Constructiveness and lattices in Lorenzen's work

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This is joint work with Thierry Coquand and Henri Lombardi.

Let $(M, \leqslant_M)$ be a preordered set.

Let us define the free meet-semilattice over M. Let us consider the set H of unordered lists of elements of M, denoted by $a = \alpha_1 \wedge \cdots \wedge \alpha_n$; we shall define a relation $\leqslant_H$ on H by the following deduction rules:

(1) if $\alpha \leqslant_M \beta$ then $\alpha \leqslant_H \beta$;

(2) if $a \leqslant_H c$ then $a \wedge b \leqslant_H c$;

(3) if $c \leqslant_H a$ and $c \leqslant_H b$ then $c \leqslant_H a \wedge b$.

It is easy to prove that the converse holds in (3) and that $\leqslant_H$ is transitive by showing the admissibility of the corresponding deduction rules. Furthermore, as (1)–(3) introduce relations only between elements one of which is a list of at least two elements of M, the converse holds as well in (1): this is a conservativity result.

A meet-semilattice with least element 0 is *pseudocomplemented* if for every b there is c such that $a \wedge b \leqslant 0$ if and only if $a \leqslant c$; the element c is denoted by $\bar{b}$. Let us define the free pseudocomplemented meet-semilattice over M. Let us consider the set H generated inductively from M as the set of unordered lists of elements of H or of formal pseudocomplements of elements of H; we shall define a relation $\leqslant_H$ on H by the deduction rules (1)–(5) with:

(4) if $a \wedge b \leqslant_H 0$ then $a \leqslant_H \bar{b}$;

(5) if $a \leqslant_H b$ then $a \wedge \bar{b} \leqslant_H c$.

It is easy to prove that the converse holds in (1), (3), (4); but it is quite difficult to prove that $\leqslant_H$ is transitive.

A meet-semilattice is *σ-complete* if for every sequence $(a_1, a_2, \ldots)$ there is a meet $\bigwedge(a_1, a_2, \ldots)$. Let us define the free σ-complete pseudocomplemented meet-semilattice over M. Let us consider the set H generated as before but with the additional inductive clause of containing the sequences $(a_1, a_2, \ldots)$ of elements of H written as formal meets $\bigwedge(a_1, a_2, \ldots)$; we shall define a relation $\leqslant_H$ on H by the deduction rules (1)–(8) with:

(6) if $a_k \wedge b \leqslant_H c$ then $\bigwedge(a_1, a_2, \ldots) \wedge b \leqslant_H c$;

(7) if $c \leqslant_H a_1$, $c \leqslant_H a_2$, ..., then $c \leqslant_H \bigwedge(a_1, a_2, \ldots)$;

(8) if $a \wedge a \wedge b \leqslant_H c$ then $a \wedge b \leqslant_H c$.

It is easy to prove that the converse holds in (1), (3), (4), (7); the proof of the transitivity of $\leqslant_H$ is much easier here because of the inclusion of the contraction rule (8) among the deduction rules.

The first two constructions appear in Paul Lorenzen's "Algebraische und logistische Untersuchungen über freie Verbände" (1951; the translation "Algebraic and logistic investigations on free lattices" is available as arXiv:1710.08138). The third one appears in his manuscript "Ein halbordnungstheoretischer Widerspruchsfreiheitsbeweis" (1944; the translation "A proof of freedom from contradiction within the theory of partial order" together with an introduction is available as arXiv:2006.08996), in which he explains why this construction is the semilattice counterpart to the proof of consistency of elementary number theory: this theory may be viewed as contained in the free σ-complete pseudocomplemented meet-semilattice over the set M of numerical propositions preordered by material implication. "The fact that the logic calculuses are semilattices or lattices permits a simple logistic application of free lattices" (Lorenzen 1951).

This talk is also an invitation to reflect upon mathematical objects (like the semilattices here) as given dynamically by rules instead of being considered statically as completed totalities.