 2 den ausgezeichneten orthokomplementären ω-vollständigen Halbverband. Wir benutzen dazu die logistischen Zeichen, also $\to$ statt $\leqslant$, & statt $\wedge$.

Zu den Formeln gehören also die Primformeln, mit $\mathfrak{A}$ und $\mathfrak{B}$ auch $\mathfrak{A}$ & $\mathfrak{B}$, mit $\mathfrak{A}$ auch $\overline{\mathfrak{A}}$. Die Konjunktion abzählbarer Folgen beschränken wir auf die Folgen der Form $\mathfrak{A}(1), \mathfrak{A}(1'), \ldots$ Diese Konjunktion bezeichnen wir durch $(\mathfrak{x})\, \mathfrak{A}(\mathfrak{x})$

Ferner führen wir noch freie Variable $\mathfrak{a} = a,\ b,\ \ldots$ ein durch folgende Schlußregel:

> sind $A(1), A(1'), \ldots$ herleitbare Relationen,
> so
> soll auch $A(\mathfrak{a})$ herleitbar sein.

Hierdurch werden die Beweise von § 2 nur unwesentlich modifiziert. Wir erhalten insgesamt einen Kalkül N mit den folgenden Schlußregeln

$$a)\ \frac{\mathfrak{C} \to \mathfrak{A} \quad \mathfrak{C} \to \mathfrak{B}}{\mathfrak{C} \to \mathfrak{A} \,\&\, \mathfrak{B}} \qquad d)\ \frac{\mathfrak{A} \to \mathfrak{C}}{\mathfrak{A} \,\&\, \mathfrak{B} \to \mathfrak{C}}$$

$$b)\ \frac{\mathfrak{A} \,\&\, \mathfrak{C} \to}{\mathfrak{C} \to \overline{\mathfrak{A}}} \qquad e)\ \frac{\mathfrak{A} \to \mathfrak{B}}{\mathfrak{A} \,\&\, \overline{\mathfrak{B}} \to \mathfrak{C}}$$

$$c)\ \frac{\mathfrak{C} \to \mathfrak{A}(1) \quad \cdots \quad \mathfrak{C} \to \mathfrak{A}(n) \quad \cdots}{\mathfrak{C} \to (\mathfrak{x})\, \mathfrak{A}(\mathfrak{x})} \qquad f)\ \frac{\mathfrak{A}(n) \,\&\, \mathfrak{B} \to \mathfrak{C}}{(\mathfrak{x})\, \mathfrak{A}(\mathfrak{x}) \,\&\, \mathfrak{B} \to \mathfrak{C}}$$

$$g)\ \frac{\mathfrak{A} \,\&\, \mathfrak{A} \,\&\, \mathfrak{B} \to \mathfrak{C}}{\mathfrak{A} \,\&\, \mathfrak{B} \to \mathfrak{C}}$$

$$h)\ \frac{\mathfrak{A} \,\&\, \mathfrak{B} \to \mathfrak{C}}{\mathfrak{B} \,\&\, \mathfrak{A} \to \mathfrak{C}} \qquad i)\ \frac{\mathfrak{A} \,\&\, (\mathfrak{B} \,\&\, \mathfrak{C}) \to \mathfrak{D}}{(\mathfrak{A} \,\&\, \mathfrak{B}) \,\&\, \mathfrak{C} \to \mathfrak{D}}$$

$$j)\ \frac{A(1) \quad \cdots \quad A(n) \quad \cdots}{A(\mathfrak{a})}$$

Die Schlußregeln $h)$ und $i)$ waren in § 2 überflüssig, da wir dort $a \wedge b \wedge c \ldots$ sofort als Zeichen für die Kombination von $a, b, c, \ldots$ eingeführt haben.

§ 3. In order to be able to prove the freedom from contradiction of elementary number theory with complete induction from the theorem proved in § 2, we use the following formalisation. As prime formulas we take the signs for number-theoretic predicates A(…), B(…), … with the numbers $1, 1', 1'', \ldots$ as arguments, e.g. $1 = 1''$, $1 + 1 = 1'$.

These prime formulas $\mathfrak{P}, \mathfrak{Q}, \ldots$ form a partially ordered set if we set $\mathfrak{P} \to \mathfrak{Q}$ in case the predicate $\mathfrak{P}$ implies the predicate $\mathfrak{Q}$. To the basic relations $\mathfrak{P} \to \mathfrak{Q}$ we are also adding the relations of the form $\to \mathfrak{P}$, $\mathfrak{P} \to$, $\to$, as far as they are correct in terms of content.

Over this partially ordered set P of the prime formulas we construct now as in § 2 the distinguished orthocomplemented ω-complete semilattice. We use for this the logistic signs, thus $\to$ instead of $\leqslant$, & instead of $\wedge$.

To the formulas belong thus the prime formulas, with $\mathfrak{A}$ and $\mathfrak{B}$ also $\mathfrak{A}$ & $\mathfrak{B}$, with $\mathfrak{A}$ also $\overline{\mathfrak{A}}$. We restrict the conjunction of countable sequences to the sequences of the form $\mathfrak{A}(1), \mathfrak{A}(1'), \ldots$. We designate this conjunction by $(\mathfrak{x})\,\mathfrak{A}(\mathfrak{x})$.

Moreover we introduce in addition free variables $\mathfrak{a} = a, b, \ldots$ by the following rule of inference:

> if $A(1), A(1'), \ldots$ are derivable relations, then
> $A(\mathfrak{a})$ is also to be derivable.

By this the proofs of § 2 are only modified unessentially. We obtain overall a calculus N with the following rules of inference

$$a)\ \frac{\mathfrak{C} \to \mathfrak{A} \quad \mathfrak{C} \to \mathfrak{B}}{\mathfrak{C} \to \mathfrak{A} \,\&\, \mathfrak{B}} \qquad d)\ \frac{\mathfrak{A} \to \mathfrak{C}}{\mathfrak{A} \,\&\, \mathfrak{B} \to \mathfrak{C}}$$

$$b)\ \frac{\mathfrak{A} \,\&\, \mathfrak{C} \to}{\mathfrak{C} \to \overline{\mathfrak{A}}} \qquad e)\ \frac{\mathfrak{A} \to \mathfrak{B}}{\mathfrak{A} \,\&\, \overline{\mathfrak{B}} \to \mathfrak{C}}$$

$$c)\ \frac{\mathfrak{C} \to \mathfrak{A}(1) \quad \cdots \quad \mathfrak{C} \to \mathfrak{A}(n) \quad \cdots}{\mathfrak{C} \to (\mathfrak{x})\,\mathfrak{A}(\mathfrak{x})} \qquad f)\ \frac{\mathfrak{A}(n) \,\&\, \mathfrak{B} \to \mathfrak{C}}{(\mathfrak{x})\,\mathfrak{A}(\mathfrak{x}) \,\&\, \mathfrak{B} \to \mathfrak{C}}$$

$$g)\ \frac{\mathfrak{A} \,\&\, \mathfrak{A} \,\&\, \mathfrak{B} \to \mathfrak{C}}{\mathfrak{A} \,\&\, \mathfrak{B} \to \mathfrak{C}}$$

$$h)\ \frac{\mathfrak{A} \,\&\, \mathfrak{B} \to \mathfrak{C}}{\mathfrak{B} \,\&\, \mathfrak{A} \to \mathfrak{C}} \qquad i)\ \frac{\mathfrak{A} \,\&\, (\mathfrak{B} \,\&\, \mathfrak{C}) \to \mathfrak{D}}{(\mathfrak{A} \,\&\, \mathfrak{B}) \,\&\, \mathfrak{C} \to \mathfrak{D}}$$

$$j)\ \frac{A(1) \quad \cdots \quad A(n) \quad \cdots}{A(\mathfrak{a})}\ .$$

The rules of inference $h)$ and $i)$ were dispensable in § 2, as we have introduced there $a \wedge b \wedge c \wedge \cdots$ at once as sign for the combination of $a, b, c, \ldots$.

The proof in § 2 yields now the following result: the calculus N is consistent, e.g. the empty relation $\to$ is not derivable, as only the relations correct in terms

Der Beweis in § 2 liefert jetzt das folgende Ergebnis: Der Kalkül N ist widerspruchsfrei, z. B. ist die leere Relation $\to$ nicht herleitbar, da nur die inhaltlich richtigen Relationen in P gelten und P ein Teil von N ist. Zu dem Kalkül N können die folgenden Schlußregeln hinzugenommen werden, ohne daß die Menge der herleitbaren Relationen vergrößert wird:

$$k)\ \frac{\mathfrak{A} \to \mathfrak{B} \quad \mathfrak{B} \to \mathfrak{C}}{\mathfrak{A} \to \mathfrak{C}}$$

$$l)\ \frac{\mathfrak{C} \to \mathfrak{A} \,\&\, \mathfrak{B}}{\mathfrak{C} \to \mathfrak{A}} \qquad m)\ \frac{\mathfrak{C} \to \mathfrak{A} \,\&\, \mathfrak{B}}{\mathfrak{C} \to \mathfrak{B}}$$

$$n)\ \frac{\mathfrak{C} \to \overline{\mathfrak{A}}}{\mathfrak{A} \,\&\, \mathfrak{C} \to} \qquad o)\ \frac{\mathfrak{C} \to (\mathfrak{x})\,\mathfrak{A}(\mathfrak{x})}{\mathfrak{C} \to \mathfrak{A}(n)}$$

Zu den Grundrelationen kann $\mathfrak{A} \to \mathfrak{A}$ hinzugenommen werden.

Dieses Ergebnis aus § 2 können wir jetzt ergänzen:

1) es kann auch die Schlußregel $p)\ \dfrac{A(\mathfrak{a})}{A(n)}$ hinzugenommen werden.

Der Beweis wird wieder durch eine *transfinite Prämisseninduktion* geführt. Ist $A(\mathfrak{a})$ herleitbar in N und ist die letzte Schlußregel dieser Herleitung nicht

$$\frac{A(1) \quad \cdots \quad A(n) \quad \cdots}{A(\mathfrak{a})}$$

so hat die Prämisse die Form $A'(\mathfrak{a})$. Nehmen wir als Induktionsvoraussetzung an, daß für jede Prämisse $A'(\mathfrak{a})$ auch $A'(n)$ herleitbar ist, so folgt sofort $A(n)$.

2) Zu den Grundrelationen darf $\overline{\overline{\mathfrak{A}}} \to \mathfrak{A}$ hinzugenommen werden.

Für jede Primformel $\mathfrak{P}$ gilt nämlich stets $\to \mathfrak{P}$ oder $\mathfrak{P} \to$. Wegen $\dfrac{\to \mathfrak{P}}{\overline{\overline{\mathfrak{P}}} \to \mathfrak{P}}$ $\dfrac{\dfrac{\mathfrak{P} \to}{\to \overline{\mathfrak{P}}}}{\overline{\overline{\mathfrak{P}}} \to \mathfrak{P}}$ ist also für jede Primformel stets $\overline{\overline{\mathfrak{P}}} \to \mathfrak{P}$ herleitbar. Hieraus folgt allgemein die Herleitbarkeit von $\overline{\overline{\mathfrak{A}}} \to \mathfrak{A}$ (vergl. etwa Hilbert-Bernays, *Grundlagen der Mathematik II*).

3) Es kann auch die *vollständige Induktion*

$$q)\ \frac{\mathfrak{A}(\mathfrak{a}) \to \mathfrak{A}(\mathfrak{a}')}{\mathfrak{A}(1) \to \mathfrak{A}(\mathfrak{b})}$$

zu den Schlußregeln hinzugenommen werden ohne die Menge der herleitbaren Relationen zu vergrößern.

of content hold in P and P is a part of N. To the calculus N the following rules of inference can be added without increasing the set the derivable relations:

$$k)\ \frac{\mathfrak{A} \to \mathfrak{B} \quad \mathfrak{B} \to \mathfrak{C}}{\mathfrak{A} \to \mathfrak{C}}$$

$$l)\ \frac{\mathfrak{C} \to \mathfrak{A} \,\&\, \mathfrak{B}}{\mathfrak{C} \to \mathfrak{A}} \qquad m)\ \frac{\mathfrak{C} \to \mathfrak{A} \,\&\, \mathfrak{B}}{\mathfrak{C} \to \mathfrak{B}}$$

$$n)\ \frac{\mathfrak{C} \to \overline{\mathfrak{A}}}{\mathfrak{A} \,\&\, \mathfrak{C} \to} \qquad o)\ \frac{\mathfrak{C} \to (\mathfrak{x})\,\mathfrak{A}(\mathfrak{x})}{\mathfrak{C} \to \mathfrak{A}(n)}$$

To the basic relations can be added $\mathfrak{A} \to \mathfrak{A}$.

This result from § 2 we can now supplement:

1) The rule of inference $p)\ \dfrac{A(\mathfrak{a})}{A(n)}$ can also be added.

The proof is again being led by a *transfinite premiss induction.* If $A(\mathfrak{a})$ is derivable in N and if the last rule of inference of this derivation is not

$$\frac{A(1) \quad \cdots \quad A(n) \quad \cdots}{A(\mathfrak{a})}$$

then the premiss has the form $A'(\mathfrak{a})$. If we assume as induction hypothesis that for every premiss $A'(\mathfrak{a})$ also $A'(n)$ is derivable, then $A(n)$ follows at once.

2) To the basic relations may be added $\overline{\overline{\mathfrak{A}}} \to \mathfrak{A}$.

For every prime formula $\mathfrak{P}$ holds in fact always either $\to \mathfrak{P}$ or $\mathfrak{P} \to$. Because of $\dfrac{\to \mathfrak{P}}{\overline{\overline{\mathfrak{P}}} \to \mathfrak{P}}$ $\dfrac{\dfrac{\mathfrak{P} \to}{\to \overline{\mathfrak{P}}}}{\overline{\overline{\mathfrak{P}}} \to \mathfrak{P}}$, $\overline{\overline{\mathfrak{P}}} \to \mathfrak{P}$ is thus always derivable for every prime formula. From this follows in general the derivability of $\overline{\overline{\mathfrak{A}}} \to \mathfrak{A}$ (cf. e.g. Hilbert-Bernays, *Grundlagen der Mathematik II*).

3) The *complete induction*

$$q)\ \frac{\mathfrak{A}(\mathfrak{a}) \to \mathfrak{A}(\mathfrak{a}')}{\mathfrak{A}(1) \to \mathfrak{A}(\mathfrak{b})}$$

can also be added to the rules of inference without increasing the set the derivable relations.

Ist nämlich $\mathfrak{A}(\mathfrak{a}) \to \mathfrak{A}(\mathfrak{a}')$ herleitbar, so auch die Relation $\mathfrak{A}(n) \to \mathfrak{A}(n')$ für jede Zahl n.

Für jede Zahl m folgt daraus durch m-malige Anwendung der Schlußregel $k)$ sofort $\mathfrak{A}(1) \to \mathfrak{A}(m)$.

Wegen $\dfrac{\mathfrak{A}(1) \to \mathfrak{A}(1) \quad \cdots \quad \mathfrak{A}(1) \to \mathfrak{A}(m) \quad \cdots}{\mathfrak{A}(1) \to \mathfrak{A}(\mathfrak{b})}$ ist also auch $\mathfrak{A}(1) \to \mathfrak{A}(\mathfrak{b})$ herleitbar.

Damit ist die Wf. der reinen Zahlentheorie bewiesen, da die insgesamt zulässigen Schlußregeln einen Kalkül definieren, der den klassischen Prädikatenkalkül ersichtlich enthält.

In fact, if $\mathfrak{A}(\mathfrak{a}) \to \mathfrak{A}(\mathfrak{a}')$ is derivable, then also the relation $\mathfrak{A}(n) \to \mathfrak{A}(n')$ for every number n.

For every number m follows therefrom at once $\mathfrak{A}(1) \to \mathfrak{A}(m)$ by m-fold application of the rule of inference k).

Because of $\dfrac{\mathfrak{A}(1) \to \mathfrak{A}(1) \quad \cdots \quad \mathfrak{A}(1) \to \mathfrak{A}(m) \quad \cdots}{\mathfrak{A}(1) \to \mathfrak{A}(\mathfrak{b})}$ also $\mathfrak{A}(1) \to \mathfrak{A}(\mathfrak{b})$ is thus derivable.

Thereby the freedom from contradiction of the elementary number theory is proved, as the overall admissible rules of inference define a calculus that obviously contains the classical calculus of predicates.