Regular entailment relations

Thierry Coquand Henri Lombardi Stefan Neuwirth

December 19, 2019

Keywords: Preordered group; unbounded entailment relation; regular entailment relation; system of ideals; equivariant system of ideals; morphism from a preordered group to a lattice-preordered group; Lorenzen-Clifford-Dieudonné theorem.

MSC 2010: Primary 06F20; Secondary 06F05, 13A15, 13B22.

Abstract

Inspired by the work of Lorenzen on the theory of preordered groups in the fifties, we define regular entailment relations and show a crucial theorem for this structure. We also describe equivariant systems of ideals à la Lorenzen and show that the remarkable regularisation process invented by him yields a regular entailment relation. By providing constructive objects and arguments, we pursue Lorenzen's aim of "bringing to light the basic, pure concepts in their simple and transparent clarity".

Introduction

Paul Lorenzen carried out an analysis of multiplicative ideal theory in terms of embeddings into an l-group in four articles. In Lorenzen 1939, he formulated the problem in the language of semigroups instead of integral domains. The endeavour of Lorenzen 1950 was to remove the condition of commutativity; the unavailability of the Grothendieck group construction led him to discover the "regularity condition" and to propose a far-reaching reformulation of embeddability into a product of linearly preordered groups in terms of "regularisation". He also arrived at the formulation of the concepts of equivariant system of ideals and entailment relation. The article Lorenzen 1952 broadened to the more general case of a monoid acting on a preordered set. Our research started as a study of Lorenzen 1953, in which he proved a result that suggested Theorem 1.11 to us.

If G is a preordered commutative group and we have a morphism $f : G \to L$ with L an l-group, then we can define a relation $A \vdash B$ between *nonempty* finite subsets of G by $\wedge f(A) \leqslant \vee f(B)$. This relation satisfies the following conditions.

(R_1) $A \vdash B$ if $A \supseteq A'$ and $B \supseteq B'$ and $A' \vdash B'$.

(R_2) $A \vdash B$ if $A, x \vdash B$ and $A \vdash B, x$.

(R_3) $a \vdash b$ if $a \leqslant b$ in G.

(R_4) $A \vdash B$ if $A + x \vdash B + x$.

(R_5) $a + x, b + y \vdash a + b, x + y$.

We are making the following abuses of notation for finite sets: we write a for the singleton consisting of a, and A, A' for the union of the sets A and A'; note that our framework requires only a naive set theory. We call *regular entailment relation* for a preordered group $(G, \leqslant)$ any relation which satisfies these conditions. The remarkable last condition is called the *regularity condition*.

Note that the converse of a regular entailment relation for $(G, \leqslant)$ is a regular entailment relation for $(G, \geqslant)$ (the group with the converse preorder). When we use this, we say that a result follows from another one "symmetrically".

Any relation satisfying the three first conditions defines in a canonical way a(n unbounded) distributive lattice L with a natural monotone map $G \to L$: see Lorenzen 1951, Satz 7; Cederquist and Coquand 2000, Theorem 1 (obtained independently).

The goal of this note is essentially to show that this distributive lattice has a (canonical) l-group structure, simplifying some arguments in Lorenzen 1953. This is done in Theorem 1.11. In Section 2, we explain how to define a regular entailment relation through a predicate on nonempty finite subsets of G. In Section 3, we define "equivariant systems of ideals" à la Lorenzen and we show how to express this notion through a predicate on nonempty finite subsets of G. In Section 4, we explain how Lorenzen "regularises" an equivariant system of ideals, which leads to the Lorenzen group of this system of ideals (Theorem 4.4). In Section 5, we explain the link with a constructive version of the Lorenzen-Clifford-Dieudonné theorem. In Section 6, we explain the link with the Prüfer way of defining the Lorenzen group of a system of ideals. In Section 7, we give a constructive version of a remarkable theorem of Lorenzen which uses the regularity condition in the noncommutative case. Finally, in Section 8, we give examples illustrating some constructions described in the paper.

The results of this research complement the ones of Coquand et al. (2019): we introduce various equivalent presentations of regular entailment relations; we also provide a noncommutative version and several examples.

1 General properties of regular entailment relations

A first consequence of regularity is the following.

Proposition 1.1. *We have $a, b \vdash a+x, b-x$ and $a+x, b-x \vdash a, b$. In particular, $a \vdash a+x, a-x$ and $a+x, a-x \vdash a$.*

Proof. By regularity, we have $x+(a-x), (b-2x)+2x \vdash x+(b-2x), (a-x)+2x$, which is $a, b \vdash a+x, b-x$. The other claim follows symmetrically. □

Corollary 1.2. *In the distributive lattice L defined by the (unbounded) entailment relation $\vdash$, $\wedge A \leqslant (\wedge(A+x)) \vee (\wedge(A-x))$.*

Proof. In L, we have $(\wedge(A+x)) \vee (\wedge(A-x)) = (\wedge_{a\in A}(a+x)) \vee (\wedge_{b\in A}(b-x)) = \wedge_{a,b\in A}((a+x) \vee (b-x))$, so that this follows from Proposition 1.1. □

Corollary 1.3. *If we have $A, A+x \vdash B$ and $A, A-x \vdash B$, then $A \vdash B$. Symmetrically, if $A \vdash B, B+x$ and $A \vdash B, B-x$ then $A \vdash B$.*

Lemma 1.4. *We have $A, A+x \vdash B$ iff $A \vdash B, B-x$.*

Proof. We assume $A, A+x \vdash B$ and we prove $A \vdash B, B-x$. By Corollary 1.3, it is enough to show $A, A-x \vdash B, B-x$, but this follows from $A, A+x \vdash B$ by translating by $-x$ and then weakening. The other direction is symmetric. □

Lemma 1.5. *If $0 \leqslant p \leqslant q$, then $a, a+qx \vdash a+px$.*

Proof. We prove this by induction on q. It holds for $q = 0$. If it holds for q, we note that we have $a, a+(q+1)x \vdash a+x, a+qx$ by regularity, and since $a, a+qx \vdash a+x$ by induction, we get $a, a+(q+1)x \vdash a+x$ by cut. By induction we have $a, a+qx \vdash a+px$ for $p \leqslant q$, and hence $a+x, a+(q+1)x \vdash a+(p+1)x$. By cut with $a, a+(q+1)x \vdash a+x$ we get $a, a+(q+1)x \vdash a+(p+1)x$. □

Given a regular entailment relation $\vdash$ and an element x, we describe now the *regular* entailment relation $\vdash_x$ for which we force $0 \vdash_x x$. This relation exists by universal algebra.

Let us define that $A \vdash_x B$ holds iff there exists p such that $A, A+px \vdash B$, iff (by Lemma 1.4) there exists p such that $A \vdash B, B-px$, and we are going to show that this is the least regular entailment relation containing $\vdash$ and such that $0 \vdash_x x$. We have $0 \vdash_x x$ since $0, x \vdash x$.

Note that, by using Lemma 1.5, if we have $A, A+px \vdash B$, we also have $A, A+qx \vdash B$ for $q \geqslant p$.

Proposition 1.6. *The relation $\vdash_x$ is a regular entailment relation. It is the least regular entailment relation containing $\vdash$ and such that $0 \vdash_x x$.*

Proof. The only complex condition is the cut rule. We assume $A, A + px \vdash B, u$ and $A, A+qx, u, u+qx \vdash B$, and we prove $A \vdash_x B$. By Lemma 1.5, we can assume $p = q$. We write $y = px$ and we have $A, A + y \vdash B, u$ and $A, A + y, u, u + y \vdash B$. We write $C = A, A + y, A + 2y$ and we prove $C \vdash B$.

We have by weakening $C \vdash B, u$ and $C, u, u + y \vdash B$ and $C \vdash B + y, u + y$. By cut, we get $C, u \vdash B, B+y$. By Lemma 1.4, this is equivalent to $C, u, C-y, u-y \vdash B$. We also have $C, u, C + y, u + y \vdash B$ by weakening $C, u, u + y \vdash B$. Hence by Lemma 1.3 we get $C, u \vdash B$. Since we also have $C \vdash B, u$, we get $C \vdash B$ by cut.

By Lemma 1.5 we have $A, A + 2y \vdash B$, which shows $A \vdash_x B$. □

Proposition 1.7. *If $A \vdash_x B$ and $A \vdash_{-x} B$ then $A \vdash B$.*

Proof. We have $A, A + px \vdash B$ and $A, A - qx \vdash B$. Using Lemma 1.5 we can assume $p = q$ and then conclude by Corollary 1.3. □

Proposition 1.7 implies that in order to prove an entailment involving some elements, we can always assume that all elements occurring in the proof are linearly preordered for the relation $a \vdash b$. This corresponds to the informal covering principle by quotients for l-groups (Lombardi and Quitté 2015, Principle XI-2.10). Here are two direct applications.

Proposition 1.8. *We have $A \vdash b_1, \dots, b_m$ iff $A - b_1, \dots, A - b_m \vdash 0$.*

Thus $A \vdash B$ iff $A - B \vdash 0$ iff $0 \vdash B - A$. The first equivalence is exactly Proposition 1.8, and the second equivalence follows symmetrically.

Proposition 1.9. *If $A + b_1, \dots, A + b_m \vdash b_j$ for $j = 1, \dots, m$, then $A \vdash 0$.*

It follows from Proposition 1.9 that if we consider the monoid of formal elements $\wedge A$ with the operation $\wedge A + \wedge B = \wedge(A + B)$, preordered by the relation $\wedge A \leqslant \wedge B$ iff $A \vdash b$ for all b in B, we get a *cancellative* monoid.

The *Grothendieck l-group* of a meet-monoid $(M, +, 0, \wedge)$ is the l-group that it freely generates. Its group structure is given by the Grothendieck group of the monoid $(M, +, 0)$.

Corollary 1.10. *The distributive lattice defined by the Grothendieck l-group of the previously defined cancellative monoid coincides with the distributive lattice defined by the relation $\vdash$.*

We have realised in this way our goal.

Theorem 1.11. *The distributive lattice V generated by a regular entailment relation has a canonical l-group structure for which the natural preorder morphism $\varphi : G \to V$ is a group morphism.*

Note that we may have $a \vdash b$ without $a \leq b$, so φ is not necessarily injective.

Here is another consequence of the fact that we can always assume that elements are linearly preordered for the relation $a \vdash b$.

Corollary 1.12. *If $a_1 + \cdots + a_n = 0$ then $a_1, \ldots, a_n \vdash 0$.*

Corollary 1.13. *If $a_1 + \cdots + a_n = b_1 + \cdots + b_n$ then $a_1, \ldots, a_n \vdash b_1, \ldots, b_n$.*

Proof. We have $\Sigma_{i,j} a_i - b_j = 0$ and we can apply the previous result and Proposition 1.8. □

2 Another presentation of regular entailment relations

It follows from Proposition 1.8 that the relation $\vdash$ is completely determined by the predicate $A \vdash 0$ on nonempty finite subsets of the group. Let us analyse the properties satisfied by this predicate $R(A) = A \vdash 0$. Firstly, it satisfies

(P_3) $R(a)$ if $a \leqslant 0$ in G.

Secondly, it is *monotone*:

(P_1) $R(A)$ if $R(A')$ and $A' \subseteq A$.

The cut rule can be stated as $R(A-B)$ if $R(A-B, x-B)$ and $R(A-B, A-x)$, so we get the following property (since we can assume $x = 0$ by translating and replace B by $-B$):

(P_2) $R(A + B)$ if $R(A + B, A)$ and $R(A + B, B)$.

Finally, the regularity condition gives $R(a-b, b-a, x-y, y-x)$ which simplifies using (P_1) into

(P_5) $R(x, -x)$.

We get in this way another presentation of a regular entailment relation as a predicate satisfying the conditions $(P_1), (P_2), (P_3), (P_5)$: if R satisfies these properties and $A \vdash B$ is defined by $R(A - B)$, then we get a regular entailment relation (we have one axiom less since the translation property "$A \vdash B$ if $A + x \vdash B + x$" is automatically satisfied).

3 Equivariant systems of ideals

Let us make the same analysis for the notion of *equivariant system of ideals*. A *system of ideals for a preordered set* G can be defined à la Lorenzen as a single-conclusion entailment relation, i.e. a relation $A \rhd x$ between nonempty finite subsets of G and elements x in G satisfying the following conditions.

(S_1) $A \rhd x$ if $A \supseteq A'$ and $A' \rhd x$.

(S_2) $A \rhd x$ if $A, y \rhd x$ and $A \rhd y$.

(S_3) $a \rhd x$ if $a \leqslant x$ in G.

A system of ideals for a preordered group G is said to be equivariant when it satisfies the condition

(S_4) $A \rhd x$ if $A + y \rhd x + y$.

When we have an equivariant system of ideals, let us consider the predicate $S(A) = A \rhd 0$. This predicate satisfies the following conditions.

(P_1) $S(A)$ if $A \supseteq A'$ and $S(A')$.

(P_2') $S(A)$ if $S(A, u)$ and $S(A - u)$.

(P_3) $S(a)$ if $a \leqslant 0$ in G.

Conversely, if S satisfies (P_1), (P_2') and (P_3) and if we define $A \rhd x$ by $S(A - x)$, then $\rhd$ is an equivariant system of ideals, so that S is just another presentation for it.

To an equivariant system of ideals S we can clearly associate the relation $A \leqslant_S B$ given by "$A \rhd b$ for all b in B", and we define thus a preordered monoid with $A + B$ as monoid operation and $A \wedge B = A, B$ as meet operation. We call the corresponding preordered monoid *the meet-monoid generated by S on G*.

Conversely, consider for a preordered group $(G, \leqslant)$ any preorder $\leq$ on the monoid of finite nonempty subsets with $a \leqslant b \Rightarrow a \leq b$, the meet operation $A \wedge B$ defined as A, B and the monoid operation $A + B$. Then we get the equivariant system of ideals $A \rhd b = A \leq b$.

4 Regularisation of an equivariant system of ideals

Note that both notions, reformulations of regular entailment relation and of equivariant system of ideals, are now predicates on nonempty finite subsets of G. We say that an equivariant system of ideals is *regular* if it satisfies (P_2) and (P_5).

The following proposition follows from Proposition 1.9.

Proposition 4.1. *Let S be an equivariant system of ideals for a preordered group G. Then the meet-monoid generated by S on G is cancellative if, and only if, S is regular.*

Proof. If S is regular, then $\leqslant_S$ is cancellative by Proposition 1.9. Conversely, if $\leqslant_S$ is cancellative, then the meet-monoid it defines embeds into its Grothendieck l-group, which is a distributive lattice. □

We always have the *least* equivariant system of ideals for a preordered group G: $S_{\mathrm{M}}(A) = A \rhd_{\mathrm{M}} 0$ iff A contains an element $\leqslant 0$ in G. It clearly satisfies (P_1) and (P_3), and it satisfies (P_2'): if $A, u \rhd_{\mathrm{M}} 0$ then either $A \rhd_{\mathrm{M}} 0$ or $u \rhd_{\mathrm{M}} 0$, and if $u \rhd_{\mathrm{M}} 0$ then $A \rhd_{\mathrm{M}} u$ implies $A \rhd_{\mathrm{M}} 0$.

Note also that equivariant systems of ideals are closed by arbitrary intersections and directed unions.

Let S be an equivariant system of ideals. We define $T_x(S)$ to be the least equivariant system of ideals Q containing S and such that $Q(x)$. We have $T_xT_y = T_yT_x$ and $T_x(S \cap S') = T_x(S) \cap T_x(S')$ directly from this definition. Lorenzen (1950, page 516) found an elegant direct description of $T_x(S)$.

Proposition 4.2. *$T_x(S)(A)$ iff there exists $k \geqslant 0$ such that $S(A, A - x, \ldots, A - kx)$.*

Proof. If we have $A, A-x, \ldots, A-kx \leqslant_S u$ and $A, A-x, \ldots, A-lx, u, u-x, \ldots, u-lx \leqslant_S v$, then we have by l cuts $A, A-x, \ldots, A-(k+l)x \leqslant_S v$. □

Note that in contradistinction with Lemma 1.5, we cannot simplify this condition to $S(A, A - kx)$ in general: see Examples 8.1 and 8.2.

We next define $U_x(S) = T_x(S) \cap T_{-x}(S)$. We have $U_xU_y = U_yU_x$.

Lemma 4.3. *If S is an equivariant system of ideals such that $U_x(S) = S$ for all x, then S is regular.*

Proof. We show that conditions (P_5) and (P_2) hold.

We have $S(x, -x)$ since we have both $T_x(S)(x, -x)$ and $T_{-x}(S)(x, -x)$. This shows (P_5).

Let us show (P_2). We assume $\wedge(A+B) \wedge \wedge B \leqslant_S 0$ and $\wedge(A+B) \wedge \wedge A \leqslant_S 0$, and we show $\wedge(A+B) \leqslant_S 0$.

Note that we have $T_a(S)(A+B)$ for any a in A by monotonicity: if we force $a \leqslant_S 0$, then $\wedge(A+B) \leqslant_{T_a(S)} \wedge B$, and so $\wedge(A+B) \leqslant_{T_a(S)} 0$ follows from $\wedge(A+B) \wedge \wedge B \leqslant_{T_a(S)} 0$.

Let T be the composition of all the T_{-a} with a in A: we force $0 \leqslant_S a$ for all a in A. We have $\wedge B \leqslant_{T(S)} \wedge(A+B)$, and so $\wedge B \leqslant_{T(S)} 0$ follows from $\wedge(A+B)\wedge\wedge B \leqslant_{T(S)} 0$. This implies $\wedge(A+B) \leqslant_{T(S)} \wedge A$, and so $\wedge(A+B) \leqslant_{T(S)} 0$ follows from $\wedge(A+B)\wedge\wedge A \leqslant_{T(S)} 0$.

We have $\wedge(A+B) \leqslant_{T_a(S)} 0$ and $\wedge(A+B) \leqslant_{T_{-a}(S)} 0$ for all a in A. Since $U_a(S) = S$, we get $\wedge(A+B) \leqslant_S 0$ as desired. □

Let us define $L(S)$ as the (directed) union of the $U_{x_1}\cdots U_{x_n}(S)$, as Lorenzen (1953, §2 and p. 23) did. We get the following theorem.

Theorem 4.4. *$L(S)$ is the least regular system containing S, in other words it is the* regularisation *of S. The l-group granted by Theorem 1.11 for $L(S)$ is called the* Lorenzen l-group *associated to the equivariant system of ideals S.*

5 Constructive version of the Lorenzen-Clifford-Dieudonné Theorem

In particular, we can start from the least equivariant system of ideals for a given preordered group G. In this case, we have $L(S_{\mathrm{M}})(A)$ iff there exist $x_1, \ldots, x_n$ such that for any choice $\epsilon_1, \ldots, \epsilon_n$ of signs ± 1 we can find $k_1, \ldots, k_n \geqslant 0$ and a in A such that $a + \epsilon_1 k_1 x_1 + \cdots + \epsilon_n k_n x_n \leqslant 0$. We clearly have by elimination: if $L(S_{\mathrm{M}})(a)$, then $na \leqslant 0$ for some $n > 0$. We can then deduce from this a constructive version of the Lorenzen-Clifford-Dieudonné Theorem.

Theorem 5.1. *For any commutative preordered group G, we can build an l-group L and a map $f : G \to L$ such that $f(a) \geqslant 0$ iff there exists $n > 0$ such that $na \geqslant 0$. More generally, we have $f(a_1) \vee \cdots \vee f(a_k) \geqslant 0$ iff there exist $n_1, \ldots, n_k \geqslant 0$ such that $n_1 a_1 + \cdots + n_k a_k \geqslant 0$ and $n_1 + \cdots + n_k > 0$.*

Note that this l-group L is the l-group freely generated by the preordered group G.

6 Prüfer's definition of the regularisation

Prüfer (1932) found the following direct definition of the regularisation, which follows directly from Proposition 4.1.

Theorem 6.1. *The regularisation R of an equivariant system of ideals S can be defined by $R(A)$ holding iff there exists B such that $A + B \leqslant_S B$.*

This gives another proof that if we have $L(S_{\mathrm{M}})(a)$ then $na \leqslant 0$ for some $n > 0$: if we have B such that $a + B \leqslant_{S_{\mathrm{M}}} B$ then we have a cycle $a + b_2 \leqslant b_1$, $\ldots$, $a + b_1 \leqslant b_n$, and then $na \leqslant 0$.

7 Noncommutative version

If G is a not necessarily commutative preordered group, we use a multiplicative notation and we define a *regular entailment relation* by the following conditions.

(R_1) $A \vdash B$ if $A \supseteq A'$ and $B \supseteq B'$ and $A' \vdash B'$.

(R_2) $A \vdash B$ if $A, x \vdash B$ and $A \vdash B, x$.

(R_3) $a \vdash b$ if $a \leqslant b$ in G.

(R_4) $A \vdash B$ if $xAy \vdash xBy$.

(R_5) $xa, by \vdash xb, ay$.

Note that (R_5) is satisfied in linearly preordered groups: if $a \leqslant b$, then $xa \wedge by \leqslant xa \leqslant xb \leqslant xb \vee ay$, and if $b \leqslant a$, then $xa \wedge by \leqslant by \leqslant ay \leqslant xb \vee ay$.

If $\vdash$ is a regular entailment relation and $(V, \leqslant)$ is the corresponding distributive lattice, then (R_4) shows that we have a left and right action of G on $\leqslant$.

We define $\leqslant_{a,b}$ to be the lattice preorder with left and right action of G on it obtained from $\leqslant$ by forcing $b \leqslant_{a,b} a$.

We define $u \leqslant^{a,b} v$ by "$xa \wedge uy \leqslant xb \vee vy$ for all x and y in G".

Lemma 7.1. *We have $xa \wedge by \leqslant xb \vee ay$ for all a and b in V and all x and y in G.*

Proof. This holds for a and b in G. Then, if we have $xa_1 \wedge by \leqslant xb \vee a_1 y$ and $xa_2 \wedge by \leqslant xb \vee a_2 y$, we get $xa \wedge by \leqslant xb \vee ay$ for $a = a_1 \wedge a_2$ and for $a = a_1 \vee a_2$. □

Proposition 7.2 (see Lorenzen 1952, Satz 3)**.** *$\leqslant^{a,b}$ defines a lattice quotient of V with left and right action of G on it such that $b \leqslant^{a,b} a$ if a and b are in G.*

Proof. We have $b \leqslant^{a,b} a$ since $xa \wedge by \leqslant xb \vee ay$ for all x and y by the previous Lemma.

If we have $u \leqslant^{a,b} v$ and $v \leqslant^{a,b} w$ then $xa \wedge uy \leqslant xb \vee vy$ and $xa \wedge vy \leqslant xb \vee wy$ for all x and y. By cut, we get $xa \wedge uy \leqslant xb \vee wy$ for all x and y, that is $u \leqslant^{a,b} w$. This shows that the relation $\leqslant^{a,b}$ is transitive. This relation is also reflexive since $xa \wedge uy \leqslant uy \leqslant xb \vee uy$ for all x and y in G.

Finally, if we have $u \leqslant^{a,b} v$, that is $xa \wedge uy \leqslant xb \vee vy$ for all x and y in G, then we also have $zut \leqslant^{a,b} zvt$, that is $xa \wedge zuty \leqslant xb \vee zvty$ for all x and y in G, since we have $z^{-1}xa \wedge uty \leqslant z^{-1}xb \vee vty$ for all x and y in G. □

By definition $u \leqslant_{a,b} v$ implies $u \leqslant^{a,b} v$ since $\leqslant_{a,b}$ is the *least* invariant preorder relation forcing $a \leqslant_{a,b} b$.

Also by definition, note that we have $u \leqslant^{a,b} v$ iff $a \leqslant^{u,v} b$ since $xa \wedge uy \leqslant xb \vee vy$ is equivalent to $x^{-1}u \wedge ay^{-1} \leqslant x^{-1}v \vee by^{-1}$.

Proposition 7.3. *$u \leqslant_{a,b} v$ and $u \leqslant_{b,a} v$ imply $u \leqslant v$.*

Proof. In fact, $u \leqslant_{a,b} v$ implies $u \leqslant^{a,b} v$ which implies $a \leqslant^{u,v} b$. But $u \leqslant_{b,a} v$ implies that u is less than or equal to v in any lattice quotient in which a is less than or equal to b: therefore $u \leqslant^{u,v} v$. So $xu \wedge uy \leqslant xv \vee vy$ for all x, y. In particular for $x = y = 1$ we have $u \leqslant v$. □

It follows from this that V admits a group structure which extends the one on G. In fact, Proposition 7.3 reduces the verification of the required equations to the case where G is linearly preordered by $x \vdash y$, for which $V = G$. This is the noncommutative analogue of Theorem 1.11.

The difference between the noncommutative case and the commutative one is the following. In the commutative case, we give an *explicit* description of the relation $\vdash_x$; then we use Proposition 1.7 to show that we can reason by case distinction, forcing $0 \leqslant x$ or $x \leqslant 0$. In the noncommutative case, we use Proposition 7.3 to show that we can reason by case distinction, forcing $a \leqslant b$ or $b \leqslant a$, without recourse to an explicit description of the relation $\leqslant_{a,b}$. The proof is shorter and very smart, but gives less information than in the commutative case.

8 Examples

Example 8.1. We illustrate here the remark made after Proposition 4.2.

Let us consider the group $\mathbf{Z} = (\mathbf{Z}, 0, +, -)$ preordered by the relation $x \preceq y$ defined as $y \in x + 60\mathbf{N}$. We consider the meet-monoid $(M, 0, +, -, \leq)$ freely generated by $(\mathbf{Z}, \preceq)$. The elements of M are formal finite meets of elements of $\mathbf{Z}$. E.g. we have in M

$$a = 10 \wedge 24 \leq b = 130 \wedge 84$$

since $10 \preceq 130$ and $24 \preceq 84$.

Now let us consider the equivariant system of ideals $\leq_7$ for M that we get by forcing $0 \leq_7 7$, i.e. $-7 \leq_7 0$ (see Proposition 4.2).

We have $3 \leq_7 b$ since

$$3 \wedge (3 + 7) \wedge (3 + 21) = 3 \wedge a \leq a \leq b.$$

But $3 \wedge (3 + 21) \not\leq b$.

On the other hand we see easily that $-1 \leq_{-1} 0$ and $-1 \leq_1 0$, so that in the regularisation of M we have $0 \leqslant 1$, which shows that the regularisation is the integer ring $\mathbf{Z}$ with the usual preorder.

Example 8.2. The following similar example is from algebraic number theory.

We consider the ring $\mathbf{Z}[x]$ with x an algebraic integer solution of $x^3 - x^2 + x + 7 = 0$. We denote by $a_1, \ldots, a_k \rhd_d b$ the Dedekind equivariant system of ideals for the divisibility group G of $\mathbf{Z}[x]$, defined as $b \in (a_1, \ldots, a_k)\mathbf{Z}[x]$ for b and the a_i's in the fraction field $\mathbf{Q}[x]$. In fact, the finitely generated fractional ideals form a meet-monoid $(M, \leq)$ extending the divisibility group G. The corresponding preorder is given by $a_1 \wedge \cdots \wedge a_k \leq b_1 \wedge \cdots \wedge b_h$ iff each b_i belongs to $(a_1, \ldots, a_k)\mathbf{Z}[x]$.

The ring $\mathbf{Z}[x]$ is not integrally closed. The element $y = \frac{1}{2}(x^2 + 1)$ of $\mathbf{Q}[x]$ is integral over $\mathbf{Z}$ and a fortiori over $\mathbf{Z}[x]$: $y^3 = y^2 - 4y + 4$, or equivalently $1 = z - 4z^2 + 4z^3$ with $z = y^{-1}$.

Let us denote by $\vdash$ the regularisation of $\leq$. Now let us consider, for $u \in M$, the equivariant system of ideals $\leq_u$ that we get by forcing $1 \leq_u u$, i.e. $u^{-1} \leq_u 1$. We see that $1 \vdash y$, i.e. $z \vdash 1$, by showing $z \leq_y 1$ (which holds by definition) and $z \leq_z 1$, which is certified (using Proposition 4.2) by $z, z^2, z^3 \leq 1$, since the fractional ideal $z\mathbf{Z}[x] + z^2\mathbf{Z}[x] + z^3\mathbf{Z}[x]$ contains 1.

Moreover we remark that $z\mathbf{Z}[x] + z^3\mathbf{Z}[x]$ does not contain 1.

Example 8.3. Let us consider the group $\mathbf{Z} = (\mathbf{Z}, 0, +, -)$ preordered by the relation $x = y$. We compute the corresponding Lorenzen l-group.

We denote by $\mathbb{Z}$ the group $\mathbf{Z}$ with the usual preorder $\leqslant$, and by sup and inf the associated supremum and infimum. We denote by $\mathbb{Z}^\circ$ the conversely preordered group.

We consider the meet-monoid $(M, 0, +, -, \leq)$ freely generated by $(\mathbf{Z}, =)$. The elements of M are formal finite meets of elements of $\mathbf{Z}$. We have $\wedge A \leq b$ iff $b \in A$, and $\wedge A \leq \wedge B$ iff $B \subseteq A$.

We denote by $\leq_n$ the equivariant system of ideals that we get by forcing $0 \leq_n n$. Note that $0 \leq_1 n$ for $n \geqslant 0$. Using Proposition 4.2, we find that $A \leq_{-1} b$ iff $b \leqslant \sup(A)$ and $A \leq_1 b$ iff $b \geqslant \inf(A)$. We deduce that the regularisation of $(M, \leq)$ can be described as the set of intervals $[\![m\,..\,n]\!]$ inside $\mathbb{Z}$ with the order by inclusion. Equivalently, it is identified as the set of pairs $(m, n) \in \mathbb{Z} \times \mathbb{Z}^\circ$ such that $m \leqslant n$. Now it is easy to see that the corresponding Grothendieck l-group is $\mathbb{Z} \times \mathbb{Z}^\circ$, where the opposite of (m, n) can be identified with $(-m, -n)$. The canonical morphism $(\mathbf{Z}, =) \to \mathbb{Z} \times \mathbb{Z}^\circ$ is $m \mapsto (m, m)$.

Note that since $(\mathbf{Z}, =)$ is the free abelian group on a singleton, we recover in this rather complicated way $\mathbb{Z} \times \mathbb{Z}^\circ$ as the free l-group on a singleton.

References

Jan Cederquist and Thierry Coquand. Entailment relations and distributive lattices. In *Logic Colloquium '98: proceedings of the annual European summer meeting of the Association for symbolic logic, held in Prague, Czech Republic,*

August 9–15, 1998, edited by Samuel R. Buss, Petr Hájek, and Pavel Pudlák, 127–139. Lecture notes in logic, 13, Association for symbolic logic, Urbana, 2000.

Thierry Coquand, Henri Lombardi and Stefan Neuwirth. Lattice-ordered groups generated by an ordered group and regular systems of ideals. *Rocky Mountain J. Math.*, 49, 1449–1489, 2019. doi:10.1216/rmj-2019-49-5-1449.

Henri Lombardi and Claude Quitté. *Commutative algebra: constructive methods. Finite projective modules*. Algebra and applications, 20, Springer, Dordrecht, 2015. Translated from the French (Calvage & Mounet, Paris, 2011, revised and extended by the authors) by Tania K. Roblot.

Paul Lorenzen. Abstrakte Begründung der multiplikativen Idealtheorie. *Math. Z.*, 45, 533–553, 1939. `http://eudml.org/doc/168865`.

Paul Lorenzen. Über halbgeordnete Gruppen. *Math. Z.*, 52, 483–526, 1950. `http://eudml.org/doc/169131`.

Paul Lorenzen. Algebraische und logistische Untersuchungen über freie Verbände. *J. Symbolic Logic*, 16, 81–106, 1951. `http://www.jstor.org/stable/2266681`. Translation by Stefan Neuwirth: *Algebraic and logistic investigations on free lattices*, `http://arxiv.org/abs/1710.08138`.

Paul Lorenzen. Teilbarkeitstheorie in Bereichen. *Math. Z.*, 55, 269–275, 1952. `http://eudml.org/doc/169251`.

Paul Lorenzen. Die Erweiterung halbgeordneter Gruppen zu Verbandsgruppen. *Math. Z.*, 58, 15–24, 1953. `http://eudml.org/doc/169331`.

Heinz Prüfer. Untersuchungen über Teilbarkeitseigenschaften in Körpern. *J. Reine Angew. Math.*, 168, 1–36, 1932. `http://eudml.org/doc/149823`.