Lorenzen's conceptions of predicativity and of constructiveness

(0) Lorenzen's first works in foundations of mathematics stem from his discovery that his work in ideal theory is formally addressing the same problem as the consistency of elementary number theory. This leads to a manuscript and an article that are now available in English. But he does not stop there and wishes to formalise the means of his proof of consistency. It takes him some time to understand that he is engaged into an infinite regress and that the only way out is to base the foundations not in an axiomatic system, but on certain elementary abilities that we learn by training, as we learn to speak and to count. These abilities can be described by calculi, and these calculi do what they ought to do. E.g., the principle of complete induction follows immediately, without further foundation. Lorenzen isolates here a specific ability, to operate, i.e. to proceed according to rules, and this is why he calls his mathematics "operational" (note that most people write "operative" today).

(1) In 1955, he publishes a book, <u>Einführung in die operative Logik und Mathematik</u> (not translated), which starts from that point and builds mathematics upon it, in particular analysis as it is needed in physics. Here are his ingredients.

<u>Abstraction</u>. As in Weyl's <u>Das Kontinuum</u>, properties are the starting point, and each property comes with its construction by the "mathematical process". The sets are the properties together with the equality of logical equivalence: an assertion about a property is an assertion about the set if it is so that it holds for the property if and only it holds for a logically equivalent property.

<u>Language layers</u>. In order to deal with the "mathematical process", Lorenzen introduces layers: he defines an elementary language of so-called "founded" and "separated" inductive definitions for relations for objects. The subsequent layers are defined by repeating this process with taking into account the new objects thus defined. This process continues along ordinals. This enables him to introduce real numbers and to obtain the usual theorems about them, e.g. he proves that a sequence of real numbers satisfying the Cauchy criterion converges to a real number by situating this real number in the layer beyond the layers of the terms of the sequence.

(2) In 1965, Lorenzen publishes another book, Differential und Integral, that differs in several aspects from the 1955 book. He remains faithful to his goal of grounding mathematics, i.e. of proceeding in a meaningful way, but he wishes to provide the most practical way of doing so.

Definite / Indefinite. The natural numbers are definite because there is a definite way of generating them. Properties of natural numbers are indefinite because every definite way of generating them may be used to produce new properties. Furthermore, there is a proof of consistency for number theory, which Lorenzen presents as a constructive explanation of the law of excluded middle, so that quantifiers over numbers may be safely used with classical logic. Quantifiers over properties, on the other hand, must be used with intuitionistic logic only. One may still structure the indefinite into language layers, but this is useless with regard to doing analysis.

This book is dedicated to the memory of Hermann Weyl, and it is impressively faithful to Das Kontinuum: the law of excluded middle plays a minor role with respect to the use of the indefinite (the transfinite as Weyl puts it); most importantly, every hierarchy is avoided inside the indefinite; analysis can proceed in a quite natural way.

(3) Three themes seem especially promising in the 1955/1965 books: (a) the emphasis on calculi, specially little calculi, as objects of mathematical investigation: this invites to think of any mathematical object as of a calculus; (b) the role given by Lorenzen to inductive definitions; (c) the use of abstraction with respect to terms and properties that allows access to their logical structure, so that they can be manipulated concretely.

It might be interesting to check whether Lorenzen's use of classical logic for definite quantifiers is essential or whether it is just a facility that allows to stay near to textbook mathematics.

Finally, note that Lorenzen barely ever uses the word "predicative", and that his terminology "finitary", "effective", "constructive" is floating: these words seem to be interchangeable for him, and he abandons the first two in favour of the last.