If X already corresponds to a theory $\mathbb{T}_X$, then the internal $\mathbb{T}$ amounts externally to an extension $\mathbb{T}_Y$, a process that is easier to work with if one uses "geometric type theory" rather than forcing theories into a standard form such as sites or first order theories.

For more discussion, see [1].

References

1 Steven Vickers. Generalized point-free spaces, pointwise. arXiv:2206.01113, 2022

2 Joyal and Tierney. An extension of the Galois theory of Grothendieck. Memoirs AMS 309, 1984

3 Steven Vickers. The double powerlocale and exponentiation Theory and Applications of Categories 12, 2004

5 Open problems

5.1 On the status of Zorn's lemma

Ingo Blechschmidt (Universität Augsburg, DE)

Over classical Zermelo-Fraenkel set theory, the axiom of choice (AC) and Zorn's lemma (ZL) are well-known to be equivalent. Dropping the law of excluded middle (LEM) allows us to distinguish these two principles: A refined analysis shows that AC = ZL + LEM. While AC implies LEM and is hence a constructive taboo, ZL can be regarded as constructively neutral.

In fact, assuming ZL in the metatheory, there are plenty of models of constructive mathematics which validate ZL, and even more which validate all bounded first-order consequences of ZL: All localic Grothendieck toposes respectively all Grothendieck toposes.

That said, in mathematical practice, applications of ZL are often followed by an appeal to LEM, and without LEM, ZL loses much of its power. But there are important results which use only ZL and not LEM, such as in commutative algebra the existence of maximal ideals, the equivalence of divisible and injective abelian groups, and the existence of enough injectives.

The talk gave a summary of this circle of ideas and invited discussion on open questions: Is there a way to extract constructive content from ZL-powered results? Are there models of constructive mathematics which validate ZL, have strong ties to a given standard model and which do not require ZL in the metatheory?

5.2 Remarks on predicativity

Stefan Neuwirth (University of Franche-Comté – Besancon, FR)

The talk by Laura Crosilla on predicativity has triggered a special interest for the "classical approach to predicativity" that culminated in the determination of the so-called limit of predicativity. This approach has much to offer in terms of technical sophistication, but it leads to the following question, both an epistemological and a sociological one: does the fascination for technique do justice to the very concept of predicativity? Predicativity is about the open-endedness of the process of mathematical creation; how could it be given a definite limit?