The Sidon constant of sets with three elements

Stefan Neuwirth

Abstract

We solve an elementary extremal problem on trigonometric polynomials and obtain the exact value of the Sidon constant for sets with three elements $\{n_0, n_1, n_2\}$: it is

$$\sec\left(\pi \gcd(n_1 - n_0, n_2 - n_0)/2\max|n_i - n_j|\right).$$

1 Introduction

Let $\Lambda = \{\lambda_0, \lambda_1, \lambda_2\}$ be a set of three frequencies and $\varrho_0, \varrho_1, \varrho_2$ three positive intensities. We solve the following extremal problem:

(†) To find $\vartheta_0, \vartheta_1, \vartheta_2$ three phases such that, putting $c_j = \varrho_j \mathrm{e}^{\mathrm{i}\vartheta_j}$, the maximum $\max_t |c_0\mathrm{e}^{\mathrm{i}\lambda_0 t} + c_1\mathrm{e}^{\mathrm{i}\lambda_1 t} + c_2\mathrm{e}^{\mathrm{i}\lambda_2 t}|$ is minimal.

This enables us to generalise a result of D. J. Newman. He solved the following extremal problem for $\Lambda = \{0, 1, 2\}$:

(‡) To find $f(t) = c_0\mathrm{e}^{\mathrm{i}\lambda_0 t} + c_1\mathrm{e}^{\mathrm{i}\lambda_1 t} + c_2\mathrm{e}^{\mathrm{i}\lambda_2 t}$ with $\|f\|_\infty = \max_t |f(t)| \leqslant 1$ such that $\|\widehat{f}\|_1 = |c_0| + |c_1| + |c_2|$ is maximal.

Note that for such an f, $\|\widehat{f}\|_1$ is the Sidon constant of Λ. Newman's argument is the following (see [6, Chapter 3]): by the parallelogram law,

$$\begin{aligned}
\max_t |f(t)|^2 &= \max_t |f(t)|^2 \vee |f(t+\pi)|^2 \\
&\geqslant \max_t \big(|f(t)|^2 + |f(t+\pi)|^2\big)/2 \\
&= \max_t \big(|c_0 + c_1\mathrm{e}^{\mathrm{i}t} + c_2\mathrm{e}^{\mathrm{i}2t}|^2 + |c_0 - c_1\mathrm{e}^{\mathrm{i}t} + c_2\mathrm{e}^{\mathrm{i}2t}|^2\big)/2 \\
&= \max_t |c_0 + c_2\mathrm{e}^{\mathrm{i}2t}|^2 + |c_1|^2 = \big(|c_0| + |c_2|\big)^2 + |c_1|^2 \\
&\geqslant \big(|c_0| + |c_1| + |c_2|\big)^2/2
\end{aligned}$$

and equality holds exactly for multiples and translates of $f(t) = 1 + 2\mathrm{i}\mathrm{e}^{\mathrm{i}t} + \mathrm{e}^{\mathrm{i}2t}$.

Let us describe this paper briefly. We use a real-variable approach: Problem (†) reduces to studying a function of form

$$\Phi(t, \vartheta) = |1 + r\mathrm{e}^{\mathrm{i}\vartheta}\mathrm{e}^{\mathrm{i}kt} + s\mathrm{e}^{\mathrm{i}lt}|^2 \text{ for } r, s > 0,\ k \neq l \in \mathbb{Z}^*$$

and more precisely $\Phi^*(\vartheta) = \max_t \Phi(t, \vartheta)$. We obtain the variations of Φ^*: the point is that we find "by hand" a local minimum of Φ^* and that any two minima of Φ^* are separated by a maximum of Φ^*, which corresponds to an extremal point of Φ and therefore has a handy description. The solution to Problem (‡) then turns out to derive easily from this.

The initial motivation was twofold. In the first place, we wanted to decide whether sets $\Lambda = \{\lambda_n\}$ such that λ_{n+1}/λ_n is bounded by some q may have a Sidon constant arbitrarily close to 1 and to find evidence among sets with three elements. That there are such sets, arbitrarily large but finite, may in fact be proven by the method of Riesz products in [2, Appendix V, §1.II]. In the second place, we wished to show that the real and complex unconditionality constants are distinct for basic sequences of characters e^{int}; we prove however that they coincide in the space $\mathscr{C}(\mathbb{T})$ for sequences with three terms.

Notation. $\mathbb{T} = \{\, z \in \mathbb{C} : |z| = 1 \,\}$ and $\mathrm{e}_\lambda(z) = z^\lambda$ for $z \in \mathbb{T}$ and $\lambda \in \mathbb{Z}$.

2 Definitions

Definition 2.1. (1) Let $\Lambda \subseteq \mathbb{Z}$. Λ is a Sidon set if there is a constant C such that for all trigonometric polynomials $f(t) = \sum_{\lambda\in\Lambda} c_\lambda \mathrm{e}^{\mathrm{i}\lambda t}$ with spectrum in Λ we have

$$\|\widehat{f}\|_1 = \sum_{\lambda\in\Lambda} |c_\lambda| \leqslant C \max_t |f(t)| = \|f\|_\infty.$$

The optimal C is called the Sidon constant of Λ.

(2) Let X be a Banach space. The sequence $(x_n) \subseteq X$ is a real (vs. complex) unconditional basic sequence in X if there is a constant C such that

$$\Big\|\sum \vartheta_n c_n x_n\Big\|_X \leqslant C\Big\|\sum c_n x_n\Big\|_X$$

for every real (vs. complex) choice of signs $\vartheta_n \in \{-1, 1\}$ (vs. $\vartheta_n \in \mathbb{T}$) and every finitely supported family of coefficients (c_n). The optimal C is the real (vs. complex) unconditionality constant of (x_n) in X.

Let us state the two following well known facts.

Proposition 2.2. (1) *The Sidon constant of* Λ *is the complex unconditionality constant of the sequence of functions* $(\mathrm{e}_\lambda)_{\lambda\in\Lambda}$ *in the space* $\mathscr{C}(\mathbb{T})$.

(2) *The complex unconditionality constant is at most* $\pi/2$ *times the real unconditionality constant.*

Proof. (1) holds because $\big\|\sum \vartheta_\lambda c_\lambda \mathrm{e}_\lambda\big\|_\infty = \sum |c_\lambda|$ for $\vartheta_\lambda = \overline{c_\lambda}/|c_\lambda|$.

(2) Because the complex unconditionality constant of the sequence (ϵ_n) of Rademacher functions in $\mathscr{C}(\{-1,1\}^\infty)$ is $\pi/2$ (see [5]),

$$\begin{aligned}
\sup_{\vartheta_n\in\mathbb{T}} \Big\|\sum \vartheta_n c_n x_n\Big\|_X &= \sup_{x^*\in B_{X^*}} \sup_{\vartheta_n\in\mathbb{T}} \sup_{\epsilon_n=\pm1} \Big|\sum \vartheta_n c_n \langle x^*, x_n\rangle \epsilon_n\Big| \\
&\leqslant \pi/2 \sup_{x^*\in B_{X^*}} \sup_{\epsilon_n=\pm1} \Big|\sum c_n \langle x^*, x_n\rangle \epsilon_n\Big| \\
&= \pi/2 \sup_{\epsilon_n=\pm1} \Big\|\sum \epsilon_n c_n x_n\Big\|_X.
\end{aligned}$$

Furthermore the real unconditionality constant of (ϵ_n) in $\mathscr{C}(\{-1,1\}^\infty)$ is 1: therefore the factor $\pi/2$ is optimal. ☐

Let us straighten out the expression of the Sidon constant. For

$$f(t) = c_0 \mathrm{e}^{\mathrm{i}\lambda_0 t} + c_1 \mathrm{e}^{\mathrm{i}\lambda_1 t} + c_2 \mathrm{e}^{\mathrm{i}\lambda_2 t},\ c_j = \varrho_j \mathrm{e}^{\mathrm{i}\vartheta_j},$$

the supremum norm $\|f\|_\infty$ of f is equal to

$$\|\varrho_0 + \varrho_1 \mathrm{e}^{\mathrm{i}\vartheta} \mathrm{e}_{\lambda_1-\lambda_0} + \varrho_2 \mathrm{e}_{\lambda_2-\lambda_0}\|_\infty,\ \vartheta = \frac{\lambda_1-\lambda_2}{\lambda_2-\lambda_0}\vartheta_0 + \vartheta_1 + \frac{\lambda_0-\lambda_1}{\lambda_2-\lambda_0}\vartheta_2 \tag{1}$$

and therefore the Sidon constant C of $\Lambda = \{\lambda_0, \lambda_1, \lambda_2\}$ may be written

$$C = \max_{r,s>0,\vartheta} (1+r+s)/\|1 + r\mathrm{e}^{\mathrm{i}\vartheta}\mathrm{e}_k + s\mathrm{e}_l\|_\infty \quad \text{with } \begin{cases} k = \lambda_1 - \lambda_0 \\ l = \lambda_2 - \lambda_0. \end{cases} \tag{2}$$

By change of variables, we may suppose w.l.o.g. that k and l are coprime.

3 A solution to Extremal problem (†)

Let us first establish

Lemma 3.1. *Let* $\lambda_1, \ldots, \lambda_k \in \mathbb{Z}^*$ *and* $\varrho_1, \ldots, \varrho_k > 0$. *Let*

$$f(t,\vartheta) = 1 + \varrho_1 \mathrm{e}^{\mathrm{i}(\lambda_1 t + \vartheta_1)} + \cdots + \varrho_{k-1} \mathrm{e}^{\mathrm{i}(\lambda_{k-1} t + \vartheta_{k-1})} + \varrho_k \mathrm{e}^{\mathrm{i}\lambda_k t}$$

and $\Phi(t,\vartheta) = |f(t,\vartheta)|^2$. *The critical points* (t,ϑ) *such that* $\nabla\Phi(t,\vartheta) = 0$ *satisfy either* $f(t,\vartheta) = 0$ *or* $\lambda_1 t + \vartheta_1 \equiv \cdots \equiv \lambda_{k-1} t + \vartheta_{k-1} \equiv \lambda_k t \equiv 0 \mod \pi$.

Proof. As $\Phi = (\Re f)^2 + (\Im f)^2$, the critical points (t,ϑ) satisfy

$$\begin{cases} \quad \Re\frac{\partial f}{\partial t}(t,\vartheta)\,\Re f(t,\vartheta) + \quad \Im\frac{\partial f}{\partial t}(t,\vartheta)\,\Im f(t,\vartheta) = 0 \\ -\sin(\lambda_i t + \vartheta_i)\,\Re f(t,\vartheta) + \cos(\lambda_i t + \vartheta_i)\,\Im f(t,\vartheta) = 0 \quad (1 \leqslant i \leqslant k-1), \end{cases}$$

which simplifies to

$$-\sin(\lambda_i t + \vartheta_i)\,\Re f(t,\vartheta) \; + \cos(\lambda_i t + \vartheta_i)\,\Im f(t,\vartheta) = 0 \quad (1 \leqslant i \leqslant k,\ \vartheta_k = 0).$$

Suppose that $f(t,\vartheta) \neq 0$: then the system above implies that

$$-\sin(\lambda_i t + \vartheta_i)\cos(\lambda_j t + \vartheta_j) + \cos(\lambda_i t + \vartheta_i)\sin(\lambda_j t + \vartheta_j) = 0 \; (1 \leqslant i,j \leqslant k,\ \vartheta_k = 0)$$

and it simplifies therefore to

$$\sin(\lambda_i t + \vartheta_i) = 0 \quad (1 \leqslant i \leqslant k,\ \vartheta_k = 0).$$

□

The following result is the core of the paper.

Lemma 3.2. *Let* $r, s > 0$, $k, l \in \mathbb{Z}^*$ *distinct and coprime. Let*

$$\begin{aligned} \Phi(t,\vartheta) &= |1 + r\mathrm{e}^{\mathrm{i}\vartheta}\mathrm{e}^{\mathrm{i}kt} + s\mathrm{e}^{\mathrm{i}lt}|^2 \\ &= 1 + r^2 + s^2 + 2r\cos(kt + \vartheta) + 2s\cos lt + 2rs\cos((l-k)t - \vartheta). \end{aligned}$$

Let $\Phi^*(\vartheta) = \max_t \Phi(t,\vartheta)$. *Then* Φ^* *is an even function with period* $2\pi/|l|$ *that decreases on* $[0, \pi/|l|]$. *Therefore* $\min_\vartheta \Phi^*(\vartheta) = \Phi^*(\pi/l)$.

Proof. Φ^* is continuous (see [4, Chapter 5.4]) and even, as $\Phi(t,-\vartheta) = \Phi(-t,\vartheta)$. Φ^* is $(2\pi/|l|)$-periodical: let $j \in \mathbb{Z}$ be such that $jk \equiv 1$ mod. l. Then

$$\Phi(t + 2j\pi/l, \vartheta) = |1 + r\mathrm{e}^{\mathrm{i}(\vartheta + 2\pi jk/l)}\mathrm{e}^{\mathrm{i}kt} + s\mathrm{e}^{\mathrm{i}lt}|^2 = \Phi(t, \vartheta + 2\pi/l).$$

Thus Φ^* attains its minimum on $[0, \pi/|l|]$. Furthermore, we have

$$\Phi(-t - 2j\pi/l, \pi/l - \vartheta) = \Phi(t + 2j\pi/l, -\pi/l + \vartheta) = \Phi(t, \pi/l + \vartheta),$$

so that Φ^* has an extremum at π/l. Now

$$\Phi^*(\pi/l + \vartheta) = \Phi^*(\pi/l) + |\vartheta| \max_{\Phi(t,\pi/l) = \Phi^*(\pi/l)} \left|\frac{\partial\Phi}{\partial\vartheta}(t,\pi/l)\right| + o(\vartheta).$$

Choose a t such that $\Phi(t,\pi/l) = \Phi^*(\pi/l)$. If $\partial\Phi/\partial\vartheta(t,\pi/l) \neq 0$, then this shows that Φ^* has a local minimum and a cusp at π/l. Let us now suppose that $\partial\Phi/\partial\vartheta(t,\pi/l) = 0$. If Φ^* had a local maximum at π/l, then $(t,\pi/l)$ would be a critical point of Φ, so that by Lemma 3.1 $\cos(kt + \pi/l) = \delta$, $\cos lt = \epsilon$, $\cos((l-k)t - \pi/l) = \delta\epsilon$ for some $\delta, \epsilon \in \{-1, 1\}$. One necessarily would have $(\delta,\epsilon) \neq (1,1)$. Furthermore,

$$\begin{aligned} \frac{\partial^2\Phi}{\partial\vartheta^2}(t,\pi/l) &= -2r\delta(1 + s\epsilon) \leqslant 0 \\ \begin{vmatrix} \partial^2\Phi/\partial t^2 & \partial^2\Phi/\partial t\partial\vartheta \\ \partial^2\Phi/\partial\vartheta\partial t & \partial^2\Phi/\partial\vartheta^2 \end{vmatrix}(t,\pi/l) &= 4rsl^2(\delta\epsilon + r\epsilon + s\delta) \geqslant 0, \end{aligned}$$

which would imply $\epsilon = -1$, $r = 0$, $s = 1$. Therefore Φ^* has a local minimum at π/l. Let us show that then Φ^* must decrease on $[0, \pi/|l|]$. Otherwise there are $0 \leqslant \vartheta_0 < \vartheta_1 \leqslant \pi/|l|$ such that $\Phi^*(\vartheta_1) > \Phi^*(\vartheta_0)$. As $\pi/|l|$ is a local minimum, there is a $\vartheta_0 < \vartheta^* < \pi/|l|$ such that

$$\Phi^*(\vartheta^*) = \max_{\vartheta_0 \leqslant \vartheta \leqslant \pi/|l|} \Phi^*(\vartheta) = \max_{\substack{0 \leqslant t < 2\pi \\ \vartheta_0 \leqslant \vartheta \leqslant \pi/|l|}} \Phi(t, \vartheta),$$

i.e., there further is some t^* such that Φ has a local maximum at (t^*, ϑ^*). But then $kt^* + \vartheta^* \equiv lt^* \equiv 0 \mod \pi$ and $\vartheta^* \equiv 0 \mod \pi/l$ and this is false. ☐

By Computation (1) and Lemma 3.2, we obtain

Theorem 3.3. *Let $\lambda_0, \lambda_1, \lambda_2 \in \mathbb{R}$ and $\varrho_0, \varrho_1, \varrho_2 > 0$. The solution to Extremal problem (†) is the following.*

- *If the smallest additive group containing $\lambda_1 - \lambda_0$ and $\lambda_2 - \lambda_0$ is dense in $\mathbb{R}$, then the maximum is independent of the phases $\vartheta_0, \vartheta_1, \vartheta_2$ and makes $\varrho_0 + \varrho_1 + \varrho_2$.*
- *Otherwise let $d = \gcd(\lambda_1 - \lambda_0, \lambda_2 - \lambda_0)$ be a generator of this group. Then the sought phases $\vartheta_0, \vartheta_1, \vartheta_2$ are given by*

$$\vartheta_0(\lambda_2 - \lambda_1) + \vartheta_1(\lambda_0 - \lambda_2) + \vartheta_2(\lambda_1 - \lambda_0) \equiv d\pi \mod 2d\pi.$$

  In particular, these phases may be chosen among 0 and π.

4 A solution to Extremal problem (‡)

There are two cases where one can make explicit computations by Lemma 3.2.

Example 4.1. The real and complex unconditionality constant of $\{0, 1, 2\}$ in $\mathscr{C}(\mathbb{T})$ is $\sqrt{2}$. Indeed, a case study shows that

$$\|1 + \mathrm{i}r\mathrm{e}_1 + s\mathrm{e}_2\|_\infty = \begin{cases} r + |s-1| & \text{if } r|s-1| \geqslant 4s \\ (1+s)(1 + r^2/4s)^{1/2} & \text{if } r|s-1| \leqslant 4s \end{cases}$$

and this permits to compute the maximum (2), which is obtained for $r = 2$, $s = 1$. This yields another proof to Newman's result presented in the Introduction.

Example 4.2. The real and complex unconditionality constant of $\{0, 1, 3\}$ in $\mathscr{C}(\mathbb{T})$ is $2/\sqrt{3}$. Indeed, a case study shows that $\|1 + r\mathrm{e}^{\mathrm{i}\pi/3}\mathrm{e}_1 + s\mathrm{e}_3\|_\infty$ makes

$$\begin{cases} 1 + r - s & \text{if } s \leqslant r/(4r+9) \\ \left(\frac{2}{27}s(r^2 + 9 + 3r/s)^{3/2} - \frac{2}{27}r^3 s + \frac{2}{3}r^2 + rs + s^2 + 1\right)^{1/2} & \text{if } s \geqslant r/(4r+9) \end{cases}$$

and this permits to compute the maximum (2), which is obtained exactly at $r = 3/2$, $s = 1/2$.

These examples are particular cases of the following theorem.

Theorem 4.3. *Let $\lambda_0, \lambda_1, \lambda_2 \in \mathbb{Z}$ be distinct. Then the Sidon constant of $\Lambda = \{\lambda_0, \lambda_1, \lambda_2\}$ is $\sec(\pi/2n)$, where $n = \max|\lambda_i - \lambda_j| / \gcd(\lambda_1 - \lambda_0, \lambda_2 - \lambda_0)$.*

Proof. We may suppose $\lambda_0 < \lambda_1 < \lambda_2$. Let $k = (\lambda_1 - \lambda_0)/\gcd(\lambda_1 - \lambda_0, \lambda_2 - \lambda_0)$ and $l = (\lambda_2 - \lambda_0)/\gcd(\lambda_1 - \lambda_0, \lambda_2 - \lambda_0)$. By Lemma 3.2, the Arithmetic-Geometric Mean Inequality bounds the Sidon constant C of $\{0, k, l\}$ in the following way:

$$\begin{aligned} C = \max_{r,s>0} \frac{1 + r + s}{\|1 + r\mathrm{e}^{\mathrm{i}\pi/l}\mathrm{e}_k + s\mathrm{e}_l\|_\infty} &\leqslant \max_{r,s>0} \frac{1 + r + s}{|1 + r\mathrm{e}^{\mathrm{i}\pi/l} + s|} \\ &= \max_{r,s>0} \left(1 - \sin^2 \frac{\pi}{2l} \frac{4r(1+s)}{(1+r+s)^2}\right)^{-1/2} \\ &\leqslant \left(1 - \sin^2(\pi/2l)\right)^{-1/2} = \sec(\pi/2l). \end{aligned}$$

This inequality is sharp: we have equality for $s = k/(l-k)$ and $r = 1+s$. In fact the derivative of $|1 + r\mathrm{e}^{\mathrm{i}\pi/l}\mathrm{e}^{\mathrm{i}kt} + s\mathrm{e}^{\mathrm{i}lt}|^2$ is then

$$\frac{8kl}{k-l}\cos\frac{kt+\pi/l}{2}\sin\frac{lt}{2}\cos\frac{(l-k)t-\pi/l}{2},$$

so that its critical points are

$$\frac{2j+1}{k}\pi - \frac{\pi}{kl},\ \frac{2j}{l}\pi,\ \frac{2j+1}{l-k}\pi + \frac{\pi}{l(l-k)} : j \in \mathbb{Z},$$

where it makes

$$4s^2\sin^2\frac{2j+1+l}{2k}\pi,\ 4r^2\cos^2\frac{2j+1}{2l}\pi,\ 4\cos^2\frac{2j+1+k}{2(l-k)}\pi : j \in \mathbb{Z}.$$

Therefore the maximum of $|1 + r\mathrm{e}^{\mathrm{i}\pi/l}\mathrm{e}^{\mathrm{i}kt} + s\mathrm{e}^{\mathrm{i}lt}|$ is $2r\cos(\pi/2l)$. □

This proof and (1) yield also the more precise

Proposition 4.4. *Let $\Lambda = \{\lambda_0, \lambda_1, \lambda_2\} \subseteq \mathbb{Z}$. The solution to Extremal problem (‡) is a multiple of*

$$f(t) = \epsilon_0\,|\lambda_1 - \lambda_2|\,\mathrm{e}^{\mathrm{i}\lambda_0 t} + \epsilon_1\,|\lambda_0 - \lambda_2|\,\mathrm{e}^{\mathrm{i}\lambda_1 t} + \epsilon_2\,|\lambda_0 - \lambda_1|\,\mathrm{e}^{\mathrm{i}\lambda_2 t}$$

with $\epsilon_0, \epsilon_1, \epsilon_2 \in \{-1, 1\}$ real signs such that

- *$\epsilon_0\epsilon_1 = -1$ if $2^j \mid \lambda_1 - \lambda_0$ and $2^j \nmid \lambda_2 - \lambda_0$ for some j;*
- *$\epsilon_0\epsilon_2 = -1$ if $2^j \nmid \lambda_1 - \lambda_0$ and $2^j \mid \lambda_2 - \lambda_0$ for some j;*
- *$\epsilon_1\epsilon_2 = -1$ otherwise.*

The Sidon constant of Λ is attained for this f. Therefore the complex and real unconditionality constants of $\{\mathrm{e}_\lambda\}_{\lambda\in\Lambda}$ in $\mathscr{C}(\mathbb{T})$ coincide for sets Λ with three elements.

5 Some consequences

Let us underline the following consequences of our computation.

Corollary 5.1. (1) *The Sidon constant of sets with three elements is at most $\sqrt{2}$.*

(2) *The Sidon constant of $\{0, n, 2n\}$ is $\sqrt{2}$, while the Sidon constant of $\{0, n+1, 2n\}$ is at most $\sec(\pi/2n) = 1 + \pi^2/8n^2 + o(n^{-2})$ and thus arbitrarily close to 1.*

(3) *The Sidon constant of $\{\lambda_0 < \lambda_1 < \lambda_2\}$ does not depend on λ_1 but on the g.c.d. of $\lambda_1 - \lambda_0$ and $\lambda_2 - \lambda_0$.*

Theorem 4.3 also shows anew that no set of integers with more than two elements has Sidon constant 1 (see [6, p. 21] or [1]). Recall now that $\Lambda = \{\lambda_n\} \subseteq \mathbb{Z}$ is a Hadamard set if there is a $q > 1$ such that $|\lambda_{n+1}/\lambda_n| \geqslant q$ for all n. By [3, Cor. 9.4], the Sidon constant of Λ is at most $1 + \pi^2/(2q^2 - 2 - \pi^2)$ if $q > \sqrt{\pi^2/2 + 1} \approx 2.44$. On the other hand Theorem 4.3 shows

Corollary 5.2. (1) *If there is an integer $q \geqslant 2$ such that $\Lambda \supseteq \{\lambda, \lambda + \mu, \lambda + q\mu\}$ for some integers λ and μ, then the Sidon constant of Λ is at least*

$$\sec(\pi/2q) > 1 + \pi^2/(8q^2).$$

(2) *In particular, we have the following bounds for the Sidon constant C of the set $\Lambda = \{q^k\}$, $q \in \mathbb{Z} \setminus \{0, \pm1, \pm2\}$:*

$$1 + \frac{\pi^2}{8\max(-q, q+1)^2} < C \leqslant 1 + \frac{\pi^2}{2q^2 - 2 - \pi^2}.$$

6 Three questions

(a) Is there a set Λ for which the real and complex unconditionality constants of $\{e_\lambda\}_{\lambda\in\Lambda}$ in $\mathscr{C}(\mathbb{T})$ differ? The same question is open in spaces $L^p(\mathbb{T})$, $1 \leqslant p < \infty$, and even for the case of three element sets if p is not a small even integer, and especially for the set $\{0, 1, 2, 3\}$ in any space but $L^2(\mathbb{T})$.

(b) Let $q > 1$. Are there infinite sets $\Lambda = \{\lambda_n\}$ such that $|\lambda_{n+1}/\lambda_n| \leqslant q$ with Sidon constant arbitrarily close to 1? What about the sequence of integer parts of the powers of a transcendental number $\sigma > 1$ (see [3, Cor. 2.10, Prop. 3.2])?

(c) The only set with more than three elements with known Sidon constant is $\{0, 1, 2, 3, 4\}$, for which it makes 2 (see [6, Chapter 3]). Can one compute the Sidon constant of sets with four elements? I conjecture that the Sidon constant of $\{0, 1, 2, 3\}$ is $5/3$.

Bibliography

[1] Donald I. Cartwright, Robert B. Howlett, and John R. McMullen. Extreme values for the Sidon constant. *Proc. Amer. Math. Soc.*, 81(4):531–537, 1981. (p. 5).

[2] Jean-Pierre Kahane and Raphaël Salem. *Ensembles parfaits et séries trigonométriques*. Hermann, 1963. (p. 1).

[3] Stefan Neuwirth. Metric unconditionality and Fourier analysis. *Studia Math.*, 131:19–62, 1998. (pp. 5 and 6).

[4] George Pólya and Gábor Szegő. *Problems and theorems in analysis. Vol. I: Series, integral calculus, theory of functions*. Springer-Verlag, 1972. (p. 3).

[5] Josef A. Seigner. Rademacher variables in connection with complex scalars. *Acta Math. Univ. Comenian. (N.S.)*, 66:329–336, 1997. (p. 2).

[6] Harold S. Shapiro. Extremal problems for polynomials and power series. Master's thesis, Massachusetts Institute of Technology, 1951. (pp. 1, 5, and 6).