Transfer of Fourier multipliers into Schur multipliers and sumsets in a discrete group

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Abstract

We inspect the relationship between relative Fourier multipliers on noncommutative Lebesgue-Orlicz spaces of a discrete group Γ and relative Toeplitz-Schur multipliers on Schatten-von-Neumann-Orlicz classes. Four applications are given: lacunary sets, unconditional Schauder bases for the subspace of a Lebesgue space determined by a given spectrum $\Lambda \subseteq \Gamma$, the norm of the Hilbert transform and the Riesz projection on Schatten-von-Neumann classes with exponent a power of 2, and the norm of Toeplitz Schur multipliers on Schatten-von-Neumann classes with exponent less than 1.

1 Introduction

Let Λ be a subset of $\mathbb{Z}$ and let x be a bounded measurable function on the circle $\mathbb{T}$ with Fourier spectrum in Λ: we write $x \in \mathrm{L}^\infty_\Lambda$, $x \sim \sum_{k\in\Lambda} x_k z^k$. The matrix of the associated operator $y \mapsto xy$ on L^2 with respect to its trigonometric basis is the Toeplitz matrix

$$(x_{r-c})_{(r,c)\in\mathbb{Z}\times\mathbb{Z}} = \begin{array}{cc} & \begin{array}{ccccc}\cdots & 1 & 0 & -1 & \cdots\end{array} \\ \begin{array}{c}\vdots\\ 1\\ 0\\ -1\\ \vdots\end{array} & \begin{pmatrix}\ddots&\ddots&\ddots&\ddots&\ddots\\ \ddots & x_0 & x_1 & x_2 & \ddots \\ \ddots & x_{-1} & x_0 & x_1 & \ddots\\ \ddots & x_{-2} & x_{-1} & x_0 & \ddots\\ \ddots&\ddots&\ddots&\ddots&\ddots\end{pmatrix}\end{array}$$

with support in $\ddot{\Lambda} = \{(r,c) : r - c \in \Lambda\}$.

This is a point of departure for the interplay of harmonic analysis and operator theory. In the general case of a discrete group Γ, the counterpart to a bounded measurable function is defined as a bounded operator on ℓ^2_Γ whose matrix has the form $(x_{rc^{-1}})_{(r,c)\in\Gamma\times\Gamma}$ for some sequence $(x_\gamma)_{\gamma\in\Gamma}$. This will be the framework of the body of this article, while the introduction sticks to the case $\Gamma = \mathbb{Z}$.

We are concerned with two kinds of multipliers. A sequence $\varphi = (\varphi_k)_{k\in\Lambda}$ defines

- the relative Fourier multiplication operator on trigonometric polynomials with spectrum in Λ by
$$\sum_{k\in\Lambda} x_k z^k \mapsto \sum_{k\in\Lambda} \varphi_k x_k z^k; \tag{1.1}$$
- the relative Schur multiplication operator on finite matrices with support in $\ddot{\Lambda}$ by
$$(x_{r,c})_{(r,c)\in\mathbb{Z}\times\mathbb{Z}} \mapsto (\ddot{\varphi}_{r,c} x_{r,c})_{(r,c)\in\mathbb{Z}\times\mathbb{Z}}, \tag{1.2}$$
where $\ddot{\varphi}_{r,c} = \varphi_{r-c}$.

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Marek Bożejko and Gero Fendler proved that these two multipliers have the same norm. The operator (1.1) is nothing but the restriction of (1.2) to Toeplitz matrices. They noted that it is automatically *completely* bounded: it has the same norm when acting on trigonometric series with operator coefficients x_k, and this permits to remove this restriction. Schur multiplication is also automatically completely bounded.

A part of this observation has been extended by Gilles Pisier to multipliers acting on a translation invariant Lebesgue space L^p_Λ and on the subspace $\mathrm{S}^p_{\mathring{\Lambda}}$ of elements of a Schatten-von-Neumann class supported by $\mathring{\Lambda}$, respectively; it yields that the complete norm of a relative Schur multiplier (1.2) remains bounded by the complete norm of the relative Fourier multiplier (1.1).

But L^p_Λ is not a subspace of $\mathrm{S}^p_{\mathring{\Lambda}}$, so a relative Fourier multiplier may not be viewed anymore as the restriction of a relative Schur multiplier to Toeplitz matrices. We point out that this difficulty may be overcome by using Szegő's limit theorem: a bounded measurable real function on $\mathbb{T}$ is the weak* limit of the normalised counting measure of eigenvalues of finite truncates of its Toeplitz matrix. This method also applies to Orlicz norms.

Theorem 1.1. *Let $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous nondecreasing function vanishing only at 0. The norm of the relative Fourier multiplication operator (1.1) on the Lebesgue-Orlicz space L^ψ_Λ is bounded by the norm of the relative Schur multiplication operator (1.2) on the Schatten-von-Neumann-Orlicz class $\mathrm{S}^\psi_{\mathring{\Lambda}}$.*

In order to deal with complete norms, we deduce a block matrix variant of Szegő's limit theorem in the style of Erik Bédos ([2]), Theorem 2.6. Note that other types of approximation are also available, as the completely positive approximation property and Reiter sequences combined with complex interpolation. They are studied in Section 3 in terms of local embeddings of L^p into S^p. They are more canonical than Szegő's limit theorem, but give no access to Orlicz norms.

Theorem 1.2. *Let $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous nondecreasing function vanishing only at 0. The norm of the following operators is equal:*

- *the relative Fourier multiplication operator (1.1) on the Lebesgue-Orlicz space $\mathrm{L}^\psi_\Lambda(\mathrm{S}^\psi)$ of S^ψ-valued trigonometric series with spectrum in Λ;*
- *the relative Schur multiplication operator (1.2) on the Schatten-von-Neumann-Orlicz class $\mathrm{S}^\psi_{\mathring{\Lambda}}(\mathrm{S}^\psi)$ of S^ψ-valued matrices with support in $\mathring{\Lambda}$.*

See Theorems 2.1 and 2.7 for the precise statement in the general case of an amenable group Γ.

An application of this theorem to the class of all unimodular Fourier multipliers yields a transfer of lacunary subsets into lacunary matrix patterns. Call Λ *unconditional in* L^p if $(z^k)_{k\in\Lambda}$ is an unconditional basis of L^p_Λ, and call $\mathring{\Lambda}$ *unconditional in* S^p if the sequence $(\mathrm{e}_q)_{q\in\mathring{\Lambda}}$ of elementary matrices is an unconditional basis of $\mathrm{S}^p_{\mathring{\Lambda}}$. These properties are also known as $\Lambda(p)$ if $p>2$ ($\Lambda(2)$ if $p<2$) and $\sigma(p)$, respectively; they have natural "complete" counterparts that are also known as $\Lambda(p)_{\mathrm{cb}}$ if $p>2$ ($\mathrm{K}(p)_{\mathrm{cb}}$ if $p\leqslant 2$) and $\sigma(p)_{\mathrm{cb}}$, respectively. (See Definitions 4.1 and 4.2).

Corollary 1.3. *Let $1\leqslant p<\infty$. If $\mathring{\Lambda}$ is unconditional in S^p, then Λ is unconditional in L^p. $\mathring{\Lambda}$ is completely unconditional in S^p if and only if Λ is completely unconditional in L^p.*

See Proposition 4.3 for the precise statement in the general case of a discrete group Γ.

The two most prominent multipliers are the Riesz projection and the Hilbert transform. The first consists in letting φ be the indicator function of nonnegative integers and transfers into the upper triangular truncation of matrices. The second corresponds to the sign function and transfers into the Hilbert matrix transform. We obtain the following partial results.

Theorem 1.4. *The norm of the matrix Riesz projection and of the matrix Hilbert transform on $\mathrm{S}^\psi(\mathrm{S}^\psi)$ coincide with their norm on S^ψ.*

- *If p is a power of 2, then the norm of the matrix Hilbert transform on S^p is $\cot(\pi/2p)$.*
- *The norm of the matrix Riesz projection on S^4 is $\sqrt{2}$.*

The transfer technique lends itself naturally to the case where Λ contains a sumset $R+C$: if subsets R' and C' are extracted so that the $r+c$ with $r \in R'$ and $c \in C'$ are pairwise distinct, they may play the role of rows and columns. Here are the consequences of the conditionality of the sequence of elementary matrices $\mathrm{e}_{r,c}$ in S^p for $p \neq 2$ and of the unboundedness of the Riesz transform on S^1 and S^∞, respectively.

Theorem 1.5. *If $(z^k)_{k\in\Lambda}$ is a completely unconditional basis of L^p_Λ with $p \neq 2$, then Λ does not contain sumsets $R+C$ of arbitrarily large sets. If either*

- *the space L^1_Λ admits some completely unconditional approximating sequence, or*
- *the space C_Λ of continuous functions with spectrum in Λ admits some unconditional approximating sequence,*

then Λ does not contain the sumset $R+C$ of two infinite sets.

The proof of the second part of this theorem consists in constructing infinite subsets R' and C' and skipped block sums $\sum(T_{k_{j+1}} - T_{k_j})$ of a given approximating sequence that act like the projection on the "upper triangular" part of $R'+C'$. See Proposition 4.8 and Theorem 7.4 for the precise statement in the general case of a discrete group Γ.

In the case of quasi-normed Schatten-von-Neumann classes S^p with $p<1$, the transfer technique yields a new proof for the following result of Alexey Alexandrov and Vladimir Peller.

Theorem 1.6. *Let $0<p<1$. The Fourier multiplier φ is contractive on L^p or on $\mathrm{L}^p(\mathrm{S}^p)$ if and only if the Schur multiplier $\check{\varphi}$ is contractive on S^p or on $\mathrm{S}^p(\mathrm{S}^p)$ if and only if the sequence φ is the Fourier transform of an atomic measure of the form $\sum a_g\delta_g$ on $\mathbb{T}$ with $\sum|a_g|^p \leqslant 1$.*

The emphasis put on *relative* Schur multipliers motivates the natural question of how the norm of an elementary Schur multiplier, that is, a rank 1 matrix $(\varrho_{r,c}) = (x_r y_c)$, gets affected when the action of ϱ is restricted to matrices with a given support. The surprising answer is the following theorem.

Theorem 1.7. *Let $I \subseteq R\times C$ and consider $(x_r)_{r\in R}$ and $(y_c)_{c\in C}$. The relative Schur multiplier on S^∞_I given by $(x_r y_c)_{(r,c)\in I}$ has norm $\sup_{(r,c)\in I}|x_r y_c|$.*

Let us finally describe the content of this article. Section 2 develops transfer techniques for Fourier and Schur multipliers provided by a block matrix Szegő limit theorem. This theorem provides local embeddings of L^ψ into S^ψ. Section 3 shows how interpolation may be used to define such embeddings for the scale of L^p spaces. Section 4 is devoted to the transfer of lacunary sets into lacunary matrix patterns; the unconditional constant of a set Λ is related to the size of the sumsets it contains. Section 5 deals with Toeplitz Schur multipliers for $p<1$ and comments on the case $p\geqslant 1$. The Riesz projection and the Hilbert transform are studied in Section 6. In Section 7, the presence of sumsets in a spectrum Λ is shown to be an obstruction for the existence of completely unconditional bases for L^p_Λ. The last section provides a norm-preserving extension for partially specified rank 1 Schur multipliers.

Notation and terminology. Let $\mathbb{T} = \{z\in\mathbb{C} : |z|=1\}$ be the circle.

Given an index set C and $c\in C$, e_c is the sequence defined on C as the indicator function $\chi_{\{c\}}$ of the singleton $\{c\}$, so that $(\mathrm{e}_c)_{c\in C}$ is the canonical Schauder basis of the

Hilbert space of square summable sequences indexed by C, denoted by ℓ^2_C. We will use the notation $\ell^2_n = \ell^2_{\{1,2,\ldots,n\}}$ and $\ell^2 = \ell^2_{\mathbb{N}}$.

Given a product set $I = R \times C$ and $q = (r, c)$, the indicator function $\mathrm{e}_q = \mathrm{e}_{r,c}$ is the *elementary matrix* identified with the linear operator from ℓ^2_C to ℓ^2_R that maps e_c to e_r and all other basis vectors to 0. The *matrix coefficient* at coordinate q of a linear operator x from ℓ^2_C to ℓ^2_R is $x_q = \operatorname{tr} \mathrm{e}_q^* x$, and its *matrix representation* is $(x_q)_{q\in R\times C} = \sum_{q\in R\times C} x_q \mathrm{e}_q$. The *support* or *pattern* of x is $\{q \in R \times C : x_q \neq 0\}$.

The space of all bounded operators from ℓ^2_C to ℓ^2_R is denoted by $\mathbb{B}(\ell^2_C, \ell^2_R)$, and its subspace of compact operators is denoted by S^∞.

Let $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous nondecreasing function vanishing only at 0. The *Schatten-von-Neumann-Orlicz class* S^ψ is the space of those compact operators x from ℓ^2_C to ℓ^2_R such that $\operatorname{tr} \psi(|x|/a) < \infty$ for some $a > 0$. If ψ is convex, then S^ψ is a Banach space for the norm given by $\|x\|_{\mathrm{S}^\psi} = \inf\{a > 0 : \operatorname{tr} \psi(|x|/a) \leqslant 1\}$. Otherwise, S^ψ is a Fréchet space for the F-norm given by $\|x\|_{\mathrm{S}^\psi} = \inf\{a > 0 : \operatorname{tr} \psi(|x|/a) \leqslant a\}$ (see [26, Chapter 3]). This space may also be constructed as the noncommutative Lebesgue-Orlicz space $\mathrm{L}^\psi(\mathrm{tr})$ associated with a corner of the von Neumann algebra $\mathbb{B}(\ell^2_C \oplus \ell^2_R)$ endowed with the normal, faithful, semifinite trace tr. If ψ is the power function $t \mapsto t^p$, this space is denoted S^p; if $p \geqslant 1$, then $\|x\|_{\mathrm{S}^p} = (\operatorname{tr} |x|^p)^{1/p}$; if $p < 1$, then $\|x\|_{\mathrm{S}^p} = (\operatorname{tr} |x|^p)^{1/(1+p)}$.

If $\# C = \# R = n$, then $\mathbb{B}(\ell^2_C, \ell^2_R)$ identifies with the space of $n \times n$ matrices denoted S^∞_n, and we write S^ψ_n for S^ψ. Let $(R_n \times C_n)$ be a sequence of finite sets such that each element of $R \times C$ eventually is in $R_n \times C_n$. Then the sequence of operators $P_n : x \mapsto \sum_{q\in R_n\times C_n} x_q \mathrm{e}_q$ tends pointwise to the identity on S^ψ.

For $I \subseteq R \times C$, we define the space S^ψ_I as the closed subspace of S^ψ spanned by $(\mathrm{e}_q)_{q\in I}$; this coincides with the subspace of those $x \in \mathrm{S}^\psi$ whose support is a subset of I.

A *relative Schur multiplier* on S^ψ_I is a sequence $\varrho = (\varrho_q)_{q\in I} \in \mathbb{C}^I$ such that the associated *Schur multiplication operator* M_ϱ defined by $\mathrm{e}_q \mapsto \varrho_q \mathrm{e}_q$ for $q \in I$ is bounded on S^ψ_I. The norm $\|\varrho\|_{\mathrm{M}(\mathrm{S}^\psi_I)}$ of ϱ is defined as the norm of M_ϱ. This norm is the supremum of the norm of its restrictions to finite rectangle sets $R' \times C'$. We used [31, 32] as a reference.

Let Γ be a discrete group with identity ϵ. The *reduced* C^**-algebra* of Γ is the closed subspace spanned by the left translations λ_γ (the linear operators defined on ℓ^2_Γ by $\lambda_\gamma \mathrm{e}_\beta = \mathrm{e}_{\gamma\beta}$) in $\mathbb{B}(\ell^2_\Gamma)$; we denote it by C, set in roman type. The *von Neumann algebra* of Γ is its weak* closure, endowed with the normal, faithful, normalised finite trace τ defined by $\tau(x) = x_{\epsilon,\epsilon}$; we denote it by L^∞. Let $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous nondecreasing function vanishing only at 0. We define the noncommutative Lebesgue-Orlicz space L^ψ of Γ as the completion of L^∞ with respect to the norm given by $\|x\|_{\mathrm{L}^\psi} = \inf\{a > 0 : \tau(\psi(|x|/a)) \leqslant 1\}$ if ψ is convex, and with respect to the F-norm given by $\|x\|_{\mathrm{L}^\psi} = \inf\{a > 0 : \tau(\psi(|x|/a)) \leqslant a\}$ otherwise. If ψ is the power function $t \mapsto t^p$, this space is denoted L^p; if $p \geqslant 1$, then $\|x\|_{\mathrm{L}^p} = \tau(|x|^p)^{1/p}$; if $p < 1$, then $\|x\|_{\mathrm{L}^p} = \tau(|x|^p)^{1/(1+p)}$. The *Fourier coefficient* of x at γ is $x_\gamma = \tau(\lambda_\gamma^* x) = x_{\gamma,\epsilon}$ and its *Fourier series* is $\sum_{\gamma\in\Gamma} x_\gamma \lambda_\gamma$. The *spectrum* of an element x is $\{\gamma \in \Gamma : x_\gamma \neq 0\}$. Let X be the C^*-algebra C or the space L^ψ and let $\Lambda \subseteq \Gamma$; then we define X_Λ as the closed subspace of X spanned by the λ_γ with $\gamma \in \Lambda$. We skip the general question of when this coincides with the subspace of those $x \in X$ whose spectrum is a subset of Λ, but note that this is the case if Γ is an amenable group (or if Γ has the AP and L^∞ has the QWEP by [15, Theorem 4.4]) and ψ is the power function $t \mapsto t^p$. Note also that our definition of X_Λ makes it a subspace of the *heart* of X: if $x \in X_\Lambda$, then $\tau(\psi(|x|/a))$ is finite for all $a > 0$.

A *relative Fourier multiplier* on X_Λ is a sequence $\varphi = (\varphi_\gamma)_{\gamma\in\Lambda} \in \mathbb{C}^\Lambda$ such that the associated Fourier multiplication operator M_φ defined by $\lambda_\gamma \mapsto \varphi_\gamma \lambda_\gamma$ for $\gamma \in \Lambda$ is bounded on X_Λ. The norm $\|\varphi\|_{\mathrm{M}(X_\Lambda)}$ of φ is defined as the norm of M_φ. Fourier multipliers on the whole of the C^*-algebra C are also called *multipliers of the Fourier algebra* $\mathrm{A}(\Gamma)$ (which may be identified with L^1); they form the set $\mathrm{M}(\mathrm{A}(\Gamma))$.

The space $\mathrm{S}^\psi(\mathrm{S}^\psi)$ is the space of those compact operators x from $\ell^2 \otimes \ell^2_C$ to $\ell^2 \otimes \ell^2_R$ such that $\|x\|_{\mathrm{S}^\psi(\mathrm{S}^\psi)} = \inf\{a : \operatorname{tr} \otimes \operatorname{tr} \psi(|x|/a) \leqslant 1\}$: it is the noncommutative Lebesgue-Orlicz space $\mathrm{L}^{\psi}(\mathrm{tr}\otimes\mathrm{tr})$ associated with a corner of the von Neumann algebra $\mathbb{B}(\ell^2)\otimes\mathbb{B}(\ell^2_C\oplus\ell^2_R)$. One may think of $\mathrm{S}^{\psi}(\mathrm{S}^{\psi})$ as the S^{ψ}-valued Schatten-von-Neumann class; we define the matrix coefficient of x at q by $x_q=(\mathrm{Id}_{\mathrm{S}^{\psi}}\otimes\mathrm{tr})\bigl((\mathrm{Id}_{\ell^2}\otimes\mathrm{e}_q^*)x\bigr)\in\mathrm{S}^{\psi}$ and its matrix representation by $\sum_{q\in R\times C}x_q\otimes\mathrm{e}_q$. The support of x and the subspace $\mathrm{S}^{\psi}_I(\mathrm{S}^{\psi})$ are defined in the same way as S^{ψ}_I.

Similarly, the space $\mathrm{L}^{\psi}(\mathrm{tr}\otimes\tau)$ is the noncommutative Lebesgue-Orlicz space associated with the von Neumann algebra $\mathbb{B}(\ell^2)\otimes\mathrm{L}^{\infty}=\mathrm{L}^{\infty}(\mathrm{tr}\otimes\tau)$. One may think of $\mathrm{L}^{\psi}(\mathrm{tr}\otimes\tau)$ as the S^{ψ}-valued noncommutative Lebesgue space; we define the Fourier coefficient of x at γ by $x_\gamma=(\mathrm{Id}_{\mathrm{S}^{\psi}}\otimes\tau)\bigl((\mathrm{Id}_{\ell^2}\otimes\lambda_\gamma^*)x\bigr)\in\mathrm{S}^{\psi}$ and its Fourier series by $\sum_{\gamma\in\Gamma}x_\gamma\otimes\lambda_\gamma$; the spectrum of x is defined accordingly. The subspace $\mathrm{L}^{\psi}_{\Lambda}(\mathrm{tr}\otimes\tau)$ is the closed subspace of $\mathrm{L}^{\psi}(\mathrm{tr}\otimes\tau)$ spanned by the $x\otimes\lambda_\gamma$ with $x\in\mathrm{S}^{\psi}$ and $\gamma\in\Lambda$.

An operator T on S^{ψ}_I is *bounded on* $\mathrm{S}^{\psi}_I(\mathrm{S}^{\psi})$ if the linear operator $\mathrm{Id}_{\mathrm{S}^{\psi}}\otimes T$ defined by $x\otimes y\mapsto x\otimes T(y)$ for $x\in\mathrm{S}^{\psi}$ and y in S^{ψ}_I on finite tensors extends to a bounded operator $\mathrm{Id}_{\mathrm{S}^{\psi}}\otimes T$ on $\mathrm{S}^{\psi}_I(\mathrm{S}^{\psi})$. The norm of a Schur multiplier ϱ on $\mathrm{S}^{\psi}_I(\mathrm{S}^{\psi})$ is defined as the norm of $\mathrm{Id}_{\mathrm{S}^{\psi}}\otimes\mathrm{M}_\varrho$. Similar definitions hold for an operator T on $\mathrm{L}^{\psi}_{\Lambda}$; the norm of a Fourier multiplier φ on $\mathrm{L}^{\psi}_{\Lambda}(\mathrm{tr}\otimes\tau)$ is the norm of $\mathrm{Id}^{\psi}_{\mathrm{S}}\otimes\mathrm{M}_\varphi$ on $\mathrm{L}^{\psi}_{\Lambda}(\mathrm{tr}\otimes\tau)$.

Let ψ be the power function $t\mapsto t^p$ with $p\geqslant 1$; the norms on $\mathrm{S}^p(\mathrm{S}^p)$ and $\mathrm{L}^p(\mathrm{tr}\otimes\tau)$ describe the canonical operator space structure on S^p and L^p, respectively (see [31, Corollary 1.4]); we should rather use the notation $\mathrm{S}^p[\mathrm{S}^p]$ and $\mathrm{S}^p[\mathrm{L}^p]$. This explains the following terminology. An operator T on S^p_I is *completely bounded* (c.b.) if $\mathrm{Id}_{\mathrm{S}^p}\otimes T$ is bounded on $\mathrm{S}^p_I(\mathrm{S}^p)$; the norm of $\mathrm{Id}_{\mathrm{S}^p}\otimes T$ is the *complete norm* of T (compare [31, Lemma 1.7]). The complete norm $\|\varrho\|_{\mathrm{M_{cb}}(\mathrm{S}^p_I)}$ of a Schur multiplier ϱ is defined as the complete norm of M_ϱ. Note that the complete norm of a Schur multiplier ϱ on S^{∞}_I is equal to its norm ([28, Theorem 3.2]): $\|\varrho\|_{\mathrm{M_{cb}}(\mathrm{S}^{\infty}_I)}=\|\varrho\|_{\mathrm{M}(\mathrm{S}^{\infty}_I)}$. The complete norm $\|\varphi\|_{\mathrm{M_{cb}}(\mathrm{L}^p_\Lambda)}$ of a Fourier multiplier φ is defined as the complete norm of M_φ. The complete norm of an operator T on C_Λ is the norm of $\mathrm{Id}_{\mathrm{S}^{\infty}}\otimes T$ on the subspace of $\mathrm{S}^{\infty}\otimes\mathrm{C}$ spanned by the $x\otimes\lambda_\gamma$ with $x\in\mathrm{S}^{\infty}$ and $\gamma\in\Lambda$. In the case $\Lambda=\Gamma$, φ is also called a *c.b. multiplier of the Fourier algebra* $\mathrm{A}(\Gamma)$ and one writes $\varphi\in\mathrm{M_{cb}}(\mathrm{A}(\Gamma))$. If Γ is amenable, the complete norm of a Fourier multiplier φ on C_Λ is equal to its norm: $\|\varphi\|_{\mathrm{M_{cb}}(\mathrm{C}_\Lambda)}=\|\varphi\|_{\mathrm{M}(\mathrm{C}_\Lambda)}$ (this follows from [7, Corollary 1.8] as shown by the proof of Theorem 2.7 (c)).

An element whose norm is at most 1 is *contractive*, and if its complete norm is at most 1, it is *completely* contractive.

If Γ is abelian, let G be its dual group and endow it with its unique normalised Haar measure m. Then the Fourier transform identifies the C^*-algebra C as the space of continuous functions on G, L^{∞} as the space of classes of bounded measurable functions on (G,m), L^{ψ} as the Lebesgue-Orlicz space of classes of ψ-integrable functions on (G,m), $\tau(x)$ as $\int_G x(g)\,\mathrm{d}m(g)$, $\mathrm{L}^{\psi}(\mathrm{tr}\otimes\tau)$ as the S^{ψ}-valued Lebesgue-Orlicz space $\mathrm{L}^{\psi}(\mathrm{S}^{\psi})$ and x_γ as $\hat{x}(\gamma)$.

2 Transfer between Fourier and Schur multipliers

Let Λ be a subset of a discrete group Γ and let φ be a relative Fourier multiplier on C_Λ, the closed subspace spanned by $(\lambda_\gamma)_{\gamma\in\Lambda}$ in the reduced C^*-algebra of Γ. Let $x\in\mathrm{C}_\Lambda$; the matrix of x is *constant down the diagonals* in the sense that for every $(r,c)\in\Gamma\times\Gamma$, $x_{r,c}=x_{rc^{-1},\epsilon}=x_{rc^{-1}}$. We say that x is a *Toeplitz* operator on ℓ^2_Γ. Furthermore, the matrix of the Fourier product $\mathrm{M}_\varphi x$ of φ with x is given by $(\mathrm{M}_\varphi x)_{r,c}=\varphi_{rc^{-1}}x_{r,c}$. This equality shows that if we set $\check{\check{\Lambda}}=\{(r,c)\in\Gamma\times\Gamma : rc^{-1}\in\Lambda\}$ and $\check{\check{\varphi}}_{r,c}=\varphi_{rc^{-1}}$, then $\mathrm{M}_\varphi x$ is the Schur product $\mathrm{M}_{\check{\check{\varphi}}}x$ of $\check{\check{\varphi}}$ with x. We have transferred the Fourier multiplier φ into the Schur multiplier $\check{\check{\varphi}}$. This proves at once that the norm of the Fourier multiplier φ on C_Λ is the norm of the Schur multiplier $\check{\check{\varphi}}$ on the subspace of Toeplitz elements of $\mathbb{B}(\ell^2_\Gamma)$ with support in $\check{\check{\Lambda}}$, and that the same holds for complete norms.

We shall now provide the means to generalise this identification to the setting of Lebesgue-Orlicz spaces L^ψ. We shall bypass the main obstacle, that L^ψ may not be considered as a subspace of S^ψ, by the Szegő limit theorem as stated by Erik Bédos ([2]).

Consider a discrete amenable group Γ; it admits a *Følner averaging net of sets* (Γ_ι), that is,

- each Γ_ι is a finite subset of Γ;
- $\#(\gamma\Gamma_\iota \Delta \Gamma_\iota) = o(\#\,\Gamma_\iota)$ for each $\gamma \in \Gamma$.

Each set Γ_ι corresponds to the orthogonal projection p_ι of ℓ^2_Γ onto its $(\#\,\Gamma_\iota)$-dimensional subspace of sequences supported by Γ_ι. The *truncate* of a selfadjoint operator $y \in \mathbb{B}(\ell^2_\Gamma)$ with respect to Γ_ι is $y_\iota = p_\iota y p_\iota^*$; it has $\#\,\Gamma_\iota$ eigenvalues α_j, counted with multiplicities, and its *normalised counting measure of eigenvalues* is

$$\mu_\iota = \frac{1}{\#\,\Gamma_\iota}\sum_{j=1}^{\#\,\Gamma_\iota}\delta_{\alpha_j}.$$

If y is a Toeplitz operator, that is, if $y \in \mathrm{L}^\infty$, Erik Bédos ([2, Theorem 10]) proved that (μ_ι) converges weak* to the *spectral measure* of y with respect to τ, which is the unique Borel probability measure μ on $\mathbb{R}$ such that

$$\tau(f(y)) = \int_{\mathbb{R}} f(\alpha)\mathrm{d}\mu(\alpha)$$

for every continuous function f on $\mathbb{R}$ that tends to zero at infinity. If Γ is abelian, then y may be identified as the class of a real-valued bounded measurable function on the group G dual to Γ and μ is the distribution of y.

Let us now state and prove the L^ψ version of the identification described at the beginning of this section.

Theorem 2.1. *Let Γ be a discrete amenable group and let $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous nondecreasing function vanishing only at 0. Let $\Lambda \subseteq \Gamma$ and $\varphi \in \mathbb{C}^\Lambda$. Consider the associated Toeplitz set $\check{\Lambda} = \{(r,c) \in \Gamma\times\Gamma : rc^{-1} \in \Lambda\}$ and the Toeplitz matrix defined by $\check{\varphi}_{r,c} = \varphi_{rc^{-1}}$. The norm of the relative Fourier multiplier φ on L^ψ_Λ is bounded by the norm of the relative Schur multiplier $\check{\varphi}$ on $\mathrm{S}^\psi_{\check{\Lambda}}$.*

Proof. A Toeplitz matrix has the form $(x_{rc^{-1}})_{(r,c)\in\check{\Lambda}}$. Our definition of the space L^ψ_Λ (in the section on Notation and terminology) ensures that we may suppose that only a finite number of the x_γ are nonzero for the computation of the norm of φ. Then $(x_{rc^{-1}})_{(r,c)\in\check{\Lambda}}$ is the matrix of the operator $x = \sum_{\gamma\in\Lambda} x_\gamma\lambda_\gamma$ for the canonical basis of ℓ^2_Γ.

Let $y = x^*x$ and let $\tilde{\psi}$ be a continuous function with compact support such that $\tilde{\psi}(t) = \psi(t)$ on $[0, \|y\|]$. By Szegő's limit theorem,

$$\frac{1}{\#\Gamma_\iota}\operatorname{tr}\psi(y_\iota) = \frac{1}{\#\Gamma_\iota}\operatorname{tr}\tilde{\psi}(y_\iota) \to \tau(\tilde{\psi}(y)) = \tau(\psi(y)).$$

We have $y_\iota = (xp_\iota^*)^*(xp_\iota^*)$; let us describe how $\check{\varphi}$ acts on xp_ι^*. Schur multiplication with $\check{\varphi}$ transforms the matrix of xp_ι^*, that is, the truncated Toeplitz matrix $(x_{rc^{-1}})_{(r,c)\in\check{\Lambda}\cap\Gamma\times\Gamma_\iota}$, into the matrix $(\varphi_{rc^{-1}}x_{rc^{-1}})_{(r,c)\in\check{\Lambda}\cap\Gamma\times\Gamma_\iota}$ so that it transforms xp_ι^* into $(\mathrm{M}_\varphi x)p_\iota^*$. □

Remark 2.2. In the case of a finite abelian group, no limit theorem is needed. This case was considered in [22, Proposition 2.5 (*b*)]; compare with [29, Chapter 6, Lemma 3.8].

Remark 2.3. Our technique proves in fact that the norm of a Fourier multiplier is the upper limit of the norm of the corresponding relative Schur multipliers on subspaces of truncated Toeplitz matrices. We ignore whether or not it is actually their supremum.

Remark 5.2 illustrates that the two norms in Theorem 2.1 are different in general. This is not so in the S^ψ-valued case because of the following argument. It has been used (first in [5], see [6, Proposition D.6]) to show that the complete norm of the Fourier multiplier φ on C_Λ bounds the complete norm of the Schur multiplier $\check{\varphi}$ on $\mathrm{S}^\infty_{\check{\Lambda}}$, so that we have in full generality $\|\varphi\|_{\mathrm{M_{cb}}(\mathrm{C}_\Lambda)} = \|\check{\varphi}\|_{\mathrm{M_{cb}}(\mathrm{S}^\infty_{\check{\Lambda}})}$.

Lemma 2.4. *Let Γ be a discrete group and let R and C be subsets of Γ. With $\Lambda \subseteq \Gamma$ associate $\check{\Lambda} = \{(r,c) \in R \times C : rc^{-1} \in \Lambda\}$; given $\varphi \in \mathbb{C}^\Lambda$, define $\check{\varphi} \in \mathbb{C}^{\check{\Lambda}}$ by $\check{\varphi}_{r,c} = \varphi_{rc^{-1}}$. Let $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous nondecreasing function vanishing only at 0. The norm of the relative Schur multiplier $\check{\varphi}$ on $\mathrm{S}^\psi_{\check{\Lambda}}(\mathrm{S}^\psi)$ is bounded by the norm of the relative Fourier multiplier φ on $\mathrm{L}^\psi_\Lambda(\mathrm{tr} \otimes \tau)$.*

Proof. We adapt the argument in [31, Lemma 8.1.4]. Let $x_q \in \mathrm{S}^\psi$, of which only a finite number are nonzero. The space $\mathrm{L}^\psi(\mathrm{tr} \otimes \mathrm{tr} \otimes \tau)$ is a left and right $\mathrm{L}^\infty(\mathrm{tr} \otimes \mathrm{tr} \otimes \tau)$-module, and $\sum_{\gamma \in \Gamma} \mathrm{e}_{\gamma\gamma} \otimes \lambda_\gamma$ is a unitary in $\mathrm{L}^\infty(\mathrm{tr} \otimes \tau)$ so that

$$
\begin{aligned}
&\Big\| \sum_{q \in \check{\Lambda}} x_q \otimes \mathrm{e}_q \Big\|_{\mathrm{S}^\psi_{\check{\Lambda}}(\mathrm{S}^\psi)} \\
&\qquad = \Big\| \Big(\mathrm{Id} \otimes \sum_{r \in R} \mathrm{e}_{r,r} \otimes \lambda_r\Big)\Big(\sum_{q \in \check{\Lambda}} x_q \otimes \mathrm{e}_q \otimes \lambda_\epsilon\Big)\Big(\mathrm{Id} \otimes \sum_{c \in C} \mathrm{e}_{c,c} \otimes \lambda_c^*\Big)\Big\|_{\mathrm{L}^\psi(\mathrm{tr} \otimes \mathrm{tr} \otimes \tau)} \\
&\qquad = \Big\| \sum_{(r,c) \in \check{\Lambda}} x_{r,c} \otimes \mathrm{e}_{r,c} \otimes \lambda_{rc^{-1}} \Big\|_{\mathrm{L}^\psi(\mathrm{tr} \otimes \mathrm{tr} \otimes \tau)} \\
&\qquad = \Big\| \sum_{\gamma \in \Lambda} \Big(\sum_{rc^{-1} = \gamma} x_{r,c} \otimes \mathrm{e}_{r,c} \Big) \otimes \lambda_\gamma \Big\|_{\mathrm{L}^\psi_\Lambda(\mathrm{tr} \otimes \mathrm{tr} \otimes \tau)}.
\end{aligned}
$$

This yields an isometric embedding of $\mathrm{S}^\psi_{\check{\Lambda}}(\mathrm{S}^\psi)$ in $\mathrm{L}^\psi_\Lambda(\mathrm{tr} \otimes \mathrm{tr} \otimes \tau)$. As $\mathrm{S}^\psi(\mathrm{S}^\psi)$ is the Schatten-von-Neumann-Orlicz class for the Hilbert space $\ell^2 \otimes \ell^2_\Gamma$, which may be identified with ℓ^2,

$$
\begin{aligned}
\Big\| \sum_{q \in \check{\Lambda}} x_q \otimes \check{\varphi}_q \mathrm{e}_q \Big\|_{\mathrm{S}^\psi_{\check{\Lambda}}(\mathrm{S}^\psi)} &= \Big\| \sum_{\gamma \in \Lambda} \Big(\sum_{rc^{-1} = \gamma} x_{r,c} \otimes \mathrm{e}_{r,c} \Big) \otimes \varphi_\gamma \lambda_\gamma \Big\|_{\mathrm{L}^\psi_\Lambda(\mathrm{tr} \otimes \mathrm{tr} \otimes \tau)} \\
&\leqslant \|\mathrm{Id}_{\mathrm{S}^\psi} \otimes \mathrm{M}_\varphi\| \Big\| \sum_{q \in \check{\Lambda}} x_q \otimes \mathrm{e}_q \Big\|_{\mathrm{S}^\psi_{\check{\Lambda}}(\mathrm{S}^\psi)}. \qquad \square
\end{aligned}
$$

Remark 2.5. This proof also shows the following transfer: let (r_i) and (c_j) be sequences in Γ, consider $\check{\Lambda} = \{(i,j) \in \mathbb{N} \times \mathbb{N} : r_i c_j \in \Lambda\}$ and define $\check{\varphi} \in \mathbb{C}^{\check{\Lambda}}$ by $\check{\varphi}(i,j) = \varphi(r_i c_j)$. Then the norm of the relative Schur multiplier $\check{\varphi}$ on $\mathrm{S}^\psi_{\check{\Lambda}}(\mathrm{S}^\psi)$ is bounded by the norm of the relative Fourier multiplier $\mathrm{Id}_{\mathrm{S}^\psi} \otimes \mathrm{M}_\varphi$ on $\mathrm{L}^\psi_\Lambda(\mathrm{tr} \otimes \tau)$ (compare with [32, Theorem 6.4]). In particular, if the $r_i c_j$ are pairwise distinct, this permits us to transfer every Schur multiplier, not just the Toeplitz ones. See [22, Section 11] for applications of this transfer.

We shall now prove that the two norms in this lemma are in fact equal. As we want to compute norms of multipliers on S^ψ-valued spaces, we shall generalise the Szegő limit theorem to the block matrix case, which was not considered in [2]. This is the analogue of the scalar case for selfadjoint elements $y \in \mathrm{S}^\infty_n \otimes \mathrm{L}^\infty$, whose S^∞_n*-valued spectral measure* μ is defined by

$$\int_{\mathbb{R}} f(\alpha) \mathrm{d}\mu(\alpha) = \mathrm{Id}_{\mathrm{S}^\infty_n} \otimes \tau(f(y))$$

for every continuous function f on $\mathbb{R}$ that tends to zero at infinity.

The orthogonal projection $\tilde{p}_\iota = \mathrm{Id}_{\ell^2_n} \otimes p_\iota$ defines the truncate $y_\iota = \tilde{p}_\iota y \tilde{p}_\iota^* \in \mathrm{S}^\infty_n \otimes \mathbb{B}(\ell^2_{\Gamma_\iota})$, and the S^∞_n*-valued normalised counting measure of eigenvalues* μ_ι by

$$\int_{\mathbb{R}} f(\alpha)\mathrm{d}\mu_\iota(\alpha) = \mathrm{Id}_{\mathrm{S}^\infty_n} \otimes \frac{\mathrm{tr}}{\#\,\Gamma_\iota}(f(y_\iota))$$

for every continuous function f on $\mathbb{R}$ that tends to zero at infinity.

Theorem 2.6 (Matrix Szegő limit theorem)**.** *Let Γ be a discrete amenable group and let (Γ_ι) be a Følner averaging net for Γ. Let y be a selfadjoint element of $\mathrm{S}^\infty_n \otimes \mathrm{L}^\infty$. The net (μ_ι) of S^∞_n-valued normalised counting measures of eigenvalues of the truncates of y with respect to Γ_ι converges in the weak* topology to the spectral measure of y:*

$$\int_{\mathbb{R}} f(\alpha)\mathrm{d}\mu_\iota(\alpha) \to \mathrm{Id}_{\mathrm{S}^\infty_n} \otimes \tau(f(y))$$

for every continuous function f on $\mathbb{R}$ that tends to zero at infinity.

Sketch of proof. We first suppose that $y = \sum_{\gamma\in\Gamma} y_\gamma \otimes \lambda_\gamma$ with only a finite number of the $y_\gamma \in \mathrm{S}^\infty_n$ nonzero. The S^∞_n-valued matrix of the truncate y_ι of y for the canonical basis of $\ell^2_{\Gamma_\iota}$ is $(y_{rc^{-1}})_{(r,c)\in\Gamma_\iota\times\Gamma_\iota}$. As the truncates y_ι of y are uniformly bounded, it suffices to prove that

$$\mathrm{Id} \otimes \frac{\mathrm{tr}}{\#\Gamma_\iota}(y_\iota^k) \to \mathrm{Id} \otimes \tau(y^k)$$

for every k. This is trivial if $k = 0$. If $k = 1$, then

$$\mathrm{Id} \otimes \frac{\mathrm{tr}}{\#\Gamma_\iota}(y_\iota) = \frac{1}{\#\Gamma_\iota}\sum_{c\in\Gamma_\iota} y_{c,c} = \mathrm{Id} \otimes \tau(y)$$

as $y_{c,c} = y_{cc^{-1}} = y_\epsilon$. If $k \geqslant 2$, the same formula holds with y^k instead of y:

$$\mathrm{Id} \otimes \tau(y^k) = \mathrm{Id} \otimes \frac{\mathrm{tr}}{\#\Gamma_\iota}(\tilde{p}_\iota y^k \tilde{p}_\iota^*),$$

so that we wish to prove

$$\mathrm{Id} \otimes \mathrm{tr}(\tilde{p}_\iota y^k \tilde{p}_\iota^* - (\tilde{p}_\iota y \tilde{p}_\iota^*)^k) = o(\#\,\Gamma_\iota).$$

Note that

$$\big\|\mathrm{Id} \otimes \mathrm{tr}\big(\tilde{p}_\iota y^k \tilde{p}_\iota^* - (\tilde{p}_\iota y \tilde{p}_\iota^*)^k\big)\big\|_{\mathrm{S}^1_n} \leqslant \|\tilde{p}_\iota y^k \tilde{p}_\iota^* - (\tilde{p}_\iota y \tilde{p}_\iota^*)^k\|_{\mathrm{S}^1(\mathrm{S}^1_n)}.$$

Lemma 5 in [2] provides the following estimate. As

$$\tilde{p}_\iota y^k \tilde{p}_\iota^* - (\tilde{p}_\iota y \tilde{p}_\iota^*)^k = \tilde{p}_\iota y^{k-1}(y\tilde{p}_\iota^* - \tilde{p}_\iota^*\tilde{p}_\iota y \tilde{p}_\iota^*) + (\tilde{p}_\iota y^{k-1}\tilde{p}_\iota^* - (\tilde{p}_\iota y \tilde{p}_\iota^*)^{k-1})\tilde{p}_\iota y \tilde{p}_\iota^*,$$

an induction yields

$$\|\tilde{p}_\iota y^k \tilde{p}_\iota^* - (\tilde{p}_\iota y \tilde{p}_\iota^*)^k\|_{\mathrm{S}^1(\mathrm{S}^1_n)} \leqslant (k-1)\|y\|^{k-1}_{\mathrm{S}^\infty_n\otimes\mathrm{L}^\infty}\|y\tilde{p}_\iota^* - \tilde{p}_\iota^*\tilde{p}_\iota y \tilde{p}_\iota^*\|_{\mathrm{S}^1(\mathrm{S}^1_n)}.$$

It suffices to consider the very last norm for each term $y_\gamma \otimes \lambda_\gamma$ of y: let $h \in \ell^2_n$ and $\beta \in \Gamma$; as

$$\big((y_\gamma \otimes \lambda_\gamma)\tilde{p}_\iota^* - \tilde{p}_\iota^*\tilde{p}_\iota(y_\gamma \otimes \lambda_\gamma)\tilde{p}_\iota^*\big)(h \otimes \mathrm{e}_\beta) = \begin{cases} y_\gamma(h)\mathrm{e}_{\gamma\beta} & \text{if } \beta \in \Gamma_\iota \text{ and } \gamma\beta \notin \Gamma_\iota \\ 0 & \text{otherwise,} \end{cases}$$

the definition of a Følner averaging net yields

$$\|(y_\gamma \otimes \lambda_\gamma)\tilde{p}_\iota^* - \tilde{p}_\iota^*\tilde{p}_\iota(y_\gamma \otimes \lambda_\gamma)\tilde{p}_\iota^*\|_{\mathrm{S}^1(\mathrm{S}^1_n)} \leqslant \#(\Gamma_\iota \setminus \gamma^{-1}\Gamma_\iota)\|y_\gamma\|_{\mathrm{S}^1_n} = o(\#\,\Gamma_\iota).$$

An approximation argument as in the proof of [2, Proposition 4] permits us to conclude for $y \in \mathrm{S}^\infty_n \otimes \mathrm{L}^\infty$. □

Here is the promised strengthening of Lemma 2.4 together with three variants.

Theorem 2.7. *Let Γ be a discrete amenable group. Let $\Lambda \subseteq \Gamma$ and $\varphi \in \mathbb{C}^{\Lambda}$. Consider the associated Toeplitz set $\breve{\Lambda} = \{(r,c) \in \Gamma \times \Gamma : rc^{-1} \in \Lambda\}$ and the Toeplitz matrix defined by $\breve{\varphi}_{r,c} = \varphi_{rc^{-1}}$.*

(*a*) *Let $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous nondecreasing function vanishing only at 0. The norm of the relative Fourier multiplier φ on $\mathrm{L}^{\psi}_{\Lambda}(\mathrm{tr} \otimes \tau)$ and the norm of the relative Schur multiplier $\breve{\varphi}$ on $\mathrm{S}^{\psi}_{\breve{\Lambda}}(\mathrm{S}^{\psi})$ are equal.*

(*b*) *Let $p \geqslant 1$. The complete norm of the relative Fourier multiplier φ on L^p_{Λ} and the complete norm of the relative Schur multiplier $\breve{\varphi}$ on $\mathrm{S}^p_{\breve{\Lambda}}$ are equal:*

$$\|\varphi\|_{\mathrm{M_{cb}}(\mathrm{L}^p_{\Lambda})} = \|\breve{\varphi}\|_{\mathrm{M_{cb}}(\mathrm{S}^p_{\breve{\Lambda}})}.$$

(*c*) *The norm of the relative Fourier multiplier φ on C_{Λ}, its complete norm, the norm of the relative Schur multiplier $\breve{\varphi}$ on $\mathrm{S}^{\infty}_{\breve{\Lambda}}$, and its complete norm are equal:*

$$\|\varphi\|_{\mathrm{M}(\mathrm{C}_{\Lambda})} = \|\varphi\|_{\mathrm{M_{cb}}(\mathrm{C}_{\Lambda})} = \|\breve{\varphi}\|_{\mathrm{M_{cb}}(\mathrm{S}^{\infty}_{\breve{\Lambda}})} = \|\breve{\varphi}\|_{\mathrm{M}(\mathrm{S}^{\infty}_{\breve{\Lambda}})}.$$

(*d*) *Suppose that $\Lambda = \Gamma$. The norm of the Fourier algebra multiplier φ, its complete norm, the norm of the Schur multiplier $\breve{\varphi}$ on S^{∞}, and its complete norm are equal:*

$$\|\varphi\|_{\mathrm{M}(\mathrm{A}(\Gamma))} = \|\varphi\|_{\mathrm{M_{cb}}(\mathrm{A}(\Gamma))} = \|\breve{\varphi}\|_{\mathrm{M_{cb}}(\mathrm{S}^{\infty})} = \|\breve{\varphi}\|_{\mathrm{M}(\mathrm{S}^{\infty})}.$$

Proof. (*a*). Combine the argument in Theorem 2.1 with the matrix Szegő limit theorem and apply Lemma 2.4.

(*c*). Recall that the complete norm of a Schur multiplier $\breve{\varphi}$ on $\mathrm{S}^{\infty}_{\breve{\Lambda}}$ is equal to its norm ([28, Theorem 3.2]). Recall also that the norm of a Fourier multiplier χ on C is equal to its complete norm, because Γ is amenable. Moreover, it coincides with the norm of χ in $\mathrm{A}(\Gamma)$ ([7, Corollary 1.8]). Let φ be a relative contractive Fourier multiplier on C_{Λ}; compose it with the trivial character of Γ to obtain a contractive form on C_{Λ}. Then, by the Hahn-Banach extension theorem, φ is the restriction of a contractive element χ in $\mathrm{A}(\Gamma)$. Now χ is a completely contractive Fourier multiplier on C, and so is φ on C_{Λ}. The conclusion follows from (*a*) and (*b*). ∎

3 Local embeddings of L^p into S^p

The proof of Theorem 2.1 can be interpreted as an embedding of L^{ψ} into an ultraproduct of finite-dimensional spaces S^{ψ}_n that intertwines Fourier and Toeplitz Schur multipliers. If we restrict ourselves to power functions $\psi\colon t \mapsto t^p$ with $p \geqslant 1$, such embeddings are well known and the proof of Theorem 2.7 does not need the full strength of the matrix Szegő limit theorem but only the existence of such embeddings. In this section, we explain two ways to obtain them by interpolation.

The first way is to extend the classical result that the reduced C*-algebra C of a discrete group Γ has the completely positive approximation property if Γ is amenable. We follow the approach of [6, Theorem 2.6.8]. Let Γ be a discrete amenable group and let Γ_ι be a Følner averaging net of sets. As above, we denote by p_ι the orthogonal projection from ℓ^2_{Γ} to $\ell^2_{\Gamma_\iota}$. Define the compression ϕ_ι and the embedding ψ_ι by

$$\begin{aligned} \phi_\iota\colon \mathrm{C} &\to \mathbb{B}(\ell^2_{\Gamma_\iota}) &\quad\text{and}\quad \psi_\iota\colon \mathbb{B}(\ell^2_{\Gamma_\iota}) &\to \mathrm{C} \\ x &\mapsto p_\iota x p_\iota^* & \mathrm{e}_{r,c} &\mapsto (1/\#\Gamma_\iota)\lambda_r\lambda_{c^{-1}}. \end{aligned} \tag{3.1}$$

If we endow $\mathbb{B}(\ell^2_{\Gamma_\iota})$ with the normalised trace, these maps are unital completely positive, trace preserving (and normal), and the net $(\psi_\iota\phi_\iota)$ converges pointwise to the identity of

C. One can therefore extend them by interpolation to completely positive contractions on the respective noncommutative Lebesgue spaces. Recall that $\mathrm{L}^p(\mathbb{B}(\ell^2_{\Gamma_\iota}), (1/\#\Gamma_\iota)\operatorname{tr})$ is $(\#\Gamma_\iota)^{-1/p}\mathrm{S}^p_{\#\Gamma_\iota}$. We get a net of complete contractions

$$\tilde\phi_\iota\colon \mathrm{L}^p \to (\#\Gamma_\iota)^{-1/p}\mathrm{S}^p_{\#\Gamma_\iota} \quad \text{and} \quad \tilde\psi_\iota\colon (\#\Gamma_\iota)^{-1/p}\mathrm{S}^p_{\#\Gamma_\iota} \to \mathrm{L}^p$$

such that $(\tilde\psi_\iota\tilde\phi_\iota)$ converges pointwise to the identity of L^p. Moreover, the definitions (3.1) show that these maps also intertwine Fourier and Toeplitz Schur multipliers.

This approach is more canonical, as it allows us to extend the transfer to vector-valued spaces in the sense of [31, Chapter 3]. Recall that for any hyperfinite semifinite von Neumann algebra M and any operator space E, one can define $\mathrm{L}^p(M,E)$. For $p=\infty$, this space is defined as $M\otimes_{\min} E$; for $p=1$, this space is defined as $M^{\mathrm{op}}_*\hat\otimes E$; these spaces form an interpolation scale for the complex method when $1\leqslant p\leqslant\infty$. For us, M will be $\mathbb{B}(\ell^2)$ or the group von Neumann algebra L^∞. As the maps ψ_ι and ϕ_ι are unital completely positive and trace preserving and normal, they define simultaneously complete contractions on M and M_*. By interpolation, the maps $\psi_\iota\otimes\mathrm{Id}_E$ and $\phi_\iota\otimes\mathrm{Id}_E$ are still complete contractions on the spaces $\mathrm{L}_p(E)$ and $\mathrm{S}^p[E]$. Let $\varphi\in\mathbb{C}^\Gamma$; the transfer shows that the norm of $\mathrm{Id}_E\otimes\mathrm{M}_\varphi$ on $\mathrm{L}^p(E)$ is bounded by the norm of $\mathrm{Id}_E\otimes\mathrm{M}_{\check\varphi}$ on $\mathrm{S}^p[E]$ and that their complete norms coincide. In formulas,

$$\|\mathrm{Id}_E\otimes\mathrm{M}_\varphi\|_{\mathbb{B}(\mathrm{L}^p(E))}\leqslant\|\mathrm{Id}_E\otimes\mathrm{M}_{\check\varphi}\|_{\mathbb{B}(\mathrm{S}^p[E])},$$
$$\|\mathrm{Id}_E\otimes\mathrm{M}_\varphi\|_{\mathrm{cb}(\mathrm{L}^p(E))}=\|\mathrm{Id}_E\otimes\mathrm{M}_{\check\varphi}\|_{\mathrm{cb}(\mathrm{S}^p[E])}.$$

The compression ϕ_ι provides a two-sided approximation of an element x, whereas the proof of Theorem 2.1 uses only a one-sided approximation. This subtlety makes a difference in our second way to obtain embeddings, a direct proof by complex interpolation.

Proposition 3.1. *Let Γ be a discrete amenable group and let (μ_ι) be a* Reiter net of means *for Γ:*

- *each μ_ι is a positive sequence summing to 1 with finite support $\Gamma_\iota\subseteq\Gamma$ and viewed as a diagonal operator from $\ell^2_{\Gamma_\iota}$ to ℓ^2_Γ, so that*

$$\|\mu_\iota\|_{\mathrm{S}^1}=\sum_{\gamma\in\Gamma_\iota}(\mu_\iota)_\gamma=1;$$

- *the net (μ_ι) satisfies, for each $\gamma\in\Gamma$,* Reiter's Property P_1*:*

$$\sum_{\beta\in\Gamma}\bigl|(\mu_\iota)_{\gamma^{-1}\beta}-(\mu_\iota)_\beta\bigr|\to 0.$$

Let $x\in\mathrm{S}^\infty_n\otimes\mathrm{L}^\infty=\mathrm{L}^\infty(\operatorname{tr}\otimes\tau)$ and $p\geqslant 1$. Then

$$\limsup\|x\mu_\iota^{1/p}\|_{\mathrm{S}^p(\mathrm{S}^p_n)}=\|x\|_{\mathrm{L}^p(\operatorname{tr}\otimes\tau)}.$$

Proof. Consider $x=\sum_{\gamma\in\Gamma}x_\gamma\otimes\lambda_\gamma$ with only a finite number of the $x_\gamma\in\mathrm{S}^\infty_n$ nonzero. As

$$\sum_{\beta\in\Gamma}\bigl|(\mu_\iota)^{1/2}_{\gamma^{-1}\beta}-(\mu_\iota)^{1/2}_\beta\bigr|^2\leqslant\sum_{\beta\in\Gamma}\bigl|(\mu_\iota)_{\gamma^{-1}\beta}-(\mu_\iota)_\beta\bigr|,$$

Property P_1 implies *Property P_2*:

$$\|\lambda_\gamma\mu_\iota^{1/2}-\mu_\iota^{1/2}\lambda_\gamma\|_{\mathrm{S}^2}\to 0,$$

so that

$$\|x\mu_\iota^{1/2}-\mu_\iota^{1/2}x\|_{\mathrm{S}^2(\mathrm{S}^2_n)}\to 0.$$

As the S^∞_n-valued matrix of x for the canonical basis of ℓ^2_Γ is $(x_{rc^{-1}})_{(r,c)\in\Gamma\times\Gamma}$,

$$\begin{aligned}\|x\mu_\iota^{1/2}\|^2_{\mathrm{S}^2(\mathrm{S}^2_n)} &= \sum_{(r,c)\in\Gamma\times\Gamma} \|x_{rc^{-1}}\|^2_{\mathrm{S}^n_2}(\mu_\iota)_c\\ &= \sum_{c\in\Gamma}(\mu_\iota)_c\sum_{r\in\Gamma}\|x_{rc^{-1}}\|^2_{\mathrm{S}^n_2}\\ &= \sum_{c\in\Gamma}(\mu_\iota)_c\|x\|^2_{\mathrm{L}^2(\mathrm{tr}\otimes\tau)} = \|x\|^2_{\mathrm{L}^2(\mathrm{tr}\otimes\tau)}.\end{aligned}$$

By density and continuity, the result extends to all $x\in\mathrm{L}^2(\mathrm{tr}\otimes\tau)$.

Let us prove now that for $x\in\mathrm{L}^\infty(\mathrm{tr}\otimes\tau)$,

$$\limsup\|x\mu_\iota\|_{\mathrm{S}^1(\mathrm{S}^1_n)}\leqslant\|x\|_{\mathrm{L}^1(\mathrm{tr}\otimes\tau)}.$$

The polar decomposition $x=u|x|$ yields a factorisation $x=ab$ with $a=u|x|^{1/2}$ and $b=|x|^{1/2}$ in $\mathrm{L}^\infty(\mathrm{tr}\otimes\tau)$ such that

$$\begin{aligned}\|a\|_{\mathrm{L}^2(\mathrm{tr}\otimes\tau)}=\|b\|_{\mathrm{L}^2(\mathrm{tr}\otimes\tau)}&=\|x\|^{1/2}_{\mathrm{L}^1(\mathrm{tr}\otimes\tau)}\\ \|a\|_{\mathrm{L}^\infty(\mathrm{tr}\otimes\tau)}&=\|x\|^{1/2}_{\mathrm{L}^\infty(\mathrm{tr}\otimes\tau)}.\end{aligned}$$

Then $x\mu_\iota=a(b\mu_\iota^{1/2}-\mu_\iota^{1/2}b)\mu_\iota^{1/2}+a\mu_\iota^{1/2}b\mu_\iota^{1/2}$, so that the Cauchy-Schwarz inequality yields

$$\begin{aligned}\|x\mu_\iota\|_{\mathrm{S}^1(\mathrm{S}^1_n)}&\leqslant\|a\|_{\mathrm{L}^\infty(\mathrm{tr}\otimes\tau)}\|(b\mu_\iota^{1/2}-\mu_\iota^{1/2}b)\mu_\iota^{1/2}\|_{\mathrm{S}^1(\mathrm{S}^1_n)}+\|a\mu_\iota^{1/2}b\mu_\iota^{1/2}\|_{\mathrm{S}^1(\mathrm{S}^1_n)}\\ &\leqslant\|a\|_{\mathrm{L}^\infty(\mathrm{tr}\otimes\tau)}\|b\mu_\iota^{1/2}-\mu_\iota^{1/2}b\|_{\mathrm{S}^2(\mathrm{S}^2_n)}+\|a\|_{\mathrm{L}^2(\mathrm{tr}\otimes\tau)}\|b\|_{\mathrm{L}^2(\mathrm{tr}\otimes\tau)}\end{aligned}$$

and therefore our claim. Now complex interpolation yields

$$\limsup\|x\mu_\iota^{1/p}\|_{\mathrm{S}^p(\mathrm{S}^p_n)}\leqslant\|x\|_{\mathrm{L}^p(\mathrm{tr}\otimes\tau)}$$

for $x\in\mathrm{L}^\infty(\mathrm{tr}\otimes\tau)$ and $p\in[1,\infty]$. In fact, consider the function $f(z)=u|x|^{pz}\mu_\iota^z$ analytic in the strip $0<\Im z<1$ and continuous on its closure; then $f(\mathrm{i}t)$ is a product of unitaries for $t\in\mathbb{R}$, so that $\|f(\mathrm{i}t)\|_{\mathrm{L}^\infty(\mathrm{tr}\otimes\tau)}=1$. Also

$$\|f(1+\mathrm{i}t)\|_{\mathrm{S}^1(\mathrm{S}^1_n)}=\||x|^p\mu_\iota\|_{\mathrm{S}^1(\mathrm{S}^1_n)}.$$

As $\mathrm{S}^p(\mathrm{S}^p_n)$ is the complex interpolation space $(\mathrm{S}^\infty(\mathrm{S}^\infty_n),\mathrm{S}^1(\mathrm{S}^1_n))_{1/p}$,

$$\|x\mu_\iota^{1/p}\|_{\mathrm{S}^p(\mathrm{S}^p_n)}=\|f(1/p)\|_{\mathrm{S}^p(\mathrm{S}^p_n)}\leqslant\||x|^p\mu_\iota\|^{1/p}_{\mathrm{S}^1(\mathrm{S}^1_n)}.$$

Then, taking the upper limit and using the estimate on $\mathrm{S}^1(\mathrm{S}^1_n)$,

$$\begin{aligned}\limsup\|x\mu_\iota^{1/p}\|_{\mathrm{S}^p(\mathrm{S}^p_n)}&\leqslant\limsup\||x|^p\mu_\iota\|^{1/p}_{\mathrm{S}^1(\mathrm{S}^1_n)}\\ &\leqslant\||x|^p\|^{1/p}_{\mathrm{L}^1(\mathrm{tr}\otimes\tau)}=\|x\|_{\mathrm{L}^p(\mathrm{tr}\otimes\tau)}.\end{aligned}$$

The reverse inequality is obtained by duality; first note that for $y\in\mathrm{L}^\infty(\mathrm{tr}\otimes\tau)$,

$$\lim\mathrm{tr}\otimes\mathrm{tr}(y\mu_\iota)=\mathrm{tr}\otimes\tau(y).$$

With the above notation and the inequality for p',

$$\begin{aligned}\|x\|^p_{\mathrm{L}^p(\mathrm{tr}\otimes\tau)}&=\tau(|x|^p)=\lim\mathrm{tr}\,|x|^p\mu_\iota=\lim\mathrm{tr}\,\mu_\iota^{1-1/p}|x|^{p-1}u^*x\mu_\iota^{1/p}\\ &\leqslant\limsup\|\mu_\iota^{1-1/p}|x|^{p-1}\|_{\mathrm{S}^{p'}(\mathrm{S}^{p'}_n)}\|x\mu_\iota^{1/p}\|_{\mathrm{S}^p(\mathrm{S}^p_n)}\\ &=\limsup\||x|^{p-1}\mu_\iota^{1-1/p}\|_{\mathrm{S}^{p'}(\mathrm{S}^{p'}_n)}\|x\mu_\iota^{1/p}\|_{\mathrm{S}^p(\mathrm{S}^p_n)}\\ &\leqslant\||x|^{p-1}\|_{\mathrm{L}^{p'}(\mathrm{tr}\otimes\tau)}\limsup\|x\mu_\iota^{1/p}\|_{\mathrm{S}^p(\mathrm{S}^p_n)},\end{aligned}$$

so that

$$\limsup\|x\mu_\iota^{1/p}\|_{\mathrm{S}^p(\mathrm{S}^p_n)}=\|x\|^p_{\mathrm{L}^p(\mathrm{tr}\otimes\tau)}.$$ □

Remark 3.2. Let μ be any positive diagonal operator with $\operatorname{tr}\mu = 1$ and $p \geqslant 2$; then $\|x\mu^{1/p}\|_{\mathrm{S}^p(\mathrm{S}^p_n)} \leqslant \|x\|_{\mathrm{L}^p}$ for all $x \in \mathrm{L}^\infty(\operatorname{tr}\otimes\tau)$. The Reiter condition is only necessary to go below exponent 2.

We could also have used interpolation with a two-sided approximation by Reiter means. We would have obtained

$$\limsup \|\mu_\iota^{1/2p} x \mu_\iota^{1/2p}\|_{\mathrm{S}^p(\mathrm{S}^p_n)} = \|x\|_{\mathrm{L}^p(\operatorname{tr}\otimes\tau)}.$$

This formula is in the spirit of the first approach of this section.

4 Transfer of lacunary sets into lacunary matrix patterns

As a first application of Theorem 2.7, let us mention that it provides a shortcut for some arguments in [12], as it permits us to transfer lacunary subsets of a discrete group Γ into lacunary matrix patterns in $\Gamma \times \Gamma$. Let us first introduce the following terminology.

Definition 4.1. Let Γ be a discrete group and $\Lambda \subseteq \Gamma$. Let X be the reduced C^*-algebra C of Γ or its noncommutative Lebesgue space L^p for $p \in [1, \infty[$.

(a) The set Λ is *unconditional* in X if the Fourier series of every $x \in X_\Lambda$ converges unconditionally; i.e., there is a constant D such that

$$\Big\| \sum_{\gamma\in\Lambda'} x_\gamma \varepsilon_\gamma \lambda_\gamma \Big\|_X \leqslant D\|x\|_X$$

for finite $\Lambda' \subseteq \Lambda$ and $\varepsilon_\gamma \in \mathbb{T}$. The minimal constant D is the *unconditional constant of Λ in X*.

(b) If $X = \mathrm{C}$, let $\tilde X = \mathrm{S}^\infty \otimes \mathrm{C}$; if $X = \mathrm{L}^p$, let $\tilde X = \mathrm{L}^p(\operatorname{tr}\otimes\tau)$. The set Λ is *completely unconditional* in X if the Fourier series of every $x \in \tilde X_\Lambda$ converges unconditionally; i.e., there is a constant D such that

$$\Big\| \sum_{\gamma\in\Lambda'} x_\gamma \otimes \varepsilon_\gamma \lambda_\gamma \Big\|_{\tilde X} \leqslant D\|x\|_{\tilde X}$$

for finite $\Lambda' \subseteq \Lambda$ and $\varepsilon_\gamma \in \mathbb{T}$. The minimal constant D is the *complete unconditional constant of Λ in X*.

Unconditional sets in L^p have been introduced as "$\Lambda(p)$ sets" in [12, Definition 1.1] for $p > 2$. If Γ is abelian, they are Walter Rudin's $\Lambda(p)$ sets if $p > 2$ and his $\Lambda(2)$ sets if $p < 2$ (see [36, 3]). Asma Harcharras ([12, Definition 1.5, Comments 1.9]) called completely unconditional sets in L^p "$\Lambda(p)_{\mathrm{cb}}$ sets" if $p \in]2, \infty[$, and "$\mathrm{K}(p)_{\mathrm{cb}}$ sets" if $p \in]1, 2]$; her definitions are equivalent to ours by the noncommutative Khinchin inequality.

Sets that are unconditional in C have been introduced as "unconditional Sidon sets" in [4]. If Γ is amenable, Fourier multipliers are automatically c.b. on C_Λ, so that such sets are automatically completely unconditional in C, and there are at least three more equivalent definitions for the counterpart of Sidon sets in an abelian group. If Γ is nonamenable, these definitions are no longer all equivalent, and our notion of completely unconditional sets in C corresponds to Marek Bożejko's "c.b. Sidon sets."

Definition 4.2. Let $1 \leqslant p \leqslant \infty$ and I be a subset of the product $R \times C$ of two index sets.

(a) The set I is *unconditional* in the Schatten-von-Neumann class S^p associated with $\mathbb{B}(\ell^2_C, \ell^2_R)$ if the matrix representation of every $x \in \mathrm{S}^p_I$ converges unconditionally; i.e., there is a constant D such that

$$\Big\| \sum_{q\in I'} x_q \varepsilon_q \mathrm{e}_q \Big\|_p \leqslant D\|x\|_p$$

for finite $I' \subseteq I$ and $\varepsilon_q \in \mathbb{T}$. The minimal constant D is the *unconditional constant of I in* S^p.

(b) The set I is *completely* unconditional in S^p if the matrix representation of every $x \in \mathrm{S}^p_I(\mathrm{S}^p)$ converges unconditionally; i.e., there is a constant D such that

$$\Big\|\sum_{q\in I'} x_q \otimes \varepsilon_q \mathrm{e}_q\Big\|_p \leqslant D\|x\|_p$$

for finite $I' \subseteq I$ and $\varepsilon_q \in \mathbb{T}$. The minimal constant D is the *complete unconditional constant of I in* S^p.

Harcharras called unconditional and completely unconditional sets in S^p "$\sigma(p)$ sets" and "$\sigma(p)_{\mathrm{cb}}$ sets", respectively ([12, Definitions 4.1 and 4.4, Remarks 4.6 (iv)]); she supposed $p < \infty$, so that her definitions are equivalent to ours by the noncommutative Khinchin inequality.

Proposition 4.3. *Let $\varGamma$ be a discrete group. Let $\varLambda \subseteq \varGamma$ and consider the associated Toeplitz set $\ddot{\varLambda} = \{(r,c) \in \varGamma \times \varGamma : rc^{-1} \in \varLambda\}$. Let $p \in [1,\infty[$.*

(a) *If $\varGamma$ is amenable, then $\varLambda$ is unconditional in L^p if $\ddot{\varLambda}$ is unconditional in S^p.*

(b) *If $\varLambda$ is completely unconditional in L^p, then $\ddot{\varLambda}$ is completely unconditional in S^p. The converse holds if $\varGamma$ is amenable.*

Proof. The first part of (b) follows by the argument of the proof of [12, Proposition 4.7]; let us sketch it. Consider the isometric embedding of the space $\mathrm{S}^p_{\ddot{\varLambda}}(\mathrm{S}^p)$ in $\mathrm{L}^p_{\varLambda}(\mathrm{tr}\otimes\mathrm{tr}\otimes\tau)$ that is given in the proof of Lemma 2.4 and apply the equivalent Definition 1.5 in [12] of the complete unconditionality of $\varLambda$: this gives the complete unconditionality of $\ddot{\varLambda}$ in the equivalent Definition 4.4 in [12].

Unconditionality in L^p expresses the uniform boundedness of relative unimodular Fourier multipliers on $\mathrm{L}^p_{\varLambda}$; complete unconditionality expresses their uniform complete boundedness. Unconditionality in S^p expresses the uniform boundedness of relative unimodular Schur multipliers on $\mathrm{S}^p_{\ddot{\varLambda}}$; complete unconditionality expresses their uniform complete boundedness. The second part of (b) follows therefore from Theorem 2.7 (b) and (a) follows from Theorem 2.1. ☐

Remark 4.4. This transfer does not pass to the limit $p = \infty$ in (b) and is void in (a). Nicholas Varopoulos proved that unconditional sets in S^∞ are finite unions of patterns whose rows or whose columns contain at most one element, and this excludes sets of the form $\ddot{\varLambda}$ for any infinite $\varLambda$ ([37, Theorem 4.2], see [22, § 5] for a reader's guide).

Remark 4.5. See [22, Remark 11.3] for an illustration of Proposition 4.3 (b) in a particular context.

Remark 4.6. Let p be an even integer greater than or equal to 4. The existence of a $\sigma(p)_{\mathrm{cb}}$ set that is not a $\sigma(q)$ set for any $q > p$ ([12, Theorem 4.9]) becomes a direct consequence of Walter Rudin's construction ([36, Theorem 4.8]) of a $\Lambda(p)$ set that is not a $\Lambda(q)$ set for any $q > p$, because this set has property $\mathrm{B}(p/2)$ ([12, Definition 2.4]) and is therefore $\Lambda(p)_{\mathrm{cb}}$ by [12, Theorem 1.13] (in fact, it is even "1-unconditional" in L^p because $\mathrm{B}(p/2)$ is "$p/2$-independence" ([22, § 11])).

Remark 4.7. In the same way, [12, Theorem 5.2] becomes a mere reformulation of [12, Proposition 3.6] if one remembers that the Toeplitz Schur multipliers are 1-complemented in the Schur multipliers for an amenable discrete group and for all classical norms. Basically, results on $\Lambda(p)_{\mathrm{cb}}$ sets produce results on $\sigma(p)_{\mathrm{cb}}$ sets.

Let us now estimate the complete unconditional constant of sumsets. In the case $\Gamma = \mathbb{Z}$, Harcharras ([12, Prop. 2.8]) proved that a completely unconditional set in L^p cannot contain the sumset of characters $A + A$ for arbitrary large finite sets A. In particular, if $\Lambda \supseteq A + A$ with A infinite, then Λ is not a completely unconditional set in L^p. Thus, her proof provided examples of $\Lambda(p)$ sets that are not $\Lambda(p)_{\mathrm{cb}}$ sets.

We generalise Harcharras' result in two directions. Compare [18, § 1.4].

Proposition 4.8. *Let Γ be a discrete group and $p \neq 2$. A completely unconditional set in L^p cannot contain the sumset of two arbitrarily large sets. More precisely, let R and C be subsets of Γ with $\# R \geqslant n$ and $\# C \geqslant n^3$. Then, for any $p \geqslant 1$, the complete unconditional constant of the sumset RC in L^p is at least $n^{|1/2-1/p|}$.*

Proof. Let $r_1, \ldots, r_n$ be pairwise distinct elements in R. We shall select inductively elements $c_1, \ldots, c_n$ in C such that the $r_i c_j$ are pairwise distinct. Assume there are $c_1, \ldots, c_{m-1}$ such that the induction hypothesis

$$\forall\, i, k \leqslant n \;\forall\, j, l \leqslant m-1 \quad (i,j) \neq (k,l) \;\Rightarrow\; r_i c_j \neq r_k c_l.$$

holds. We are looking for an element $c_m \in C$ such that

$$\forall\, i, k \leqslant n \;\forall\, l \leqslant m-1 \quad r_i c_m \neq r_k c_l.$$

Such an element exists as long as $m \leqslant n$, because the set $\{r_i^{-1} r_k c_l : i, k \leqslant n,\, l \leqslant m-1\}$ has at most $\bigl(n(n-1)+1\bigr)(m-1) < n^3$ elements.

The end of the proof is the same as Harcharras'. The unconditional constant of the canonical basis of elementary matrices in S^p_n is $n^{|1/2-1/p|}$; in particular, there is an unimodular Schur multiplier $\breve{\varphi}$ on S^p_n of norm $n^{|1/2-1/p|}$ (which is also its complete norm, by the way; see [31, Lemma 8.1.5]). Let Λ be the sumset $\{r_i c_j : i, j \leqslant n\}$; as the $r_i c_j$ are pairwise distinct, we may define a sequence $\varphi \in \mathbb{C}^\Lambda$ by $\varphi_{r_i c_j} = \breve{\varphi}_{i,j}$. By Remark 2.5, the complete norm of the Fourier multiplier φ on L^p_Λ is bounded below by the complete norm of the Schur multiplier $\breve{\varphi}$ on S^p_I. ☐

Example 4.9. $\Lambda = \{2^i - 2^j : i > j\}$ does not form a complete $\Lambda(p)$ set for any $p \neq 2$. Indeed, $\{2^i - 2^j\} = \Lambda \cup -\Lambda$ does not, and if Λ did, then so would $-\Lambda$ and $\Lambda \cup -\Lambda$.

5 Toeplitz Schur multipliers on S^p for $p < 1$

When $0 < p < 1$, a complete characterisation of bounded Schur multipliers of Toeplitz type has been obtained by Alexey Alexandrov and Vladimir Peller in [1, Theorem 5.1]. This result was an easy consequence of their deep results on Hankel Schur multipliers. The transfer approach provides a direct proof.

Corollary 5.1. *Let $0 < p < 1$. Let Γ be a discrete abelian group with dual group G. Let φ be a sequence indexed by Γ and define the associated Toeplitz matrix $\check{\varphi} \in \mathbb{C}^{\check{\Lambda}}$ by $\check{\varphi}(r,c) = \varphi(rc^{-1})$ for $(r,c) \in \Gamma \times \Gamma$. Then the following are equivalent:*

(*a*) *the sequence φ is the Fourier transform of an atomic measure $\mu = \sum a_g \delta_g$ on G with $\sum |a_g|^p \leqslant 1$;*

(*b*) *the Fourier multiplier φ is contractive on L^p;*

(*c*) *the Fourier multiplier φ is contractive on $\mathrm{L}^p(\mathrm{S}^p)$;*

(*d*) *the Schur multiplier $\check{\varphi}$ is contractive on S^p;*

(*e*) *the Schur multiplier $\check{\varphi}$ is contractive on $\mathrm{S}^p(\mathrm{S}^p)$.*

Proof. The implication $(d) \Rightarrow (b)$ follows from Theorem 2.1. The equivalence $(c) \Leftrightarrow (e)$ follows from Theorem 2.7 (a). The characterisation $(a) \Leftrightarrow (b)$ is an old result of Daniel Oberlin ([23]). It is plain that $(e) \Rightarrow (d)$. At last, $(a) \Rightarrow (c)$ is obvious by the p-triangular inequality. □

Remark 5.2. As a consequence, we get that the norm of a Toeplitz Schur multiplier on $\mathrm{S}^p(\mathrm{S}^p)$ coincides with its norm on S^p when $p < 1$. If $p \in \{1, 2, \infty\}$, this holds for every Schur multiplier. Let $p \in]1, 2[\cup]2, \infty[$. Then we still do not know whether Schur multipliers are automatically c.b. on S^p. But from [31, Proposition 8.1.3], we know that (b) and (c) are not equivalent: if Γ is an infinite abelian group, there is a bounded Fourier multiplier on L^p that is not c.b. This counterexample is easy to describe: if an infinite set $A \subseteq \Gamma$ is lacunary enough, the sumset $A+A$ is unconditional in L^p (see [18, Theorem 5.13]). By Proposition 4.8, it cannot be completely unconditional. In particular, this shows that in Remark 2.3 we cannot remove the restriction to truncated Toeplitz matrices in the computation of the Schur multiplier norm; that is, $(b) \Rightarrow (d)$ does not hold.

Remark 5.3. Our questions may also be addressed in the case of a compact group like $\mathbb{T}$. A measurable function φ on $\mathbb{T}$ defines

- the Fourier multiplier on measurable functions on $\mathbb{T}$ by $x \mapsto \varphi x$;
- the Schur multiplier on integral operators on $\mathrm{L}^2(\mathbb{T})$ with kernel a measurable function x on $\mathbb{T} \times \mathbb{T}$ by $x \mapsto \check{\varphi} x$, where $\check{\varphi}(z, w) = \varphi(zw^{-1})$.

Victor Olevskii ([25]) constructed a continuous function φ that defines a bounded Fourier multiplier on the space of functions with p-summable Fourier series endowed with the norm given by $\|x\| = \left(\sum |\hat{x}(n)|^p\right)^{1/p}$ for every $p \in]1, \infty[$, while the corresponding Schur multiplier is not bounded on the Schatten-von-Neumann class S^p of operators on $\mathrm{L}^2(\mathbb{T})$ for any $p \in]1, 2[\cup]2, \infty[$.

6 The Riesz projection and the Hilbert transform

In this section, we concentrate on $\Gamma = \mathbb{Z}$, the dual group of $\mathbb{T}$.

Proposition 6.1. *Let ϱ be a linear combination of the identity and the upper triangular projection of $\mathbb{N} \times \mathbb{N}$; i.e., there are $z, w \in \mathbb{C}$ so that $\varrho_{i,j} = z$ if $i \leqslant j$ and $\varrho_{i,j} = w$ if $i > j$. Then the norm of the Schur multiplier ϱ on S^ψ coincides with the norm of the Schur multiplier ϱ on $\mathrm{S}^\psi(\mathrm{S}^\psi)$.*

Proof. Let $a \in \mathrm{S}^\psi_m(\mathrm{S}^\psi_n)$; a may be considered as an $m \times m$ matrix (a_{ij}) whose entries a_{ij} are $n \times n$ matrices, and may be identified with the block matrix

$$\tilde{a} = \begin{pmatrix} 0 & a_{11} & 0 & a_{12} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & a_{21} & 0 & a_{22} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

In this identification, $\mathrm{Id}_{\mathrm{S}^\psi_n} \otimes \mathrm{M}_\varrho(a)$ is $\mathrm{M}_\varrho(\tilde{a})$. □

The Hilbert transform $\mathscr{H}$ is the Schur multiplier obtained by choosing $z = -1$ and $w = 1$. The upper triangular operators in S^p can be seen as a noncommutative H^p space, and $\mathscr{H}$ corresponds exactly to the Hilbert transform in this setting (see [33, 19]). Using classical results on H^p spaces, all Hilbert transforms are c.b. for $1 < p < \infty$ (see [38, 33, 19]).

On the circle $\mathbb{T}$, the classical Hilbert transform H corresponds to the Fourier multiplier given by the sign function (with the convention $\mathrm{sgn}(0) = 1$), and its norm on L^p is

$\cot(\pi/2\max(p,p')) = \csc(\pi/p) + \cot(\pi/p)$ for $1 < p < \infty$. The story of the computation of this norm starts with a paper by Israel Gohberg and Naum Krupnik ([10]) for p a power of 2. The remaining cases were handled by Stylianos Pichorides ([30]) and Brian Cole (see [8]) independently. The best results in this subject are those of Brian Hollenbeck, Nigel Kalton, and Igor Verbitsky ([13]), but they rely on complex variable methods that are not available in the operator-valued case. When p is a power of 2 (or its conjugate), a combination of arguments of Gohberg and Krupnik ([9]) with some of László Zsidó ([38]) yields the following result.

Theorem 6.2. *Let $p \in]1,\infty[$. The norm and the complete norm of the Hilbert transform $\mathscr{H}$ on S^p coincide with the complete norm of the Hilbert transform H on L^p: if $\tilde{\tilde{\operatorname{sgn}}}(i,j) = \operatorname{sgn}(i-j)$ for $i, j \geqslant 1$,*

$$\|\,\tilde{\tilde{\operatorname{sgn}}}\,\|_{\mathrm{M}(\mathrm{S}^p)} = \|\,\tilde{\tilde{\operatorname{sgn}}}\,\|_{\mathrm{M_{cb}}(\mathrm{S}^p)} = \|\operatorname{sgn}\|_{\mathrm{M_{cb}}(\mathrm{L}^p)}.$$

If p is a power of 2, then these norms coincide with the norm of H on L^p:

$$\|\,\tilde{\tilde{\operatorname{sgn}}}\,\|_{\mathrm{M}(\mathrm{S}^p)} = \|\,\tilde{\tilde{\operatorname{sgn}}}\,\|_{\mathrm{M_{cb}}(\mathrm{S}^p)} = \|\operatorname{sgn}\|_{\mathrm{M_{cb}}(\mathrm{L}^p)} = \|\operatorname{sgn}\|_{\mathrm{M}(\mathrm{L}^p)} = \cot(\pi/2p).$$

Proof. Let $p \geqslant 2$. The norm of H on L^p is $\cot(\pi/2p)$ and the three other norms are equal by the transfer theorem 2.7 and the above proposition. We only need to compute the complete norm of H. Let $\tilde{H} = \mathrm{Id}_{\mathrm{S}^p} \otimes H$ be the Hilbert transform on $\mathrm{L}^p(\mathrm{tr}\otimes\tau)$. We shall use Mischa Cotlar's trick to go from L^p to L^{2p}: the equality $(\operatorname{sgn} i \operatorname{sgn} j) + 1 = \operatorname{sgn}(i+j)(\operatorname{sgn} i + \operatorname{sgn} j)$ shows that

$$(\tilde{H}f)(\tilde{H}g) + fg = \tilde{H}\big((\tilde{H}f)g + f(\tilde{H}g)\big). \tag{6.1}$$

Step 1. The function sgn is not odd, because of its value in 0; this can be fixed in the following way. Let $\Lambda = 2\mathbb{Z}+1$. The norm of $\tilde{H}$ on $\mathrm{L}^p(\mathrm{tr}\otimes\tau)$ is equal to its norm on $\mathrm{L}^p_\Lambda(\mathrm{tr}\otimes\tau)$. In fact, let D be defined by $Df(z) = zf(z^2)$; D is a complete isometry on L^p with range L^p_Λ that commutes with H.

Step 2. Let S be the real subspace of $\mathrm{L}^p_\Lambda(\mathrm{tr}\otimes\tau)$ consisting of functions with values in S^p so that $f(z)$ is selfadjoint for almost all $z \in \mathbb{T}$. Let us apply Vern Paulsen's off-diagonal trick ([27, Lemma 8.1]) to show that the norm of $\tilde{H}$ on L^p is equal to its norm on S. Let $f \in \mathrm{L}^p_\Lambda(\mathrm{tr}\otimes\tau)$. Identifying $\mathrm{S}^p_2(\mathrm{S}^p)$ with S^p,

$$g(z) = \begin{pmatrix} 0 & f(z) \\ f(z)^* & 0 \end{pmatrix}$$

defines an element of S. As the adjoint operation is isometric on S^p,

$$\|g\|_S = 2^{1/p}\|f\|_{\mathrm{L}^p(\mathrm{tr}\otimes\tau)}.$$

Let us now consider

$$\tilde{H}g = \begin{pmatrix} 0 & \tilde{H}f \\ \tilde{H}(f^*) & 0 \end{pmatrix}.$$

As $0 \notin \Lambda$ by Step 1, the equality $\operatorname{sgn}(-i) = -\operatorname{sgn} i$ holds for $i \in \Lambda$: this yields that $\tilde{H}(f^*) = -(\tilde{H}f)^*$. Therefore

$$\|\tilde{H}g\|_S = 2^{1/p}\|\tilde{H}f\|_{\mathrm{L}^p(\mathrm{tr}\otimes\tau)}.$$

Step 3. Let u_p be the norm of $\tilde{H}$ on $\mathrm{L}^p(\mathrm{tr}\otimes\tau)$; then $u_{2p} \leqslant u_p + \sqrt{1+u_p}$. It suffices to prove this estimate for $f \in S$, and by approximation we may suppose that f is a finite linear combination of terms $a_i \otimes z^i + a_i^* \otimes z^{-i}$ with a_i finite matrices. Note that $\tilde{H}f = -(\tilde{H}f)^*$. Formula (6.1) with $f = g$ combined with Hölder's inequality yields

$$\|(\tilde{H}f)^2\|_{\mathrm{L}^p(\mathrm{tr}\otimes\tau)} \leqslant \|f^2\|_{\mathrm{L}^p(\mathrm{tr}\otimes\tau)} + 2u_p\|f\|_{\mathrm{L}^{2p}(\mathrm{tr}\otimes\tau)}\|\tilde{H}f\|_{\mathrm{L}^{2p}(\mathrm{tr}\otimes\tau)}.$$

Since f and $\tilde{H}f$ take normal values,

$$\|f^2\|_{\mathrm{L}^p(\mathrm{tr}\otimes\tau)} = \|f\|^2_{\mathrm{L}^{2p}(\mathrm{tr}\otimes\tau)}$$
$$\|(\tilde{H}f)^2\|_{\mathrm{L}^p(\mathrm{tr}\otimes\tau)} = \|\tilde{H}f\|^2_{\mathrm{L}^{2p}(\mathrm{tr}\otimes\tau)}.$$

Therefore, if $\|f\|_{\mathrm{L}^{2p}(\mathrm{tr}\otimes\tau)} = 1$, $\|\tilde{H}f\|_{\mathrm{L}^{2p}(\mathrm{tr}\otimes\tau)}$ must be smaller than the bigger root of $t^2 - 2u_p t - 1$; that is,

$$\|\tilde{H}f\|^2_{\mathrm{L}^{2p}(\mathrm{tr}\otimes\tau)} \leqslant u_p + \sqrt{u_p^2+1} \text{ and } u_{2p} \leqslant u_p + \sqrt{u_p^2+1}.$$

Step 4. The multiplier H is an isometry on $\mathrm{L}^2(\mathrm{tr}\otimes\tau)$, so that $u_2 = 1 = \cot(\pi/4)$. As $\cot(\vartheta/2) = \cot\vartheta + \sqrt{\cot^2\vartheta + 1}$ for $\vartheta \in]0,\pi[$, we conclude by induction. □

Unfortunately, we cannot deal with other values of $p > 2$ by this method.

The Riesz projection $\mathscr{T}$ is the Schur multiplier obtained by choosing $z = 0$ and $w = 1$ in Proposition 6.1. It is the projection on the upper triangular part. On the circle, the classical Riesz projection T, that is the projection onto the analytic part, corresponds to the Fourier multiplier given by the indicator function $\chi_{\mathbb{Z}^+}$ of nonnegative integers; its norm on L^p, as computed by Hollenbeck and Verbitsky ([14]), is $\csc(\pi/p)$. As for the Hilbert transform, we know that the norm and the complete norm of $\mathscr{T}$ on S^p are equal and coincide with the complete norm of T on L^p, but, to the best of our knowledge, there is no simple formula like (6.1) to go from exponent p to $2p$. We only obtained the following computation.

Proposition 6.3. *Let $p \in]1,\infty[$. The norm and the complete norm of the Riesz projection $\mathscr{T}$ on S^p coincide with the complete norm of the Riesz projection T on L^p: if $\check{\chi}_{\mathbb{Z}^+}(i,j) = \chi_{\mathbb{Z}^+}(i-j)$ for $i,j \geqslant 1$,*

$$\|\check{\chi}_{\mathbb{Z}^+}\|_{\mathrm{M}(\mathrm{S}^p)} = \|\check{\chi}_{\mathbb{Z}^+}\|_{\mathrm{M_{cb}}(\mathrm{S}^p)} = \|\chi_{\mathbb{Z}^+}\|_{\mathrm{M_{cb}}(\mathrm{L}^p)}.$$

If $p = 4$, then these norms coincide with the norm of T on L^p:

$$\|\check{\chi}_{\mathbb{Z}^+}\|_{\mathrm{M}(\mathrm{S}^4)} = \|\check{\chi}_{\mathbb{Z}^+}\|_{\mathrm{M_{cb}}(\mathrm{S}^4)} = \|\chi_{\mathbb{Z}^+}\|_{\mathrm{M_{cb}}(\mathrm{L}^4)} = \|\chi_{\mathbb{Z}^+}\|_{\mathrm{M}(\mathrm{L}^4)} = \sqrt{2}.$$

Proof. We shall compute the norm of $\mathscr{T}$ on S^4. Let x be a finite upper triangular matrix and let y be a finite strictly lower triangular matrix. We have to prove that

$$\sqrt{2}\|x+y\|_{\mathrm{S}^4} \geqslant \|x\|_{\mathrm{S}^4}.$$

Let us make the obvious estimates on S^2 and use the fact that the adjoint operation is isometric:

$$\|\mathscr{T}(xx^*)\|_{\mathrm{S}^2} = \|\mathscr{T}((x+y)x^*)\|_{\mathrm{S}^2} \leqslant \|x+y\|_{\mathrm{S}^4}\|x\|_{\mathrm{S}^4},$$

and similarly,

$$\|(\mathrm{Id}-\mathscr{T})(xx^*)\|_{\mathrm{S}^2} = \|(\mathrm{Id}-\mathscr{T})(x(x+y)^*)\|_{\mathrm{S}^2} \leqslant \|x\|_{\mathrm{S}^4}\|x+y\|_{\mathrm{S}^4}.$$

As $\mathscr{T}$ and $\mathrm{Id}-\mathscr{T}$ have orthogonal ranges,

$$\|x\|^4_{\mathrm{S}^4} = \|xx^*\|^2_{\mathrm{S}^2} = \|(\mathrm{Id}-\mathscr{T})(xx^*)\|^2_{\mathrm{S}^2} + \|\mathscr{T}(xx^*)\|^2_{\mathrm{S}^2} \leqslant 2\|x\|^2_{\mathrm{S}^4}\|x+y\|^2_{\mathrm{S}^4}.$$ □

7 Unconditional approximating sequences

The following definition makes sense for general operator spaces, but we choose to state it only in our specific context.

Definition 7.1. Let Γ be a discrete group and $\Lambda \subseteq \Gamma$. Let X be the reduced C*-algebra of Γ or its noncommutative Lebesgue space L^p for $p \in [1,\infty[$.

(*a*) A sequence (T_k) of operators on X_Λ is an *approximating sequence* if each T_k has finite rank and $T_k x \to x$ for every $x \in X_\Lambda$. It is a *complete* approximating sequence if the T_k are uniformly c.b. If X_Λ admits a complete approximating sequence, then X_Λ enjoys the *c.b. approximation property.*

(*b*) The *difference sequence* (ΔT_k) of a sequence (T_k) is given by $\Delta T_1 = T_1$ and $\Delta T_k = T_k - T_{k-1}$ for $k \geqslant 2$. An approximating sequence (T_k) is *unconditional* if the operators

$$\sum_{k=1}^{n} \varepsilon_k \Delta T_k \quad \text{with } n \geqslant 1 \text{ and } \varepsilon_k \in \{-1, 1\} \tag{7.1}$$

are uniformly bounded on X_Λ; then X_Λ enjoys the *unconditional approximation property.*

(*c*) An approximating sequence (T_k) is *completely* unconditional if the operators in (7.1) are uniformly c.b. on X_Λ; then X_Λ enjoys the *complete* unconditional approximation property. The minimal uniform bound of these operators is the *complete unconditional constant* of X_Λ.

We may always suppose that a complete approximating sequence on C_Λ is a Fourier multiplier sequence (see [11, Theorem 2.1]). We may also do so on L^p_Λ if L^∞ has the so-called QWEP (see [15, Theorem 4.4]). More precisely, the following proposition holds.

Proposition 7.2. *Let Γ be a discrete group and $\Lambda \subseteq \Gamma$. Let X either be its reduced* C^*-*algebra or its noncommutative Lebesgue space L^p, where $p \in [1, \infty[$ and L^∞ has the QWEP. If X_Λ enjoys the completely unconditional approximation property with constant D, then for every $D' > D$ there is a complete approximating sequence of Fourier multipliers (φ_k) that realises the completely unconditional approximation property with constant D': the Fourier multipliers $\sum_{k=1}^{n} \varepsilon_k \Delta\varphi_k$ are uniformly completely bounded by D' on X_Λ.*

Let us now describe how to skip blocks in an approximating sequence in order to construct an operator that acts like the Riesz projection on the sumset of two infinite sets. The following trick will be used in the induction below (compare with the proof of [20, Theorem 4.2]):

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) - \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right) + \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Lemma 7.3. *Let Γ be a discrete group and $\Lambda \subseteq \Gamma$. Suppose that Λ contains the sumset RC of two infinite sets R and C. Let (T_k) be either an approximating sequence on L^p_Λ with $p \in [1, \infty[$, or an approximating sequence of Fourier multipliers on C_Λ. Let $\varepsilon > 0$. There is a sequence (r_i) in R, a sequence (c_i) in C, and there are indices $l_1 < k_2 < l_2 < k_3 < \dots$ such that, for every n, the* skipped block sum

$$U_n = T_{l_1} + (T_{l_2} - T_{k_2}) + \dots + (T_{l_n} - T_{k_n}) \tag{7.2}$$

acts, up to ε, as the Riesz projection on the sumset $\{r_i c_j\}_{i,j \leqslant n}$:

$$\begin{cases} \|U_n(\lambda_{r_i c_j}) - \lambda_{r_i c_j}\| < \varepsilon & \text{if } i \leqslant j \leqslant n, \\ \|U_n(\lambda_{r_i c_j})\| < \varepsilon & \text{if } j < i \leqslant n. \end{cases}$$

Proof. Let us construct the sequences and indices by induction. If $n = 1$, let r_1 and c_1 be arbitrary; there is l_1 such that $\|T_{l_1}(\lambda_{r_1 c_1}) - \lambda_{r_1 c_1}\| < \varepsilon$. Suppose that $r_1, \dots, r_n$, $c_1, \dots, c_n$, $l_1, \dots, l_n$, and $k_2, \dots, k_n$ have been constructed. Let $\delta > 0$ be chosen later.

- The operator U_n defined by Equation (7.2) has finite rank. If it is a Fourier multiplier, one can choose an element $r_{n+1} \in R$ such that $U_n(\lambda_{r_{n+1}c_j}) = 0$ for $j \leqslant n$. If it acts on L^p_Λ with $p \in [1, \infty[$, one can choose an element $r_{n+1} \in R$ such that $\|U_n(\lambda_{r_{n+1}c_j})\| < \delta$ for $j \leqslant n$ because $(\lambda_\gamma)_{\gamma\in\Gamma}$ is weakly null in L^p.
- There is $k_{n+1} > l_n$ such that $\|T_{k_{n+1}}(\lambda_\gamma) - \lambda_\gamma\| < \delta$ for $\gamma \in \{r_i c_j : 1 \leqslant i \leqslant n+1, 1 \leqslant j \leqslant n\}$.
- Again, choose $c_{n+1} \in C$ such that $\|(U_n - T_{k_{n+1}})(\lambda_{r_i c_{n+1}})\| < \delta$ for $i \leqslant n+1$.
- Again, choose $l_{n+1} > k_{n+1}$ such that $\|T_{l_{n+1}}(\lambda_\gamma) - \lambda_\gamma\| < \delta$ for $\gamma \in \{r_i c_j : 1 \leqslant i, j \leqslant n+1\}$.

Let $U_{n+1} = U_n + (T_{l_{n+1}} - T_{k_{n+1}})$. If $i \leqslant n+1$ and $j \leqslant n$, then

$$\|\Delta U_{n+1}(\lambda_{r_i c_j})\| \leqslant \|T_{l_{n+1}}(\lambda_{r_i c_j}) - \lambda_{r_i c_j}\| + \|\lambda_{r_i c_j} - T_{k_{n+1}}(\lambda_{r_i c_j})\| < 2\delta,$$

so that

$$\begin{aligned} \|U_{n+1}(\lambda_{r_i c_j}) - \lambda_{r_i c_j}\| &< \varepsilon + 2\delta \quad \text{if } i \leqslant j \leqslant n \\ \|U_{n+1}(\lambda_{r_i c_j})\| &< \varepsilon + 2\delta \quad \text{if } j < i \leqslant n \\ \|U_{n+1}(\lambda_{r_{n+1} c_j})\| &< 3\delta \quad \text{if } j \leqslant n. \end{aligned}$$

If $i \leqslant n+1$, then

$$\begin{aligned} &\|U_{n+1}(\lambda_{r_i c_{n+1}}) - \lambda_{r_i c_{n+1}}\| \\ &\qquad \leqslant \|(U_n - T_{k_{n+1}})(\lambda_{r_i c_{n+1}})\| + \|T_{l_{n+1}}(\lambda_{r_i c_{n+1}}) - \lambda_{r_i c_{n+1}}\| < 2\delta. \end{aligned}$$

This shows that our choice of $r_{n+1}, c_{n+1}, k_{n+1}$ and l_{n+1} is adequate if δ is small enough. □

This construction will provide an obstacle to the unconditionality of sumsets.

Theorem 7.4. *Let Γ be a discrete group and $\Lambda \subseteq \Gamma$. Suppose that Λ contains the sumset RC of two infinite sets R and C.*

(a) *Let $1 < p < \infty$. The complete unconditional constant of any approximating sequence for L^p is bounded below by the norm of the Riesz projection on S^p, and thus by $\csc \pi/p$.*

(b) *The spaces L^1_Λ and C_Λ do not enjoy the complete unconditional approximation property.*

(c) *If Γ is amenable, then the space C_Λ does not enjoy the unconditional approximation property.*

Proof. Let (T_k) be an approximating sequence on L^p_Λ. By Lemma 7.3, for every $\varepsilon > 0$ and every n, there are elements $r_1, \ldots, r_n \in R$, $c_1, \ldots, c_n \in C$ such that the Fourier multiplier φ given by the indicator function of $\{r_i c_j\}_{i \leqslant j}$ is near to a skipped block sum U_n of (T_k) in the sense that $\|U_n(\lambda_{r_i c_j}) - \varphi_{r_i c_j}\lambda_{r_i c_j}\| < \varepsilon$. But U_n is the mean of two operators of the form (7.1): its complete norm will provide a lower bound for the complete unconditional constant of X_Λ. Let us repeat the argument of Lemma 2.4 with $x \in \mathrm{S}^p_n$. As

$$\begin{aligned} &\Big\|\sum_{i,j=1}^n x_{i,j}\mathrm{e}_{i,j}\Big\|_{\mathrm{S}^p_n} \\ &\qquad = \Big\|\Big(\sum_{i=1}^n \mathrm{e}_{i,i} \otimes \lambda_{r_i}\Big)\Big(\sum_{i,j=1}^n x_{i,j}\mathrm{e}_{i,j} \otimes \lambda_\epsilon\Big)\Big(\sum_{j=1}^n \mathrm{e}_{j,j} \otimes \lambda_{c_j}\Big)\Big\|_{\mathrm{L}^p(\mathrm{tr}\otimes\tau)} \\ &\qquad = \Big\|\sum_{i=1}^n x_{i,j}\mathrm{e}_{i,j} \otimes \lambda_{r_i c_j}\Big\|_{\mathrm{L}^p(\mathrm{tr}\otimes\tau)} \end{aligned}$$

and

$$\Big\|\sum_{i=1}^{n} x_{i,j}\mathrm{e}_{i,j}\otimes(U_n(\lambda_{r_ic_j})-\varphi_{r_ic_j}\lambda_{r_ic_j})\Big\|_{\mathrm{L}^p(\mathrm{tr}\otimes\tau)} < n^2\varepsilon\|x\|_{\mathrm{S}^p_n},$$

the complete norm of U_n is nearly bounded below by the norm of the Riesz projection on S^p_n:

$$\begin{aligned}\Big\|\sum_{i=1}^{n} x_{i,j}\mathrm{e}_{i,j}\otimes U_n(\lambda_{r_ic_j})\Big\|_{\mathrm{L}^p(\mathrm{tr}\otimes\tau)} &> \Big\|\sum_{i\leqslant j} x_{i,j}\mathrm{e}_{i,j}\otimes\lambda_{r_ic_j}\Big\|_{\mathrm{L}^p(\mathrm{tr}\otimes\tau)} - n^2\varepsilon\|x\|_{\mathrm{S}^p_n}\\ &= \|\mathscr{T}(x)\|_{\mathrm{S}^p_n} - n^2\varepsilon\|x\|_{\mathrm{S}^p_n}.\end{aligned}$$

This proves (a) as well as the first assertion in (b), because the Riesz projection is unbounded on S^1. Let (T_k) be an approximating sequence on C_Λ; by Lemma 7.2, we may suppose that (T_k) is a sequence of Fourier multipliers. Thus the second assertion in (b) follows from Lemma 7.3 combined with the preceding argument (where S^p_n is replaced by S^∞_n and $\mathrm{L}^p(\mathrm{tr}\otimes\tau)$ by $\mathrm{S}^\infty_n\otimes\mathrm{C}$) and the unboundedness of the Riesz projection on S^∞. For (c), note that the Fourier multipliers T_k are automatically c.b. on C_Λ if Γ is amenable (proof of Theorem 2.7 (c)). ☐

Theorem 7.4 (b) was originally devised to prove that the Hardy space H^1, corresponding to the case $\Lambda=\mathbb{N}\subseteq\mathbb{Z}$ and $p=1$, admits no completely unconditional basis (see [34, 35]). Theorem 7.4 (c) both generalises the fact that a sumset cannot be a Sidon set (see [18, §§ 1.4, 6.6] for two proofs and historical remarks, or [17, Proposition IV.7]) and Daniel Li's result [16, Corollary 13] that the space C_Λ does not have the "metric" unconditional approximation property if Γ is abelian and Λ contains a sumset. Li ([16, Theorem 10]) also constructed a set $\Lambda\subseteq\mathbb{Z}$ such that C_Λ has this property, while Λ contains the sumset of arbitrarily large sets. This theorem also provides a new proof that the disc algebra has no unconditional basis and answers [21, Question 6.1.6].

Example 7.5. Neither the span of products $\{r_ir_j\}$ of two Rademacher functions in the space of continuous functions on $\{-1,1\}^\infty$ nor the span of products $\{s_is_j\}$ of two Steinhaus functions in the space of continuous functions on $\mathbb{T}^\infty$ has an unconditional basis.

8 Relative Schur multipliers of rank one

Let ϱ be an *elementary* Schur multiplier on S^∞, that is,

$$\varrho = x\otimes y = (x_ry_c)_{(r,c)\in R\times C}.$$

Then its norm is $\sup_{r\in R}|x_r|\sup_{c\in C}|y_c|$. How is this norm affected if ϱ is only partially specified, that is, if the action of ϱ is restricted to matrices with a given support?

Theorem 8.1. *Let $I\subseteq R\times C$ and consider $(x_r)_{r\in R}$ and $(y_c)_{c\in C}$. The relative Schur multiplier on S^∞_I given by $(x_ry_c)_{(r,c)\in I}$ has norm $\sup_{(r,c)\in I}|x_ry_c|$.*

Note that the norm of the Schur multiplier $(x_ry_c)_{(r,c)\in I}$ is bounded by $\sup_{r\in R}|x_r|\times\sup_{c\in C}|y_c|$ because the matrix $(x_ry_c)_{(r,c)\in R\times C}$ is a trivial extension of $(x_ry_c)_{(r,c)\in I}$; the proof below provides a constructive nontrivial extension of this Schur multiplier that is a composition of ampliations of the Schur multiplier in the following lemma.

Lemma 8.2. *The Schur multiplier $\begin{pmatrix}\bar z & w\\ \bar w & z\end{pmatrix}$ has norm $\max(|z|,|w|)$ on S^∞_2.*

Proof. This follows from the decomposition

$$\begin{pmatrix}\bar z & w\\ \bar w & z\end{pmatrix} = \frac{|z|+|w|}{2}\begin{pmatrix}\bar tu\\ t\bar u\end{pmatrix}\otimes\begin{pmatrix}\overline{tu} & tu\end{pmatrix} + \frac{|z|-|w|}{2}\begin{pmatrix}\bar tu\\ -t\bar u\end{pmatrix}\otimes\begin{pmatrix}\overline{tu} & -tu\end{pmatrix},$$

where $t,u\in\mathbb{T}$ are chosen so that $z=|z|t^2$ and $w=|w|u^2$. ☐

Proof of Theorem 8.1. We may suppose that C is the finite set $\{1, \dots, m\}$ and that R is the finite set $\{1, \dots, n\}$, that each y_c is nonzero, and that each row in R contains an element of I. We may also suppose that $(|x_r|)_{r\in R}$ and $(|y_c|)_{c\in C}$ are nonincreasing sequences. For each $r \in R$ let c_r be the least column index of elements of I in or above row r; in other words,

$$c_r = \min_{r'\leqslant r} \min\{c : (r', c) \in I\}.$$

The sequence $(c_r)_{r\in R}$ is nonincreasing. Let us define its inverse $(r_c)_{c\in C}$ in the sense that $r_c \leqslant r \Leftrightarrow c_r \leqslant c$. For each $c \in C$, let $r_c = \min\{r : c_r \leqslant c\}$. Given r, let $r' \leqslant r$ be such that $(r', c_r) \in I$; then $|x_r y_{c_r}| \leqslant |x_{r'} y_{c_r}|$, so that $\sup_{r\in R}|x_r y_{c_r}| \leqslant \sup_{(r,c)\in I}|x_r y_c|$ and the rank 1 Schur multiplier

$$\varrho_0 = (x_r y_{c_r})_{(r,c)\in R\times C}$$

with pairwise equal columns is bounded by $\sup_{(r,c)\in I}|x_r y_c|$ on S^∞_n. We will now "correct" ϱ_0 without increasing its norm so as to make it an extension of $(x_r y_c)_{(r,c)\in I}$. Let $r \in R$ and $c' \geqslant c_r$; then

$$\begin{aligned} x_r y_{c'} = x_r y_{c_r} \frac{y_{c_r+1}}{y_{c_r}} \cdots \frac{y_{c'}}{y_{c'-1}} &= x_r y_{c_r} \prod_{c_r\leqslant c\leqslant c'-1} \frac{y_{c+1}}{y_c} \\ &= x_r y_{c_r} \prod_{\substack{r\geqslant r_c \\ c'\geqslant c+1}} \frac{y_{c+1}}{y_c}. \end{aligned}$$

This shows that it suffices to compose the Schur multiplier ϱ_0 with the $m-1$ rank 2 Schur multipliers with block matrix

$$\varrho_c = \begin{array}{cc} & \begin{matrix} 1 \cdots c & \qquad c+1 \cdots m \end{matrix} \\ \begin{matrix} 1 \\ \vdots \\ r_c-1 \\ r_c \\ \vdots \\ n \end{matrix} & \left(\begin{array}{c|c} \left(\dfrac{y_{c+1}}{y_c}\right) & 1 \\ \hline 1 & \dfrac{y_{c+1}}{y_c} \end{array}\right) \end{array},$$

each of which has norm 1 on S^∞_n by Lemma 8.2. □

Remark 8.3. We learned after submitting this article that Timur Oikhberg proved Theorem 8.1 independently and gave some applications to it; see [24].

Remark 8.4. As an illustration, let $C = R = \{1, \dots, n\}$ and $I = \{(r, c) : r \geqslant c\}$, and let a_i be an increasing sequence of positive numbers. Take $x_r = a_r$ and $y_c = 1/a_c$. Then the relative Schur multiplier $(a_r/a_c)_{r\leqslant c}$ has norm 1. The above proof actually constructs the norm 1 extension $(\min(a_r/a_c, a_c/a_r))_{(r,c)}$. If we put $a_i = \mathrm{e}^{x_i}$, we recover that $(\mathrm{e}^{-|x_r-x_c|})_{(r,c)}$ is positive definite, that is, $|\cdot|$ is a conditionally negative function on $\mathbb{R}$.

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