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Kurt Hensel between Kronecker and Hasse

Hensel (1861–1941) is known as Leopold Kronecker's best student. Although I do not know of places where he spelled out his viewpoint on mathematics, his activity remained faithful to Kronecker's (and Gauss') affirmation that a concept, in order to be meaningful, must be expressed in finite terms. Hasse noted this in his obituary but he gives it the status of a private affair: « he added with a lovely meticulosity the computations behind a conceptual presentation».

Hensel is best known in mathematics for having introduced p-adic numbers. For about 30 years they remained a mathematical curiosity, and their place in mathematics today owes much to his best advocate, Hasse. In 1940, Weyl could write in his book on algebraic number theory that only p-adic numbers provide the means to deal with the subject. Hasse points out that Hensel's work contains <u>all</u> the ideas of valuation theory without ever using this concept, which has been introduced by Kürschak (1912) and developed by Ostrowski (1918). How come?

In fact, a famous error by Hensel made p-adic numbers suspicious: he wanted to show that p-adic numbers can prove the transcendance of the <u>real</u> number e by considering the p-adic number $\sum_{n=0}^{\infty} \frac{p^n}{n!}$ and giving lower bounds for its degree. Following Giusti (1980), what was the necessity of this error? It might be the following. For Hensel, following Kronecker, form is of utmost importance, and it is by forms that an object can be described (note here the fundamental theme of a common ground for function theory and for number theory), and he <u>had</u> to get mixed up by the simultaneous consideration of the realm of real numbers and of p-adic numbers.

The historical solution has been the introduction of valuations already mentioned. With them, each realm is precisely separated from the other: absolute value for real numbers, p-adic valuations, each giving rise to a specific completion and algebraic closure (and again completion). I believe that Hensel never followed this solution because these concepts, completion and algebraic closure, lack a computational description. (Note here that Roquette points out that Fränkel (the other famous student of Hensel besides Hasse) also introduced valuations in 1907 but stated in his 1967 memoirs that he did so only in a «formal» way while Kürschak's is in terms of content»).

This leads to the following question: how can one replace completion and algebraic closure by meaningful constructions? This has been done in recent years by Marieni Alonso, Franz-Viktor Kuhlmann, Henri Lombardi, Hervé Perdry thanks to a profound understanding of Hensel zeros: in order to obtain ~~the~~ desired object, it is enough to provide for every "approximate zero" of a polynomial belonging to a specific class (called Nagata polynomials), an exact zero, and for this, to add this zero formally in a universal way. In the case of valued fields, the consideration of Newton (-Puiseux) polygons provides the correct and consistent extension of the valuation to the extension by the formal zero.

The advantage of this approach w.r.t. Kürschak's is that we obtain a precise description of the objects considered in terms of form without ever relying on the impredicative concepts of completion and closure, which in a sense have to suppose that what is to be conceived is already "out there" (in set theory).