Lacunary matrices

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Abstract

We study unconditional subsequences of the canonical basis (e_{rc}) of elementary matrices in the Schatten class S^p. They form the matrix counterpart to Rudin's $\Lambda(p)$ sets of integers in Fourier analysis. In the case of p an even integer, we find a sufficient condition in terms of trails on a bipartite graph. We also establish an optimal density condition and present a random construction of bipartite graphs. As a byproduct, we get a new proof for a theorem of Erdős on circuits in graphs.

1 Introduction

We study the following question on the Schatten class S^p.

(†) How many matrix coefficients of an operator $x \in S^p$ must vanish so that the norm of x has a bounded variation if we change the sign of the remaining nonzero matrix coefficients?

Let C be the set of columns and R be the set of rows for coordinates in the matrix, in general two copies of $\mathbb{N}$. Let $I \subseteq R \times C$ be the set of matrix coordinates of the remaining nonzero matrix coefficients of x. Property (†) means that the subsequence $(\mathrm{e}_{rc})_{(r,c)\in I}$ of the canonical basis of elementary matrices is an unconditional basic sequence in S^p: I forms a $\sigma(p)$ set in the terminology of [5, §4].

It is natural to wonder about the operator valued case, where the matrix coefficients are themselves operators in S^p. As the proof of our main result carries over to that case, we shall state it in the more general terms of complete $\sigma(p)$ sets.

We show that for our purpose, a set of matrix entries $I \subseteq R \times C$ is best understood as a bipartite graph. Its two vertex classes are C and R, whose elements will respectively be termed "column vertices" and "row vertices". Its edges join only row vertices $r \in R$ with column vertices $c \in C$, this occurring exactly if $(r, c) \in I$.

We obtain a generic condition for $\sigma(p)$ sets in the case of even p (Theorem 3.2) that generalises [5, Prop. 6.5]. These sets reveal in fact as a matrix counterpart to Rudin's $\Lambda(p)$ sets and we are able to transfer Rudin's proof of [9, Theorem 4.5(b)] to a non-commutative context: his number $r_s(E, n)$ is replaced by the numbers of Def. 2.4(b) and we count trails between given vertices instead of representations of an integer.

We also establish an upper bound for the intersection of a $\sigma(p)$ set with a finite product set $R' \times C'$ (Theorem 4.2): this is a matrix counterpart to Rudin's [9, Theorem 3.5]. In terms of bipartite graphs, this intersection is the subgraph induced by the vertex subclasses $C' \subseteq C$ and $R' \subseteq R$.

The bound of Theorem 4.2 provides together with Theorem 3.2 a generalisation of a theorem by Erdős [4, p. 33] on graphs without circuits of a given even length. In the last part of this article, we present a random construction of maximal $\sigma(p)$ sets for even integers p.

Terminology. C is the set of columns and R is the set of rows, in general both indexed by $\mathbb{N}$. The set V of all vertices is their disjoint union $R \amalg C$. An edge on V is a pair $\{v, w\} \subseteq V$. A graph on V is given by its set of edges E. A bipartite graph on V with vertex classes C and R has only edges $\{r, c\}$ such that $c \in C$ and $r \in R$ and may therefore be described alternatively by the set $I = \{\, (r, c) \in R \times C : \{r, c\} \in E \,\}$. A trail of length s in a graph is a sequence $(v_0, \ldots, v_s)$ of $s + 1$ vertices such that $\{v_0, v_1\}, \ldots, \{v_{s-1}, v_s\}$ are pairwise distinct edges of the graph. A trail is a path if its vertices are pairwise distinct. A circuit of length p in a graph is a sequence $(v_1, \dots, v_p)$ of p vertices such that $\{v_1, v_2\}, \dots, \{v_{p-1}, v_p\}, \{v_p, v_1\}$ are pairwise distinct edges of the graph. A circuit is a cycle if its vertices are pairwise distinct.

Notation. $\mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}$. Let $q = (r, c) \in R \times C$. The transpose of q is $q^* = (c, r)$. The entry (elementary matrix) $\mathrm{e}_q = \mathrm{e}_{rc}$ is the operator on ℓ_2 that maps the cth basis vector on the rth basis vector and all other basis vectors on 0. The matrix coefficient at coordinate q of an operator x on ℓ_2 is $x_q = \operatorname{tr} \mathrm{e}_q^* x$ and its matrix representation is $(x_q)_{q \in R\times C} = \sum_{q\in R\times C} x_q \mathrm{e}_q$. The Schatten class S^p, $1 \leqslant p < \infty$, is the space of those compact operators x on ℓ_2 such that $\|x\|_p^p = \operatorname{tr} |x|^p = \operatorname{tr}(x^*x)^{p/2} < \infty$. For $I \subseteq R \times C$, the entry space S^p_I is the space of those $x \in S^p$ whose matrix representation is supported by I: $x_q = 0$ if $q \notin I$. S^p_I is also the closed subspace of S^p spanned by $(\mathrm{e}_q)_{q\in I}$. The S^p-valued Schatten class $S^p(S^p)$ is the space of those operators x from ℓ_2 to S^p such that $\|x\|_p^p = \operatorname{tr}(\operatorname{tr}|x|^p) < \infty$, where the inner trace is the S^p-valued analogue of the usual trace. The S^p-valued entry space $S^p_I(S^p)$ is the closed subspace spanned by the $x_q \mathrm{e}_q$ with $x_q \in S^p$ and $q \in I$: $x_q = \operatorname{tr} \mathrm{e}_q^* x$ is the operator coefficient of x at matrix coordinate q. Thus, for even integers p and $x = (x_q)_{q\in I} = \sum_{q\in I} x_q \mathrm{e}_q$ with $x_q \in S^p$ and I finite,

$$\|x\|_p^p = \sum_{q_1,\dots q_p \in I} \operatorname{tr} x_{q_1}^* x_{q_2} \dots x_{q_{p-1}}^* x_{q_p} \operatorname{tr} \mathrm{e}_{q_1}^* \mathrm{e}_{q_2} \dots \mathrm{e}_{q_{p-1}}^* \mathrm{e}_{q_p}.$$

A Schur multiplier T on S^p_I associated to $(\mu_q)_{q\in I} \in \mathbb{C}^I$ is a bounded operator on S^p_I such that $T\mathrm{e}_q = \mu_q \mathrm{e}_q$ for $q \in I$. T is furthermore completely bounded (c.b. for short) if T is bounded as the operator on $S^p_I(S^p)$ defined by $T(x_q \mathrm{e}_q) = \mu_q x_q \mathrm{e}_q$ for $x_q \in S^p$ and $q \in I$.

We shall stick to this harmonic analysis type notation; let us nevertheless show how these objects are termed with tensor products: $S^p(S^p)$ is also $S^p(\ell_2 \otimes_2 \ell_2)$ endowed with $\|x\|_p^p = \operatorname{tr} \otimes \operatorname{tr} |x|^p$; one should write $x_q \otimes \mathrm{e}_q$ instead of $x_q \mathrm{e}_q$; here $x_q = \mathrm{Id}_{S^p} \otimes \operatorname{tr}((\mathrm{Id}_{\ell_2} \otimes \mathrm{e}_q^*)x)$; T is c.b. if $\mathrm{Id}_{S^p} \otimes T$ is bounded on $S^p(\ell_2 \otimes_2 \ell_2)$.

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2 Definitions

We use the notion of unconditionality in order to define the matrix analogue of Rudin's "commutative" $\Lambda(p)$ sets.

Definition 2.1. Let X be a Banach space. The sequence $(y_n) \subseteq X$ is an unconditional basic sequence in X if there is a constant D such that

$$\Big\| \sum \vartheta_n c_n y_n \Big\|_X \leqslant D \Big\| \sum c_n y_n \Big\|_X$$

for every real (vs. complex) choice of signs $\vartheta_n \in \{-1, 1\}$ (vs. $\vartheta_n \in \mathbb{T}$) and every finitely supported sequence of scalar coefficients (c_n). The optimal D is the real (vs. complex) unconditionality constant of (y_n) in X.

Real and complex unconditionality are isomorphically equivalent: the complex unconditionality constant is at most $\pi/2$ times the real one. The notions of unconditionality and multipliers are intimately connected: we have

Proposition 2.2. *Let $(y_n) \subseteq X$ be an unconditional basic sequence in X and let Y be the closed subspace of X spanned by (y_n). The real (vs. complex) unconditionality constant of (y_n) in X is exactly the least upper bound for the norms $\|T\|_{\mathscr{L}(Y)}$, where T is the multiplication operator defined by $Ty_n = \mu_n y_n$, and the μ_n range over all real (vs. complex) numbers with $|\mu_n| \leqslant 1$.*

Let us encompass the notions proposed in Question (†).

Definition 2.3. Let $I \subseteq R \times C$ and $p > 2$.

(*a*) [5, Def. 4.1] I is a $\sigma(p)$ set if $(\mathrm{e}_q)_{q\in I}$ is an unconditional basic sequence in S^p. This amounts to the uniform boundedness of the family of all relative Schur multipliers by signs

$$T_\vartheta \colon S^p_I \to S^p_I,\ x = (x_q)_{q\in I} \mapsto T_\vartheta x = (\vartheta_q x_q)_{q\in I} \text{ with } \vartheta_q \in \{-1,1\}. \tag{1}$$

By [5, Lemma 0.5], this means that there is a constant D such that for every finitely supported operator $x = (x_q)_{q\in I} = \sum_{q\in I} x_q \mathrm{e}_q$ with $x_q \in \mathbb{C}$

$$D^{-1}\|x\|_p \leqslant |\!|\!|x|\!|\!|_p \leqslant \|x\|_p, \tag{2}$$

where the second inequality is a convexity inequality that is always satisfied (see [10, Theorem 8.9]) and

$$|\!|\!|x|\!|\!|_p^p = \sum_c \Bigl(\sum_r |x_{rc}|^2\Bigr)^{p/2} \vee \sum_r \Bigl(\sum_c |x_{rc}|^2\Bigr)^{p/2}. \tag{3}$$

(*b*) [5, Def. 4.4] I is a complete $\sigma(p)$ set if the family of all relative Schur multipliers by signs (1) is uniformly c.b. By [5, Lemma 0.5], I is completely $\sigma(p)$ if and only if there is a constant D such that for every finitely supported operator valued operator $x = (x_q)_{q\in I} = \sum_{q\in I} x_q \mathrm{e}_q$ with $x_q \in S^p$

$$D^{-1}\|x\|_p \leqslant |\!|\!|x|\!|\!|_p \leqslant \|x\|_p, \tag{4}$$

where the second inequality is a convexity inequality that is always satisfied and

$$|\!|\!|x|\!|\!|_p^p = \sum_c \Bigl\|\Bigl(\sum_r x_{rc}^* x_{rc}\Bigr)^{1/2}\Bigr\|_p^p \vee \sum_r \Bigl\|\Bigl(\sum_c x_{rc} x_{rc}^*\Bigr)^{1/2}\Bigr\|_p^p.$$

The notion of a complete $\sigma(p)$ set is stronger than that of a $\sigma(p)$ set: Inequality (2) amounts to Inequality (4) tested on operators of the type $x = \sum_{q\in I} x_q \mathrm{e}_q$ with each x_q acting on the same one-dimensional subspace of ℓ_2. It is an important open problem to decide whether the notions differ. An affirmative answer would solve Pisier's conjecture about completely bounded Schur multipliers [8, p. 113].

Notorious examples of 1-unconditional basic sequences in all Schatten classes S^p are single columns, single rows, single diagonals and single anti-diagonals — and more generally "column sets" (vs. "row sets") I such that for each $(r,c) \in I$, no other element of I is in the column c (vs. row r). These sets are called sections in [11, Def. 4.3]

We shall try to express these notions in terms of trails on bipartite graphs. We proceed as announced in the Introduction: then each example above is a union of disjoint star graphs in which one vertex of one class is connected to some vertices of the other class: trails in a star graph have at most length 2.

Definition 2.4. Let $I \subseteq R \times C$ and $s \geqslant 1$ an integer. We consider I as a bipartite graph: its vertex set is $V = R \amalg C$ and its edge set is $E = \{\, \{r,c\} \subseteq V : (r,c) \in I \,\}$.

(*a*) The sets of trails of length s on the graph I from the column (vs. row) vertex v_0 to the vertex v_s are respectively

$$\mathscr{C}^s(I; v_0, v_s) = \{\, (v_0, \ldots, v_s) \in V^{s+1} : v_0 \in C \;\&\; \text{all } \{v_i, v_{i+1}\} \in E \text{ are distinct} \,\},$$

$$\mathscr{R}^s(I; v_0, v_s) = \{\, (v_0, \ldots, v_s) \in V^{s+1} : v_0 \in R \;\&\; \text{all } \{v_i, v_{i+1}\} \in E \text{ are distinct} \,\}.$$

(*b*) We define the Rudin numbers of trails starting respectively with a column vertex and a row vertex by $c_s(I; v_0, v_s) = \#\mathscr{C}^s(I; v_0, v_s)$ and $r_s(I; v_0, v_s) = \#\mathscr{R}^s(I; v_0, v_s)$.

Remark 2.5. In other words, for an integer $l \geqslant 1$,

$$c_{2l-1}(I; v_0, v_{2l-1}) \;=\; \#\begin{bmatrix} (r_1,c_1),(r_1,c_2),(r_2,c_2),(r_2,c_3),\ldots,(r_l,c_l) \\ \text{pairwise distinct in } I : c_1 = v_0, r_l = v_{2l-1} \end{bmatrix}$$

$$c_{2l}(I; v_0, v_{2l}) \;=\; \#\begin{bmatrix} (r_1,c_1),(r_1,c_2),\ldots,(r_l,c_l),(r_l,c_{l+1}) \\ \text{pairwise distinct in } I : c_1 = v_0, c_{l+1} = v_{2l} \end{bmatrix}$$

and similarly for $r_s(I; v_0, v_s)$. If s is odd, then $c_s(I; v_0, v_s) = r_s(I; v_s, v_0)$ for all $(v_0, v_s) \in C \times R$. But if s is even, one Rudin number may be bounded while the other is infinite: see [5, Rem. 6.4(*ii*)].

3 $\sigma(p)$ sets as matrix $\Lambda(p)$ sets

We claim the following result.

Theorem 3.1. *Let $I \subseteq R \times C$ and $p = 2s$ be an even integer. If I is a union of sets $I_1, \dots, I_l$ such that one of the Rudin numbers $c_s(I_j; v_0, v_s)$ or $r_s(I_j; v_0, v_s)$ is a bounded function of (v_0, v_s), for each j, then I is a complete $\sigma(p)$ set.*

This follows from Theorem 3.2 below: the union of two complete $\sigma(p)$ sets is a complete $\sigma(p)$ set by [5, Rem. after Def. 4.4]; furthermore the transposed set $I^* = \{ q^* : q \in I \} \subseteq C \times R$ is a complete $\sigma(p)$ set provided I is. Note that the case of $\sigma(\infty)$ sets (see [5, Rem. 4.6(iii)]) provides evidence that Theorem 3.1 might be a characterisation of complete $\sigma(p)$ sets for even p.

Theorem 3.2. *Let $I \subseteq R \times C$ and $p = 2s$ be an even integer. If the Rudin number $c_s(I; v_0, v_s)$ is a bounded function of (v_0, v_s), then I is a complete $\sigma(p)$ set.*

This is proved for $p = 4$ in [5, Prop. 6.5]. We wish to emphasise that the proof below follows the scheme of the proof of [5, Theorem 1.13]. In particular, we make crucial use of Pisier's idea to express repetitions by dependent Rademacher variables ([5, Prop. 1.14]).

Proof. Let $x = \sum_{q \in I} x_q \mathrm{e}_q$ with $x_q \in S^p$. We have the following expression for $\|x\|_p$.

$$\|x\|_p^p = \operatorname{tr} \otimes \operatorname{tr}(x^* x)^s = \|y\|_2^2 \quad \text{with} \quad y = \overbrace{x^* x x^* \cdots x^{(*)}}^{s \text{ terms}},$$

i.e., y is the product of s terms which are alternatively x^* and x, and we set $x^{(*)} = x$ for even s, $x^{(*)} = x^*$ for odd s. Set $C^{(*)} = C$ for even s and $C^{(*)} = R$ for odd s. Let $(v_0, v_s) \in C \times C^{(*)}$ and $y_{v_0 v_s} = \operatorname{tr} \mathrm{e}^*_{v_0 v_s} y$ be the matrix coefficient of y at coordinate (v_0, v_s). Then we obtain by the rule of matrix multiplication

$$y = \sum_{q_1, \dots, q_s \in I} (x^*_{q_1} \mathrm{e}^*_{q_1})(x_{q_2} \mathrm{e}_{q_2}) \dots (x^{(*)}_{q_s} \mathrm{e}^{(*)}_{q_s})$$

$$y_{v_0 v_s} = \sum_{(v_1, v_0), (v_1, v_2), \dots \in I} x^*_{v_1 v_0} x_{v_1 v_2} x^*_{v_3 v_2} \cdots x^{(*)}_{(v_{s-1}, v_s)^{(*)}}. \tag{5}$$

Let $\mathscr{E}$ be the set of equivalence relations on $\{1, \dots, s\}$. Then

$$y = \sum_{\sim \in \mathscr{E}} \sum_{i \sim j \Leftrightarrow q_i = q_j} (x^*_{q_1} \mathrm{e}^*_{q_1})(x_{q_2} \mathrm{e}_{q_2}) \dots (x^{(*)}_{q_s} \mathrm{e}^{(*)}_{q_s}). \tag{6}$$

We shall bound the sum above in two steps.

(a) Let $\sim$ be equality and consider the corresponding term in the sum (6). The number of terms in the sum (5) such that $\{v_{i-1}, v_i\} \neq \{v_{j-1}, v_j\}$ if $i \neq j$ is $c_s(I; v_0, v_s)$. If c is an upper bound for $c_s(I; v_0, v_s)$, we have by the expression of the Hilbert–Schmidt norm and the Arithmetic-Quadratic Mean Inequality

$$\begin{aligned}
&\Big\| \sum_{\substack{q_1, \dots, q_s \\ \text{pairwise distinct}}} (x^*_{q_1} \mathrm{e}^*_{q_1})(x_{q_2} \mathrm{e}_{q_2}) \dots (x^{(*)}_{q_s} \mathrm{e}^{(*)}_{q_s}) \Big\|_2^2 \\
&= \sum_{(v_0, v_s) \in C \times C^{(*)}} \Big\| \sum_{v \in \mathscr{C}^s(I; v_0, v_s)} x^*_{v_1 v_0} x_{v_1 v_2} x^*_{v_3 v_2} \cdots x^{(*)}_{(v_{s-1}, v_s)^{(*)}} \Big\|_2^2 \\
&\leqslant c \sum_{(v_0, v_s) \in C \times C^{(*)}} \sum_{v \in \mathscr{C}^s(I; v_0, v_s)} \Big\| x^*_{v_1 v_0} x_{v_1 v_2} x^*_{v_3 v_2} \cdots x^{(*)}_{(v_{s-1}, v_s)^{(*)}} \Big\|_2^2 \\
&= c \sum_{\substack{q_1, \dots, q_s \\ \text{pairwise distinct}}} \big\| (x^*_{q_1} \mathrm{e}^*_{q_1})(x_{q_2} \mathrm{e}_{q_2}) \dots (x^{(*)}_{q_s} \mathrm{e}^{(*)}_{q_s}) \big\|_2^2 \\
&\leqslant c \sum_{q_1, \dots, q_s} \big\| (x^*_{q_1} \mathrm{e}^*_{q_1})(x_{q_2} \mathrm{e}_{q_2}) \dots (x^{(*)}_{q_s} \mathrm{e}^{(*)}_{q_s}) \big\|_2^2 \\
&= c \Big\| \sum_{q_1, \dots, q_s} |(x^*_{q_1} \mathrm{e}^*_{q_1})(x_{q_2} \mathrm{e}_{q_2}) \dots (x^{(*)}_{q_s} \mathrm{e}^{(*)}_{q_s})|^2 \Big\|_1
\end{aligned}$$

Now this last expression may be bounded accordingly to [5, Cor. 0.9] by

$$c\Big(\Big\|\sum(x_q^*\mathrm{e}_q^*)(x_q\mathrm{e}_q)\Big\|_s \vee \Big\|\sum(x_q\mathrm{e}_q)(x_q^*\mathrm{e}_q^*)\Big\|_s\Big)^s = c\|x\|_p^p : \tag{7}$$

see [5, Lemma 0.5] for the last equality.

(b) Let $\sim$ be distinct from equality. The corresponding term in the sum (6) cannot be bounded directly. Consider instead

$$\Psi(\sim) = \Big\| \sum_{i\sim j \Rightarrow q_i = q_j} (x_{q_1}^*\mathrm{e}_{q_1}^*)(x_{q_2}\mathrm{e}_{q_2})\dots(x_{q_s}^{(*)}\mathrm{e}_{q_s}^{(*)}) \Big\|_2 = \Big\| \sum_{i\sim j \Rightarrow q_i = q_j} \prod_{i=1}^s f_i(q_i) \Big\|_2$$

with $f_i(q) = x_q\mathrm{e}_q$ for even i and $f_i(q) = x_q^*\mathrm{e}_q^*$ for odd i. We may now apply Pisier's Lemma [5, Prop. 1.14]: let $0 \leqslant r \leqslant s-2$ be the number of one element equivalence classes modulo $\sim$; then

$$\Psi(\sim) \leqslant \|x\|_p^r (B\|x\|_p)^{s-r}, \tag{8}$$

where B is the constant arising in Lust-Piquard's non-commutative Khinchin inequality. In order to finish the proof, one does an induction on the number of atoms of the partition induced by $\sim$, along the lines of step 2 of the proof of [5, Theorem 1.13]. □

The Moebius inversion formula for partitions enabled Pisier [7] to obtain the following explicit bounds in the computation above:

$$\|y\|_2 \leqslant c^{1/2}\|x\|_p^s + \sum_{0\leqslant r\leqslant s-2} \binom{s}{r} (s-r)!\|x\|_p^r \big((3\pi/4)\|x\|_p\big)^{s-r}$$
$$\|x\|_p \leqslant \big((4c)^{1/p} \vee 9\pi p/8\big)\|x\|_p. \tag{9}$$

Let us also record the following consequence of his study of p-orthogonal sums. The family $(x_q\mathrm{e}_q)_{q\in I}$ is p-orthogonal in the sense of [7] if and only if the graph associated to I does not contain any circuit of length p, so that we have by [7, Theorem 3.1]:

Theorem 3.3. *Let $p \geqslant 4$ be an even integer. If I does not contain any circuit of length p, then I is a complete $\sigma(p)$ set with constant at most $3\pi p/2$.*

Remark 3.4. Pisier proposed to us the following argument to deduce a weaker version of Theorem 3.2 from [5, Theorem 1.13]. Let $\Gamma = \mathbb{T}^V$ and z_v denote the vth coordinate function on Γ. Associate to I the set $\Lambda = \{\, z_r z_c : (r,c) \in I \,\}$. Let still $p = 2s$ be an even integer. Then I is a complete $\sigma(p)$ set if Λ is a complete $\Lambda(p)$ set as defined in [5, Def. 1.5], which in turn holds if Λ has property $Z(s)$ as given in [5, Def. 1.11]. It turns out that this condition implies the uniform boundedness of

$$c_t(I; v_0, v_t) \vee r_t(I; v_0, v_t) \quad \text{for } t \leqslant s,\ v_0, v_t \in V.$$

For $p \geqslant 8$, this implication is strict: in fact, the countable union of disjoint cycles of length 4 ("quadrilaterals")

$$I = \bigcup\nolimits_{i\geqslant 0} \big\{(2i,2i),(2i,2i+1),(2i+1,2i+1),(2i+1,2i)\big\}$$

satisfies $c_t(I; v_0, v_t) \vee r_t(I; v_0, v_t) \leqslant 2$ whereas Λ does not satisfy $Z(s)$ for any $s \geqslant 4$.

Remark 3.5. Theorem 3.1 is especially useful to construct c.b. Schur multipliers: by [5, Rem. 4.6(ii)], if I is a complete $\sigma(p)$ set, there is a constant D (the constant D in (4)) such that for every sequence $(\mu_q) \in \mathbb{C}^{R\times C}$ supported by I and every operator $T_\mu : (x_q) \mapsto (\mu_q x_q)$ we have

$$\|T_\mu\|_{\mathscr{L}(S^p(S^p))} \leqslant D \sup_{q\in I} |\mu_q|.$$

4 The intersection of a $\sigma(p)$ set with a finite product set

Let $I \subseteq R \times C$ considered as a bipartite graph as in the Introduction and let $I' \subseteq I$ be the subgraph induced by the vertex set $C' \amalg R'$, with $C' \subseteq C$ a set of m column vertices and $R' \subseteq R$ a set of n row vertices. In other words, $I' = I \cap R' \times C'$. Let $d(v)$ be the degree of the vertex $v \in C' \amalg R'$ in I': in other words,

$$\forall c \in C' \quad d(c) = \#[I' \cap R' \times \{c\}],$$
$$\forall r \in R' \quad d(r) = \#[I' \cap \{r\} \times C'].$$

Let us recall that the dual norm of (3) is

$$\|x\|_{p'} = \inf_{\substack{\alpha,\beta \in S^{p'} \\ \alpha+\beta=x}} \Big(\sum_c \Big(\sum_r |\alpha_{rc}|^2\Big)^{p'/2}\Big)^{1/p'} + \Big(\sum_r \Big(\sum_c |\beta_{rc}|^2\Big)^{p'/2}\Big)^{1/p'},$$

where $p \geqslant 2$ and $1/p + 1/p' = 1$ (see [5, Rem. after Lemma 0.5]).

Lemma 4.1. *Let $1 \leqslant p' \leqslant 2$ and $x = \sum_{q \in I'} x_q$. Then*

$$\|x\|_{p'}^{p'} \geqslant \sum\sum_{(r,c)\in I'} \Big(\max\big(d(c), d(r)\big)^{1/2-1/p'} |x_{rc}|\Big)^{p'}.$$

Proof. By the p'-Quadratic Mean Inequality and by Minkowski's Inequality,

$$\begin{aligned}
&\Big(\sum_{c\in C'}\Big(\sum_{(r,c)\in I'} |\alpha_{rc}|^2\Big)^{p'/2}\Big)^{1/p'} + \Big(\sum_{r\in R'}\Big(\sum_{(r,c)\in I'} |\beta_{rc}|^2\Big)^{p'/2}\Big)^{1/p'} \\
\geqslant\ & \Big(\sum_{c\in C'} d(c)^{p'/2-1} \sum_{(r,c)\in I'} |\alpha_{rc}|^{p'}\Big)^{1/p'} + \Big(\sum_{r\in R'} d(r)^{p'/2-1} \sum_{(r,c)\in I'} |\beta_{rc}|^{p'}\Big)^{1/p'} \\
\geqslant\ & \Big(\sum\sum_{(r,c)\in I'} \big(d(c)^{1/2-1/p'}|\alpha_{rc}| + d(r)^{1/2-1/p'}|\beta_{rc}|\big)^{p'}\Big)^{1/p'}
\end{aligned}$$

The lemma follows by taking the infimum over all α, β with $\alpha_q + \beta_q = x_q$ for $q \in I'$ as one can suppose that $\alpha_q = \beta_q = 0$ if $q \notin I$; note further that $1/2 - 1/p' \leqslant 0$. ☐

Theorem 4.2. *If I is a $\sigma(p)$ set with constant D as in (2), then the size $\#I'$ of any subgraph I' induced by m column vertices and n row vertices, in other words the cardinal of any subset $I' = I \cap R' \times C'$ with $\#C' = m$ and $\#R' = n$, satisfies*

$$\begin{aligned}
\#I' &\leqslant D^2\big(m^{1/p}n^{1/2} + m^{1/2}n^{1/p}\big)^2 \qquad (10)\\
&\leqslant 4D^2 \min(m,n)^{2/p} \max(m,n).
\end{aligned}$$

The exponents in this inequality are optimal even for a complete $\sigma(p)$ set I in the following cases:

(a) if m or n is fixed;

(b) if p is an even integer and $m = n$.

Bound (10) holds *a fortiori* if I is a complete $\sigma(p)$ set. Density conditions thus do not so far permit to distinguish $\sigma(p)$ sets and complete $\sigma(p)$ sets. One may conjecture that Inequality (10) is also optimal for p not an even integer and $m = n$: this would be a matrix counterpart to Bourgain's theorem [3] on maximal $\Lambda(p)$ sets.

Proof. If (2) holds, then $\|x|_{I'}\|_p \leqslant D\|x\|_p$ for all $x \in S^p$ by Remark 3.5 applied to (μ_q) the indicator function of I', and by duality $\|x|_{I'}\|_{p'} \leqslant D\|x\|_{p'}$ for all $x \in S^{p'}$ (compare with [5, Rem. 4.6(*iv*)]). Let

$$y = \sum\sum_{(r,c)\in I'} d(c)^{1/p'-1/2} \mathrm{e}_{rc},$$
$$z = \sum\sum_{(r,c)\in I'} d(r)^{1/p'-1/2} \mathrm{e}_{rc},$$

Then the n rows of y are all equal, as well as the m columns of z: y and z have rank 1 and a single singular value. By the norm inequality followed by the $(2/p'-1)$-Arithmetic Mean Inequality,

$$\begin{aligned}\|y+z\|_{p'} &\leqslant \|y\|_{p'} + \|z\|_{p'}\\ &= n^{1/2}\Big(\sum_{c\in C'} d(c)^{2/p'-1}\Big)^{1/2} + m^{1/2}\Big(\sum_{r\in R'} d(r)^{2/p'-1}\Big)^{1/2}\\ &\leqslant n^{1/2}m^{1-1/p'}(\#I')^{1/p'-1/2} + m^{1/2}n^{1-1/p'}(\#I')^{1/p'-1/2}.\end{aligned}$$

We used that $\sum_{c\in C'} d(c) = \sum_{r\in R'} d(r) = \#I'$. By Lemma 4.1 applied to $x = y+z$,

$$(\#I')^{1/p'} \leqslant D(n^{1/2}m^{1-1/p'} + m^{1/2}n^{1-1/p'})(\#I')^{1/p'-1/2},$$

and we get therefore the first part of the theorem.

Let us show optimality in the given cases.

(a) Suppose that n is fixed and $C' = C$: $I' = R' \times C$ is a complete $\sigma(p)$ set for any p as a union of n rows and $\#I' = n \cdot m$.

(b) is proved in [5, Theorem 4.8]. ☐

Remark 4.3. If $n \not\approx m$, the method used in [5, Theorem 4.8] does not provide optimal $\sigma(p)$ sets but the following lower bound. Let $p = 2s$ with $s \geqslant 2$ an integer. Consider a prime q and let $k = s^{s-1}q^s$. By [9, 4.7] and [5, Theorem 2.5], there is a subset $F \subseteq \{0,\dots,k-1\}$ with q elements whose complete $\Lambda(2s)$ constant is independent of q. Let $m \geqslant k$ and $0 \leqslant n \leqslant m$ and consider the Hankel set

$$I = \{\,(r,c) \in \{0,\dots,n-1\} \times \{0,\dots,m-1\} : r+c \in F+m-k\,\}.$$

Then the complete $\sigma(p)$ constant of I is independent of q by [5, Prop. 4.7] and

$$\#I \geqslant \begin{cases} nq & \text{if } n \leqslant m-k+1\\ (m-k+1)q & \text{if } n \geqslant m-k+1.\end{cases}$$

If we choose $m = (s+1)k-1$, this yields

$$\#I \geqslant \frac{s^{1/s}}{(s+1)^{1+1/s}} \min(n,m)\max(m,n)^{1/s}.$$

Random construction 6.1 provides bigger sets than this deterministic construction; however, it also does not provide sets that would show the optimality of Inequality (10) unless $s = 2$.

5 Circuits in graphs

Non-commutative methods yield a new proof to a theorem of Erdős [4, p. 33]. Note that its generalisation by Bondy and Simonovits [2] is stronger than Theorem 5.1 below as it deals with cycles instead of circuits. By Theorem 3.3 and (10)

Theorem 5.1. *Let $p \geqslant 4$ be an even integer. If G is a nonempty graph on v vertices with e edges without circuit of length p, then*

$$e \leqslant 18\pi^2p^2\, v^{1+2/p}.$$

If G is furthermore a bipartite graph whose two vertex classes have respectively m and n elements, then

$$e \leqslant 9\pi^2p^2 \min(m,n)^{2/p}\max(m,n). \tag{11}$$

Proof. For the first assertion, recall that a graph G with e edges contains a bipartite subgraph with more than $e/2$ edges (see [1, p. xvii]). ☐

Remark 5.2. Łuczak showed to us that (11) cannot be optimal if m and n are of very different order of magnitude. In particular, let p be a multiple of 4. Let e' be the maximal number of edges of a graph on n vertices without circuit of length $p/2$. If $m > pe'$, he shows that (11) may be replaced by $e < 3m$.

We also get the following result, which enables us to conjecture a generalisation of the theorems of Erdős and Bondy and Simonovits.

Theorem 5.3. *Let G be a nonempty graph on v vertices with e edges. Let $s \geqslant 2$ be an integer.*

(i) If

$$e > 8D^2\, v^{1+1/s} \quad \text{with } D > 9\pi s/4,$$

then one may choose two vertices v_0 and v_s such that G contains more than $D^{2s}/4$ pairwise distinct trails from v_0 to v_s, each of length s and with pairwise distinct edges.

(ii) One may draw the same conclusion if G is a bipartite graph whose two vertex classes have respectively m and n elements and

$$e > 4D^2 \min(m,n)^{1/s} \max(m,n) \quad \text{with } D > 9\pi s/4.$$

Proof. (i) According to [1, p. xvii], the graph G contains a bipartite subgraph with more than $e/2$ edges, so that we may apply (ii).

(ii) Combining inequalities (9) and (10), if $D > 9\pi s/4$, then there are vertices v_0 and v_s such that the number c of pairwise distinct trails from v_0 to v_s, each of length s and with pairwise distinct edges, satisfies $(4c)^{1/2s} > D$. □

Two paths with equal endvertices are called independent if they have only their endvertices in common.

Question 5.4. Let G be a graph on v vertices with e edges. Let $s, l \geqslant 2$ be integers. Is it so that there is a constant D such that if $e > Dv^{1+1/s}$, then G contains l pairwise independent paths of length s with equal endvertices?

Remark 5.5. Note that by Theorem 4.2, the exponent $1 + 1/s$ is optimal in Theorem 5.3(i), whereas optimality of the exponent $1 + 2/p$ in Theorem 5.1 is an important open question in Graph Theory (see [6]).

One may also formulate Theorem 5.3(ii) in the following way.

Theorem 5.6. *If a bipartite graph $G_2(n,m)$ with n and m vertices in its two classes avoids any union of c pairwise distinct trails along s pairwise distinct edges between two given vertices as a subgraph, where the class of the first vertex is fixed, then the size e of the graph satisfies*

$$e \leqslant 4 \max((4c)^{1/2s}, 9\pi s/4)) \min(m,n)^{1/s} \max(m,n).$$

6 A random construction of graphs

Let us precise our construction of a random graph.

Random construction 6.1. *Let C, R be two sets such that $\#C = m$ and $\#R = n$. Let $0 \leqslant \alpha \leqslant 1$. A random bipartite graph on $V = C \amalg R$ is defined by selecting independently each edge in $E = \{\{r,c\} \subseteq V : (r,c) \in R \times C\}$ with the same probability α. The resulting random edge set is denoted by $E' \subseteq E$ and $I' \subseteq R \times C$ denotes the associated random subset.*

Our aim is to construct large sets while keeping down the Rudin number c_s.

Theorem 6.2. *For each $\varepsilon > 0$ and for each integer $s \geqslant 2$, there is an α such that Random construction 6.1 yields subsets $I' \subseteq R \times C$ with size*

$$\#I' \sim \min(m,n)^{1/2+1/s} \max(m,n)^{1/2-\varepsilon}$$

and with $\sigma(2s)$ constant independent of m and n for $mn \to \infty$.

Proof. Let us suppose without loss of generality that $m \geqslant n$. We want to estimate the Rudin number of trails in I'. Set $C^{(*)} = C$ for even s, $C^{(*)} = R$ for odd s and let $(v_0, v_s) \in C \times C^{(*)}$. Let $l \geqslant 1$ be a fixed integer. Then

$$\begin{aligned}\mathbb{P}[c_s(I'; v_0, v_s) \geqslant l] &= \mathbb{P}[\exists\ l \text{ distinct trails } (v_0^j, \ldots, v_s^j) \text{ in } \mathscr{C}^s(I'; v_0, v_s)] \\ &= \mathbb{P}\big[\, E' \supseteq \big\{\{v_{i-1}^j, v_i^j\}\big\}_{i,j} : \{(v_0^j, \ldots, v_s^j)\}_{j=1}^l \subseteq \mathscr{C}^s(R \times C; v_0, v_s)\,\big] \\ &\leqslant \sum_{k=\lceil l^{1/s}\rceil}^{ls} \#A_k \cdot \alpha^k,\end{aligned}$$

where A_k is the following set of l-element subsets of trails in $\mathscr{C}^s(R \times C; v_0, v_s)$ built with k pairwise distinct edges

$$A_k = \Big\{ \{(v_0^j, \ldots, v_s^j)\}_{j=1}^l \subseteq \mathscr{C}^s(R \times C; v_0, v_s) : \#\big\{\{v_{i-1}^j, v_i^j\}\big\}_{i,j} = k \Big\};$$

the lower limit of summation is $\lceil l^{1/s}\rceil$ because one can build at most k^s pairwise distinct trails of length s with k pairwise distinct edges.

In order to estimate $\#A_k$, we now have to bound the number of pairwise distinct vertices and the number of pairwise distinct column vertices in each set of l trails $\{(v_0^j, \ldots, v_s^j)\}_{j=1}^l \in A_k$. We claim that

$$\#\{\, v_i^j : 1 \leqslant i \leqslant s-1, 1 \leqslant j \leqslant l \,\} \quad \leqslant \quad k(s-1)/s, \tag{12}$$

$$\#\{\, v_{2i}^j : 1 \leqslant i \leqslant \lceil s/2\rceil - 1, 1 \leqslant j \leqslant l \,\} \quad \leqslant \quad k/2. \tag{13}$$

The second estimate is trivial, because each column vertex v_{2i}^j accounts for two distinct edges $\{v_{2i-1}^j, v_{2i}^j\}$ and $\{v_{2i}^j, v_{2i+1}^j\}$. For the first estimate, note that each maximal sequence of h consecutive pairwise distinct vertices $(v_{a+1}^j, \ldots, v_{a+h}^j)$ accounts for $h+1$ pairwise distinct edges

$$\{v_a^j, v_{a+1}^j\}, \{v_{a+1}^j, v_{a+2}^j\}, \ldots, \{v_{a+h}^j, v_{a+h+1}^j\};$$

as $h \leqslant s-1$, $h+1 \geqslant hs/(s-1)$. By (12) and (13),

$$\#A_k \leqslant m^{k/2} n^{k/2-k/s} (k - k/s)^{ls-l} \leqslant (ls)^{ls} m^{k/2} n^{k/2-k/s} :$$

each element of A_k is obtained by a choice of at most $k - k/s$ vertices, of which at most $k/2$ are column vertices, and the choice of an arrangement with repetitions of $ls - l$ out of at most $k - k/s$ vertices.

Put $\alpha = m^{-1/2} n^{-1/2+1/s} (\#C \cdot \#C^{(*)})^{-\varepsilon}$. Then

$$\begin{aligned}\mathbb{P}[\sup_{(v_0,v_s)} c_s(I'; v_0, v_s) \geqslant l] \quad &\leqslant \quad \#C \cdot \#C^{(*)} \cdot (ls)^{ls} \sum_{k=\lceil l^{1/s}\rceil}^{ls} (\#C \cdot \#C^{(*)})^{-k\varepsilon} \\ &\leqslant \quad (ls)^{ls} \frac{(\#C \cdot \#C^{(*)})^{1-\lceil l^{1/s}\rceil\varepsilon}}{1 - (\#C \cdot \#C^{(*)})^{-\varepsilon}}.\end{aligned}$$

Choose l such that $\lceil l^{1/s}\rceil\varepsilon > 1$. Then this probability is little for mn large. On the other hand, $\#I'$ is of order $mn\alpha$ with probability close to 1. □

Remark 6.3. This construction yields much better results for $s = 2$. Keeping the notation of the proof above and $m \geqslant n$, we get $k = 2l$, $A_k = \binom{n}{l}$ and

$$\mathbb{P}[\sup_{(v_0,v_2)\in C\times C} c_2(I'; v_0, v_2) \geqslant l] \leqslant m^2 \binom{n}{l} \alpha^{2l}.$$

Let $l \geqslant 2$ and $\alpha = m^{-1/l} n^{-1/2}$. This yields sets $I' \subseteq R \times C$ with size

$$\#I' \sim n^{1/2} m^{1-1/l}$$

and with $\sigma(4)$ constant independent of m and n. This case has been extensively studied in Graph theory as the "Zarankiewicz problem:" if $c_2(I'; v_0, v_2) \leqslant l$ for all $v_0, v_2 \in C$, then the graph I' does not contain a complete bipartite subgraph on any two column vertices v_0, v_2 and $l+1$ row vertices. Reiman (see [1, Theorem VI.2.6]) showed that then

$$\#I' \leqslant \left(lnm(m-1) + n^2/4\right)^{1/2} + n/2 \sim l^{1/2}n^{1/2}m.$$

With use of finite projective geometries, he also showed that this bound is optimal for

$$n = l\frac{q^{r+1}-1}{q^2-1}\frac{q^r-1}{q-1} \quad , \quad m = \frac{q^{r+1}-1}{q-1}$$

with q a prime power and $r \geqslant 2$ an integer, and thus with $m \leqslant n$: there seems to be no constructive example of extremal graphs with $c_2(I'; v_0, v_2) \leqslant l$ and $m > n$ besides the trivial case of complete bipartite graphs with $m > n = l - 1$.

Remark 6.4. In the case $s = 3$, our result cannot be improved just by refining the estimation of $\#A_k$. If we consider first l distinct paths that have their second vertex in common and then l independent paths, we get

$$\#A_{2l+1} \geqslant \binom{m}{l} n \quad , \quad \#A_{3l} \geqslant \binom{m}{l}\binom{n}{l}.$$

Therefore any choice of α as a monomial $m^{-t}n^{-u}$ in the proof above must satisfy $t \geqslant (l+1)/(2l+1)$, $t + u \geqslant (2l+2)/(3l)$ and this yields sets with

$$\#I' \preccurlyeq m^{1/2-1/2(4l+2)} n^{5/6-(7l+6)/(12l^2+6l)}.$$

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