The free Lorenzen algebra generated by a preordered set

1 The free Lorenzen algebra in one big step

Let M be a semilattice. Let H_0 be the set M. Let N_i be the set of terms $\bar{a}$ out of elements a of H_i. Let H_{i+1} be the set of finite unordered concatenations $b_1 {}^\frown \cdots {}^\frown b_n$ out of elements $b_1, \dots, b_n$ of $H_i \cup N_i$; if $n = 0$, the concatenation $[\,]$ is "empty". We define 0 as the term $\overline{[\,]}$. Let H be the union of the sets H_i $(i = 0, 1, \dots)$. Thus the elements of H are defined recursively as finite unordered lists (that is, multisets) whose terms are elements of M or negations of elements of H.

We shall exploit below the natural associativity of concatenation, so that we do not need brackets anywhere.

By definition, $b_{\sigma(1)} {}^\frown \cdots {}^\frown b_{\sigma(n)}$ is just another writing for $b_1 {}^\frown \cdots {}^\frown b_n$ for any permutation σ: the sequential arrangement plays no rôle.

The elements of H we call shortly "formulas," the elements of M "prime formulas". Then the following "formula induction" holds: if a claim holds

1. for each prime formula,
2. for $\bar{a}$ if for a,
3. for the unordered concatenation $b_1 {}^\frown \cdots {}^\frown b_n$ if for each of the formulas $b_1, \dots, b_n$,

then it holds for each formula.

In H we define constructively a relation $\leqslant$ by:

(1) $$\frac{}{a_1 {}^\frown \cdots {}^\frown a_n \leqslant b} \ \leqslant\texttt{-axiom} \quad \text{if } a_1 \wedge \cdots \wedge a_n \leqslant b \text{ holds in } M;$$

(2) $$\frac{a \leqslant b}{a {}^\frown c \leqslant b} \ {}^\frown\!\leqslant \quad \text{for } c \text{ not the empty formula;}$$

(3) $$\frac{c \leqslant a \text{ for each term } a \text{ of the formula } b}{c \leqslant b} \ \leqslant{}^\frown \quad \text{for } b \text{ not a one-term formula;}$$

(4) $$\frac{a {}^\frown b \leqslant 0}{a \leqslant \bar{b}} \ \leqslant{}^- \quad \text{for } b \text{ not the empty formula;}$$

(5) $$\frac{a \leqslant b}{a {}^\frown \bar{b} \leqslant c} \ {}^-\!\leqslant.$$

The first two rules import axioms of a semilattice into the left side of the relation. Rule $\leqslant$`-axiom` provides the axioms of the relation by importing relations from M into H. Rule ${}^\frown\!\leqslant$ is thinning on the left.

Rules $\leqslant{}^\frown$ and $\leqslant{}^-$ import the axiom of a Lorenzen algebra into the right side.

Rule ${}^-\!\leqslant$ has a more opaque rôle: it imports negation into the left side. The computations below show that one could replace it with the weaker rule $a \leqslant a \to a {}^\frown \bar{a} \leqslant c$ and establish rules (6), (7), (9) and (10), but we could not establish the $\leqslant$`-cut` rule (8) with this weaker rule. On the other hand, rule ${}^-\!\leqslant$ easily follows from this weaker rule and rule `extended` $\leqslant$`-cut` in (15).

If $a \leqslant b$ holds, then we call the formula pair a, b a "theorem." The theorems produced by the $\leqslant$`-axiom` rule we call "prime theorems." In the rules we call the formula pairs above the line the "premisses" and the formula pair below the line the "conclusion." The following "theorem induction" holds: if a claim holds

1. for each prime theorem,
2. for the conclusion of each rule whose premisses are theorems, if for these premisses,

then it holds for each theorem.

We now show that H is a Lorenzen algebra with $[]$ as greatest element and 0 as least element. For this we have to prove the following five rules.

(6) $$\frac{}{0 \leqslant c} \quad \frac{}{c \leqslant []}.$$

(7) $$\frac{}{c \leqslant c}.$$

(8) $$\frac{a \leqslant c \quad c \leqslant b}{a \leqslant b} \ \leqslant\text{-cut}.$$

(9) $$\frac{c \leqslant b}{c \leqslant a} \ \texttt{reverse} \leqslant{}^\frown \quad \text{for each term } a \text{ of a formula } b.$$

(10) $$\frac{a \leqslant \bar{b}}{a {}^\frown b \leqslant 0} \ \texttt{reverse} \leqslant{}^-.$$

(8) we prove—because of the difficulty—last.

(6). As $\bigwedge \emptyset \leqslant 1$ in M, we have

$$\frac{}{[] \leqslant 1} \leqslant\text{-axiom} \qquad \frac{}{c \leqslant []} \leqslant{}^\frown \qquad \frac{[] \leqslant []}{\overline{[]} \leqslant c} {}^- \leqslant .$$

Therefore $[]$ is the greatest element of H, $[] \equiv 1$ and 0 is the least element of H.

We prove (7) by formula induction. For prime formulas c holds $c \leqslant c$ because it holds in M. Further holds

$$\cfrac{\cfrac{c \leqslant c}{c {}^\frown \bar{c} \leqslant 0} {}^- \leqslant}{\bar{c} \leqslant \bar{c}} \leqslant{}^-$$

and

$$\frac{\cfrac{c_1 \leqslant c_1}{c_1 {}^\frown \cdots {}^\frown c_n \leqslant c_1} {}^\frown \leqslant \quad \cdots \quad \cfrac{c_n \leqslant c_n}{c_n {}^\frown c_1 {}^\frown \cdots {}^\frown c_{n-1} \leqslant c_1} {}^\frown \leqslant}{c_1 {}^\frown \cdots {}^\frown c_n \leqslant c_1 {}^\frown \cdots {}^\frown c_n} \leqslant{}^\frown .$$

For the proof of (9) we make it the claim of a theorem induction. If $c \leqslant b$ is a prime theorem, then b is a one-term list a, so that $c \leqslant a$ holds. Let now be $c \leqslant b$ conclusion of a rule, whose premisses are theorems, and let the claim hold for the premisses. There are then only the following possibilities.

- $c \leqslant b$ is conclusion with a premiss $c' \leqslant b$ of rule ${}^\frown \leqslant$: then holds for each term a of the formula b

$$\cfrac{\cfrac{c' \leqslant b}{c' \leqslant a} \text{claim}}{c \leqslant a} {}^\frown \leqslant .$$

- $c \leqslant b$ is conclusion of rule $\leqslant{}^\frown$ with the premiss that $c \leqslant a$ for each term a of b: this premiss is what we were looking for.
- One has $c = c' {}^\frown \overline{c''}$, and $c \leqslant b$ is conclusion of rule ${}^- \leqslant$ with the premiss $c' \leqslant c''$: then holds for each term a of the formula b

$$\frac{c' \leqslant c''}{c \leqslant a} {}^- \leqslant .$$

Thereby (9) is proved by theorem induction.

Next (10) results from a theorem induction: let (10) be its induction claim. For prime theorems there is nothing to prove as there are no prime theorems of the form $a \leqslant \bar{b}$. Let now be $a \leqslant \bar{b}$ conclusion of a rule, whose premisses are theorems, and let the claim hold for the premisses. There are then only the following possibilities.

- One has $a = a' \frown a''$ and $a \leqslant \bar{b}$ is conclusion of rule $\frown\leqslant$ with a premiss $a' \leqslant \bar{b}$: then holds

$$\cfrac{\cfrac{a' \leqslant \bar{b}}{a' \frown b \leqslant 0}\ \text{claim}}{a \frown b \leqslant 0}\ {\frown\leqslant} \quad .$$

- $a \leqslant \bar{b}$ is conclusion of rule $\leqslant^-$ with the premiss that $a \frown b \leqslant 0$: the premiss is what we were looking for.

- One has $a = a' \frown \overline{a''}$, and $a \leqslant \bar{b}$ is conclusion of rule ${}^-\leqslant$ with the premiss $a' \leqslant a''$: then holds

$$\cfrac{\cfrac{a' \leqslant a''}{a \leqslant 0}\ {}^-\leqslant}{a \frown b \leqslant 0}\ {\frown\leqslant} \quad .$$

Of (8) may be proved immediately the principle of explosion by theorem induction on a:

(11) $$\frac{a \leqslant 0}{a \leqslant b}\ \leqslant\text{-p.o.e.}$$

Let (11) be the claim of a theorem induction. For prime theorems, there is nothing to prove as $a \leqslant 0$ is not a prime theorem. Let now be $a \leqslant 0$ conclusion of a rule, whose premisses are theorems, and let the claim hold for the premisses. There are then only the following possibilities.

- $a \leqslant 0$ is conclusion with a premiss $a' \leqslant 0$ of rule $\frown\leqslant$: then holds

$$\cfrac{\cfrac{a' \leqslant 0}{a' \leqslant b}\ \text{claim}}{a \leqslant b}\ {\frown\leqslant} \quad .$$

- One has $a = a' \frown \overline{a''}$, and $a \leqslant 0$ is conclusion of rule ${}^-\leqslant$ with the premiss $a' \leqslant a''$: then holds

$$\frac{a' \leqslant a''}{a \leqslant b}\ {}^-\leqslant \quad .$$

Thereby (11) is proved by theorem induction.

For the general case we need three lemmas:

(12) $$\frac{a \frown \bar{d} \leqslant b}{a \frown c \leqslant b}\ {}^=\leqslant \quad \text{for each term } \bar{c} \text{ of the formula } d.$$

Proof by theorem induction. Let (12) be the claim of a theorem induction. For prime theorems, there is nothing to prove as $a \frown \bar{d} \leqslant b$ is not a prime theorem. Let now be $a \frown \bar{d} \leqslant b$ conclusion of a rule, whose premisses are theorems, and let the claim hold for the premisses. Let $\bar{c}$ be a term of the formula d. There are then only the following possibilities.

- $a \frown \bar{d} \leqslant b$ is conclusion of rule $\frown \leqslant$, where $a = a' \frown a''$ and the premiss is $a' \frown \bar{d} \leqslant b$: then holds

$$\dfrac{\dfrac{a' \frown \bar{d} \leqslant b}{a' \frown c \leqslant b}\,\text{claim}}{a \frown c \leqslant b}\,{\frown}\leqslant \quad .$$

- $a \frown \bar{d} \leqslant b$ is conclusion of rule $\frown \leqslant$, where $a = a' \frown a''$ and the premiss is $a' \leqslant b$: then holds

$$\dfrac{a' \leqslant b}{a \frown c \leqslant b}\,{\frown}\leqslant \, .$$

- $a \frown \bar{d} \leqslant b$ is conclusion of rule $^- \leqslant$ with the premiss $a \leqslant d$: then holds

$$\dfrac{\dfrac{\dfrac{a \leqslant d}{a \leqslant \bar{c}}\,\texttt{reverse} \leqslant \frown}{a \frown c \leqslant 0}\,\texttt{reverse} \leqslant ^-}{a \frown c \leqslant b}\,\leqslant\texttt{-p.o.e.} \quad .$$

(13)
$$\dfrac{a \frown \bar{c} \leqslant p}{a \leqslant p} \text{ or } \dfrac{a \frown \bar{c} \leqslant p}{a \frown \bar{c} \leqslant 0} \quad \text{for a prime formula } p.$$

Proof by theorem induction. Let (13) be the claim of a theorem induction. For prime theorems, there is nothing to prove as $a \frown \bar{c} \leqslant p$ is not a prime theorem. Let now be $a \frown \bar{c} \leqslant p$ conclusion of a rule, whose premisses are theorems, and let the claim hold for the premisses. There are then only the following possibilities.

- $a \frown \bar{c} \leqslant p$ is conclusion of rule $\frown \leqslant$, where $a = a' \frown a''$ and the premiss is $a' \frown \bar{c} \leqslant p$: then holds

$$\dfrac{\dfrac{a' \frown \bar{c} \leqslant p}{a' \leqslant p}\,\text{claim}}{a \leqslant p}\,{\frown}\leqslant \quad \text{or} \quad \dfrac{\dfrac{a' \frown \bar{c} \leqslant p}{a' \frown \bar{c} \leqslant 0}\,\text{claim}}{a \frown \bar{c} \leqslant 0}\,{\frown}\leqslant \quad .$$

- $a = a' \frown a''$, and $a \frown \bar{c} \leqslant p$ is conclusion of rule $\frown \leqslant$ with a premiss $a' \leqslant p$: then holds

$$\dfrac{a' \leqslant p}{a \leqslant p}\,{\frown}\leqslant \, .$$

- $a \frown \bar{c} \leqslant p$ is conclusion of rule $^- \leqslant$ with the premiss $a \leqslant c$: then holds

$$\dfrac{a \leqslant c}{a \frown \bar{c} \leqslant 0}\,{^-}\leqslant \, .$$

(14)
$$\dfrac{a \frown c \frown c \leqslant b}{a \frown c \leqslant b}\,\leqslant\texttt{-contraction} \quad \text{for } c \text{ not the empty formula.}$$

We use a formula induction on c. For prime formulas c, let (14) be the claim of a theorem induction. For prime theorems $a \frown c \frown c \leqslant b$, $a \wedge c \wedge c \leqslant b$ holds in M, so that $a \wedge c \leqslant b$ holds in M and

$$\dfrac{}{a \frown c \leqslant b}\,\leqslant\texttt{-axiom} \, .$$

Let now be $a \frown c \frown c \leqslant b$ conclusion of a rule, whose premisses are theorems, and let the claim hold for the premisses. There are then only the following possibilities.

- $a \frown c \frown c \leqslant b$ is conclusion of rule ${\frown}{\leqslant}$, where $a = a' \frown a''$ and the premiss is either $a' \frown c \frown c \leqslant b$, or $a' \frown c \leqslant b$, or $a' \leqslant b$: then holds

$$\text{either}\quad \dfrac{\dfrac{a' \frown c \frown c \leqslant b}{a' \frown c \leqslant b}\ \text{claim}}{a \frown c \leqslant b}\ {\frown}{\leqslant} \qquad \text{or} \quad \dfrac{a' \frown c \leqslant b}{a \frown c \leqslant b}\ {\frown}{\leqslant} \qquad \text{or} \quad \dfrac{a' \leqslant b}{a \frown c \leqslant b}\ {\frown}{\leqslant}\ .$$

- $a \frown c \frown c \leqslant b$ is conclusion of rule ${\leqslant}{\frown}$ with the premiss that $a \frown c \frown c \leqslant b'$ for each term b' of b. Then holds

$$\dfrac{\dfrac{a \frown c \frown c \leqslant b'}{a \frown c \leqslant b'}\ \text{claim} \qquad \text{for each term } b' \text{ of } b}{a \frown c \leqslant b}\ {\leqslant}{\frown}\ .$$

- $b = \overline{b'}$ and $a \frown c \frown c \leqslant b$ is conclusion of rule ${\leqslant}^{-}$ with the premiss that $a \frown c \frown c \frown b' \leqslant 0$. Then holds

$$\dfrac{\dfrac{a \frown c \frown c \frown b' \leqslant 0}{a \frown c \frown b' \leqslant 0}\ \text{claim}}{a \frown c \leqslant b}\ {\leqslant}^{-}\ .$$

- $a = a' \frown \overline{a''}$ and $a \frown c \frown c \leqslant b$ is conclusion of rule ${}^{-}{\leqslant}$ with the premiss that $a' \frown c \frown c \leqslant a''$. Then holds

$$\dfrac{\dfrac{a' \frown c \frown c \leqslant a''}{a' \frown c \leqslant a''}\ \text{claim}}{a \frown c \leqslant b}\ {}^{-}{\leqslant}\ .$$

Suppose that (14) holds for each of the terms $c_1, \ldots, c_n$ instead of c. Then holds

$$\begin{array}{c} \dfrac{a \frown c_1 \frown \cdots \frown c_n \frown c_1 \frown \cdots \frown c_n \leqslant b}{a \frown c_1 \frown \cdots \frown c_n \frown c_2 \frown \cdots \frown c_n \leqslant b}\ \leqslant\texttt{-contraction} \\ \vdots\ \leqslant\texttt{-contraction}^{n-1} \\ a \frown c_1 \frown \cdots \frown c_n \leqslant b \end{array}$$

For formulas $\bar{c}$ we first note that c is an unordered concatenation of prime formulas p_μ of M and of negations $\overline{c_1}, \ldots, \overline{c_n}$ of elements of H such that (14) holds for each of the c_ν instead of c as induction hypothesis of our formula induction. We shall prove (14) for $\bar{c}$ instead of c by theorem induction. All steps are the same as in the proof of (14) for prime formulas c above, except that in the last possibility a new case arises: $a \frown \bar{c} \frown \bar{c} \leqslant b$ is conclusion of rule ${}^{-}{\leqslant}$ with the premiss that $a \frown \bar{c} \leqslant c$. Then holds

$$\dfrac{a \frown \bar{c} \leqslant c}{a \frown \bar{c} \leqslant p_\mu}\ \texttt{reverse}\ {\leqslant}{\frown}\ \text{for each } p_\mu$$

and

$$\dfrac{a \frown \bar{c} \leqslant c}{a \frown \bar{c} \leqslant \overline{c_\nu}}\ \texttt{reverse}\ {\leqslant}{\frown}\ \text{for each } c_\nu.$$

Then holds by (13)

$$\dfrac{a \frown \bar{c} \leqslant p_\mu}{a \leqslant p_\mu} \ \text{or}\ \dfrac{a \frown \bar{c} \leqslant p_\mu}{a \frown \bar{c} \leqslant 0}$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{a \frown \bar{c} \leqslant \overline{c_\nu}}{a \frown c_\nu \leqslant \overline{c_\nu}}\ {}^{=}{\leqslant}}{a \frown c_\nu \frown c_\nu \leqslant 0}\ \texttt{reverse}\ {\leqslant}^{-}}{a \frown c_\nu \leqslant 0}\ \text{formula induction claim}}{a \leqslant \overline{c_\nu}}\ {\leqslant}^{-}$$

Thus

$$\dfrac{\dfrac{a \leqslant p_\mu \text{ for all } p_\mu \quad a \leqslant \overline{c_\nu} \text{ for all } c_\nu}{a \leqslant c}\ \leqslant^\frown}{a \frown \bar{c} \leqslant b}\ {}^-\!\leqslant \qquad \text{or} \qquad \dfrac{a \frown \bar{c} \leqslant 0}{a \frown \bar{c} \leqslant b}\ \leqslant\texttt{-p.o.e.}.$$

With the help of (14) we can now instead of (8) even prove

(15) $$\dfrac{a \leqslant c \quad d \frown c \leqslant b}{a \frown d \leqslant b}\ \texttt{extended } \leqslant\texttt{-cut}.$$

by formula induction for each formula c.

For prime formulas c, let (15) be the claim of a theorem induction. Let now $a \leqslant c$ and $d \frown c \leqslant b$ be prime theorems or conclusions of a rule, whose premisses are theorems, and let the claim hold for the premisses.

- If they are both prime theorems, $a \leqslant c$ and $d \wedge c \leqslant b$ hold in M, so that

$$\dfrac{\dfrac{\dfrac{a \leqslant c}{a \wedge d \leqslant c}}{a \wedge d \leqslant d \wedge c} \quad d \wedge c \leqslant b}{a \wedge d \leqslant b} \qquad \text{and} \qquad \dfrac{}{a \frown d \leqslant b}\ \leqslant\texttt{-axiom}.$$

- Suppose that $a \leqslant c$ is a prime theorem or conclusion of rule ${}^\frown\!\leqslant$, where $a = a' \frown a''$ and the premiss is $a' \leqslant c$.
  - Suppose that $d \frown c \leqslant b$ is a prime theorem or conclusion of rule ${}^\frown\!\leqslant$, where $d = d' \frown d''$ and the premiss is $d' \frown c \leqslant b$: then holds

$$\dfrac{a' \leqslant c \quad d' \frown c \leqslant b}{a' \frown d' \leqslant b}\ \text{claim} \\ \vdots \\ a \frown d \leqslant b.$$

  - Suppose that $d \frown c \leqslant b$ is conclusion of rule ${}^\frown\!\leqslant$, where $d = d' \frown d''$ and the premiss is $d' \leqslant b$. Then holds

$$\dfrac{d' \leqslant b}{d \frown a \leqslant b}\ {}^\frown\!\leqslant.$$

  - Suppose that $d \frown c \leqslant b$ is conclusion of rule $\leqslant^\frown$, where the premiss is that $d \frown c \leqslant b'$ for each term b' of b. Then holds

$$\dfrac{\dfrac{a' \leqslant c \quad d \frown c \leqslant b' \text{ for each term } b' \text{ of } b}{a' \frown d \leqslant b' \text{ for each term } b' \text{ of } b}\ \text{claim}}{a' \frown d \leqslant b}\ \leqslant^\frown \\ \vdots \\ a \frown d \leqslant b.$$

  - Suppose that $d \frown c \leqslant b$ is conclusion of rule $\leqslant^-$, where $b = \overline{b'}$ and the premiss is that $d \frown c \frown b' \leqslant 0$. Then holds

$$\dfrac{\dfrac{a' \leqslant c \quad d \frown c \frown b' \leqslant 0}{a' \frown d \frown b' \leqslant 0}\ \text{claim}}{a' \frown d \leqslant b}\ \leqslant^- \\ \vdots \\ a \frown d \leqslant b.$$

- Suppose that $d \frown c \leqslant b$ is conclusion of rule $\bar{\ } \leqslant$, where $d = d' \frown \overline{d''}$ and the premiss is that $d' \frown c \leqslant d''$. Then holds

$$\dfrac{\dfrac{a' \leqslant c \quad d' \frown c \leqslant d''}{a' \frown d' \leqslant d''}\;\text{claim}}{a' \frown d \leqslant b}\;\bar{\ }\leqslant$$
$$\vdots$$
$$a \frown d \leqslant b \qquad .$$

- Suppose that $a \leqslant c$ is conclusion of rule $\bar{\ } \leqslant$, where $a = a' \frown \overline{a''}$ and the premiss is $a' \leqslant a''$. Then holds

$$\dfrac{\dfrac{a' \leqslant a''}{a \leqslant b}\;\bar{\ }\leqslant}{a \frown d \leqslant b}\;\frown\leqslant .$$

Let us check that (15) holds for empty c:

$$\dfrac{\dfrac{a \leqslant [\,] \quad d \leqslant b}{d \leqslant b}}{d \frown a \leqslant b}\;\frown\leqslant .$$

From

$$\dfrac{a \leqslant c_1 \quad d \frown c_1 \leqslant b}{a \frown d \leqslant b} \quad \cdots \quad \dfrac{a \leqslant c_n \quad d \frown c_n \leqslant b}{a \frown d \leqslant b}$$

follows

$$\cdots \quad \dfrac{\dfrac{\dfrac{a \leqslant c_1 \frown \cdots \frown c_n}{a \leqslant c_2}\;\texttt{reverse} \leqslant\frown \qquad \dfrac{\dfrac{a \leqslant c_1 \frown \cdots \frown c_n}{a \leqslant c_1}\;\texttt{reverse} \leqslant\frown \quad d \frown c_1 \frown \cdots \frown c_n \leqslant b}{a \frown d \frown c_2 \frown \cdots \frown c_n \leqslant b}}{a \frown a \frown d \frown c_3 \frown \cdots \frown c_n \leqslant b}}{a \frown d \frown c_3 \frown \cdots \frown c_n \leqslant b}\;\leqslant\texttt{-contraction}$$
$$\vdots$$
$$a \frown d \leqslant b$$

Now let (15) hold for c. We prove the validity for $\bar{c}$ by theorem induction for all theorems $d \frown \bar{c} \leqslant b$. For prime theorems, there is nothing to prove as $d \frown \bar{c} \leqslant b$ is not a prime theorem. Let now $d \frown \bar{c} \leqslant b$ be conclusion of a rule, whose premisses are theorems, and let the claim hold for the premisses. There are then only the following possibilities.

- $d \frown \bar{c} \leqslant b$ is conclusion of rule $\frown \leqslant$, where $d = d' \frown d''$ and the premiss is $d' \frown \bar{c} \leqslant b$: then holds

$$\dfrac{\dfrac{a \leqslant \bar{c} \quad d' \frown \bar{c} \leqslant b}{a \frown d' \leqslant b}\;\text{claim}}{a \frown d \leqslant b}\;\frown\leqslant \qquad .$$

- $d \frown \bar{c} \leqslant b$ is conclusion of rule $\frown \leqslant$, where $d = d' \frown d''$ and the premiss is $d' \leqslant b$. Then holds

$$\dfrac{d' \leqslant b}{d \frown a \leqslant b}\;\frown\leqslant .$$

- $d \frown \bar{c} \leqslant b$ is conclusion of rule $\leqslant \frown$, where the premiss is that $d \frown \bar{c} \leqslant b'$ for each term b' of b. Then holds

$$\dfrac{\dfrac{a \leqslant \bar{c} \quad d \frown \bar{c} \leqslant b' \text{ for each term } b' \text{ of } b}{a \frown d \leqslant b' \text{ for each term } b' \text{ of } b}\;\text{claim}}{a \frown d \leqslant b}\;\leqslant\frown \qquad .$$

- $d \frown \bar{c} \leqslant b$ is conclusion of rule $\leqslant^-$, where $b = \overline{b'}$ and the premiss is that $d \frown \bar{c} \frown b' \leqslant 0$. Then holds

$$\dfrac{\dfrac{a \leqslant \bar{c} \quad d \frown \bar{c} \frown b' \leqslant 0}{a \frown d \frown b' \leqslant 0}\,\text{claim}}{a \frown d \leqslant b}\,\leqslant^- .$$

- $d \frown \bar{c} \leqslant b$ is conclusion of rule $^- \leqslant$, where $d = d' \frown \overline{d''}$ and the premiss is that $d' \frown \bar{c} \leqslant d''$. Then holds

$$\dfrac{\dfrac{a \leqslant \bar{c} \quad d' \frown \bar{c} \leqslant d''}{a \frown d' \leqslant d''}\,\text{claim}}{a \frown d \leqslant b}\,^- \leqslant .$$

- $d \frown \bar{c} \leqslant b$ is conclusion of rule $^- \leqslant$, where the premiss is that $d \leqslant c$. Then holds

$$\dfrac{\dfrac{\dfrac{a \leqslant \bar{c}}{a \frown c \leqslant 0}\,\texttt{reverse } \leqslant^-}{a \frown c \leqslant b}\,\leqslant\texttt{-p.o.e.} \qquad d \leqslant c}{d \frown a \leqslant b}\,\text{formula induction claim} .$$

Thereby (15) is proved and especially (8).

2 The free Lorenzen algebra in two small steps

Let M be a partially ordered set. Let us define the following inductively the negations and conjunctions of elements of M. Let H_0 be the set M. Let H_{i+1} be the union of H_i with the set of formal negations $\bar{\varphi}$ and formal conjunctions $\varphi \wedge \psi$ of elements of H_i. The following formula induction holds: if a claim holds

- for each *prime* formula, that is, for each element of H_0,
- for the negation of a formula if for the formula,
- for the conjunction of two formulas if for the formulas,

then it holds for each formula.

2.1 A calculus of sequents on the left side

We start by defining a calculus in which all theorems have the form $\Gamma \vdash$, where Γ is a finite sequent, that is, a multiset of formulas $\varphi_1, \ldots, \varphi_n$. Their meaning is that Γ is contradictory and their proof explains why they are. This is inspired by Beth's method of tableaux.

Here are the axioms and the three rules of our calculus. All three rules will turn out to be reversible.

$$\dfrac{}{p, \bar{q}, \Delta \vdash}\,\texttt{left } \vdash\texttt{-axiom} \quad \text{if } p, q \text{ are prime and } p \leqslant q \text{ in } M$$

$$\dfrac{\varphi, \psi, \Gamma \vdash}{\varphi \wedge \psi, \Gamma \vdash}\,\wedge \vdash$$

$$\dfrac{\varphi, \Gamma \vdash}{\bar{\bar{\varphi}}, \Gamma \vdash}\,^= \vdash$$

$$\dfrac{\bar{\varphi}, \Gamma \vdash \quad \bar{\psi}, \Gamma \vdash}{\overline{\varphi \wedge \psi}, \Gamma \vdash}\,\overline{\wedge} \vdash$$

The following theorem induction holds: if a claim holds

- for each *prime* theorem, that is, for each `left` $\vdash$`-axiom`,
- for the conclusion of each rule whose premisses are theorems if for these premisses,

then it holds for each theorem.

Note that each of the three rules produces a formula in its conclusion that cannot be produced by any of the two other rules, respectively a conjunction, a double negation and the negation of a conjunction.

In particular, this implies that each rule is reversible, as shows a theorem induction:

- the conclusion of each rule is not a prime theorem;
- if it is a theorem, then either it follows by this very rule from the very premisses we are wishing for, or it follows by another rule
  - that does not touch respectively the conjunction, the double negation or the negation of a conjunction,
  - from premisses to which we may apply the induction hypothesis,

  and then we get the premisses we are looking for by applying the other rule.

Let us now derive three rules:

$$\frac{\Gamma \vdash}{\Gamma, \Delta \vdash}\ \texttt{left thinning}$$

$$\frac{\Gamma, \Delta, \Delta \vdash}{\Gamma, \Delta \vdash}\ \texttt{left }\vdash\texttt{-contraction}$$

$$\frac{\bar{\varphi}, \Gamma \vdash \quad \varphi, \Delta \vdash}{\Gamma, \Delta \vdash}\ \texttt{left }\vdash\texttt{-cut}$$

Left thinning. Proof by theorem induction. If Γ is a prime theorem, Γ has the form $p, \bar{q}, \Gamma'$ with p, q prime and $p \leqslant q$. Then holds

$$\frac{}{p, \bar{q}, \Gamma', \Delta \vdash}\ \texttt{left }\vdash\texttt{-axiom}$$

If $\Gamma \vdash$ is conclusion of a rule, let the claim hold for the premisses. There are three cases, one for each rule, and in each case, it suffices to apply the claim to the premisses and then the rule.

Left contraction. It suffices to prove

$$\frac{\varphi, \varphi, \Gamma \vdash}{\varphi, \Gamma \vdash}\ \texttt{left }\vdash\texttt{-contraction}$$

Proof by formula induction. Suppose φ is a prime formula p and let us do a proof by theorem induction. If $p, p, \Gamma \vdash$ is a prime theorem, there are two cases.

- Γ has the form $p', \bar{q}, \Gamma'$ with p', q prime formulas and $p' \leqslant q$. Then holds

$$\frac{}{p', \bar{q}, \Gamma', p \vdash}\ \texttt{left }\vdash\texttt{-axiom}$$

- Γ has the form $\bar{q}, \Gamma'$ with q a prime formula and $p \leqslant q$. Then holds

$$\frac{}{p, \bar{q}, \Gamma' \vdash}\ \texttt{left }\vdash\texttt{-axiom}$$

If $p, p, \Gamma \vdash$ is conclusion of a rule, let the claim hold for the premisses. There are three cases as before and one concludes similarly.

Now suppose that contraction holds for formulas in H_i. Let us prove by theorem induction that it holds for negations of formulas φ in H_i. If $\bar{\varphi}, \bar{\varphi}, \Gamma \vdash$ is a prime theorem, then Γ has the form $p, \bar{q}, \Gamma'$ with p, q prime formulas and $p \leqslant q$, so that holds

$$\frac{}{p, \bar{q}, \Gamma', \bar{\varphi} \vdash}\ \texttt{left }\vdash\texttt{-axiom}$$

Suppose that $\bar{\varphi}, \bar{\varphi}, \Gamma \vdash$ is conclusion of a rule and let the claim hold for the premisses. There are three cases.

- $\bar{\varphi}, \bar{\varphi}, \Gamma \vdash$ is a theorem that follows from an application of a rule that does not touch $\bar{\varphi}, \bar{\varphi}$. It suffices to apply the claim to the premisses, and then the rule.
- φ has the form $\bar{\psi}$. Then holds

$$\dfrac{\dfrac{\dfrac{\dfrac{\bar{\bar{\psi}}, \bar{\bar{\psi}}, \Gamma \vdash}{\psi, \bar{\bar{\psi}}, \Gamma \vdash}\ \texttt{reverse}\ \bar{\bar{\ }}\vdash}{\psi, \psi, \Gamma \vdash}\ \texttt{reverse}\ \bar{\bar{\ }}\vdash}{\psi, \Gamma \vdash}\ \text{formula induction claim}}{\bar{\bar{\psi}}, \Gamma \vdash}\ \bar{\bar{\ }}\vdash$$

- φ has the form $\psi \wedge \chi$. Then holds

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{\psi \wedge \chi}, \overline{\psi \wedge \chi}, \Gamma \vdash}{\bar{\psi}, \overline{\psi \wedge \chi}, \Gamma \vdash}\ \texttt{reverse}\ \overline{\wedge}\vdash}{\bar{\psi}, \bar{\psi}, \Gamma \vdash}\ \texttt{reverse}\ \overline{\wedge}\vdash}{\bar{\psi}, \Gamma \vdash}\ \text{formula induction claim} \qquad \dfrac{\dfrac{\dfrac{\overline{\psi \wedge \chi}, \overline{\psi \wedge \chi}, \Gamma \vdash}{\bar{\chi}, \overline{\psi \wedge \chi}, \Gamma \vdash}\ \texttt{reverse}\ \overline{\wedge}\vdash}{\bar{\chi}, \bar{\chi}, \Gamma \vdash}\ \texttt{reverse}\ \overline{\wedge}\vdash}{\bar{\chi}, \Gamma \vdash}\ \text{formula induction claim}}{\overline{\psi \wedge \chi}, \Gamma \vdash}\ \overline{\wedge}\vdash$$

Let us prove that contraction holds for conjunctions of formulas φ, ψ in H_i.

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi \wedge \psi, \varphi \wedge \psi, \Gamma \vdash}{\varphi, \psi, \varphi \wedge \psi, \Gamma \vdash}\ \texttt{reverse}\ \wedge\vdash}{\varphi, \psi, \varphi, \psi, \Gamma \vdash}\ \texttt{reverse}\ \wedge\vdash}{\varphi, \psi, \psi, \Gamma \vdash}\ \text{formula induction claim}}{\varphi, \psi, \Gamma \vdash}\ \text{formula induction claim}$$

Left cut. Proof by formula induction. Suppose φ is a prime formula p and let us do a proof by double theorem induction. If $\bar{\varphi}, \Gamma \vdash$ and $\varphi, \Delta \vdash$ are prime theorems, there are three cases.

- Γ has the form $p', \overline{q'}, \Gamma'$ or Δ has the form $p', \overline{q'}, \Delta'$, where p', q' are prime formulas such that $p' \leqslant q'$. Then holds

$$\dfrac{}{p', \overline{q'}, \Gamma', \Delta \vdash}\ \texttt{left}\ \vdash\texttt{-axiom} \qquad \text{or} \qquad \dfrac{}{p', \overline{q'}, \Delta', \Gamma \vdash}\ \texttt{left}\ \vdash\texttt{-axiom}$$

- Γ has the form p', Γ' and Δ has the form $\overline{q'}, \Delta'$, where p', q' are prime formulas such that $p' \leqslant p \leqslant q'$. Then holds

$$\dfrac{}{p', \overline{q'}, \Gamma', \Delta' \vdash}\ \texttt{left}\ \vdash\texttt{-axiom}$$

Now suppose that cut holds for formulas in H_i. Then it holds for their negation:

$$\dfrac{\dfrac{\bar{\bar{\varphi}}, \Gamma \vdash}{\varphi, \Gamma \vdash}\ \texttt{reverse}\ \bar{\bar{\ }}\vdash \qquad \bar{\varphi}, \Delta \vdash}{\Gamma, \Delta}\ \texttt{left}\ \vdash\texttt{-cut}$$

and for their concatenation:

$$\dfrac{\dfrac{\dfrac{\overline{\varphi \wedge \psi}, \Gamma \vdash}{\bar{\psi}, \Gamma \vdash}\ \texttt{reverse}\ \overline{\wedge}\vdash \qquad \dfrac{\dfrac{\overline{\varphi \wedge \psi}, \Gamma \vdash}{\bar{\varphi}, \Gamma \vdash}\ \texttt{reverse}\ \overline{\wedge}\vdash \qquad \dfrac{\varphi \wedge \psi, \Delta \vdash}{\varphi, \psi, \Delta \vdash}\ \texttt{reverse}\ \wedge\vdash}{\Gamma, \psi, \Delta \vdash}\ \texttt{left}\ \vdash\texttt{-cut}}{\Gamma, \Gamma, \Delta \vdash}\ \texttt{left}\ \vdash\texttt{-cut}}{\Gamma, \Delta \vdash}\ \vdash\texttt{-contraction}$$

2.2 Extension of the calculus to the right side

We shall now extend this calculus by adding axioms, two rules that inject the previous calculus (remind that p.o.e. stands for principle of explosion) and a rule for conjunctions on the right. The last two rules will turn out to be reversible:

$$\frac{}{p, \Gamma \vdash q}\ \texttt{right}\ \vdash\texttt{-axiom} \quad \text{if } p, q \text{ are prime and } p \leqslant q \text{ in } M$$

$$\frac{\Gamma \vdash}{\Gamma \vdash p}\ \vdash\texttt{-p.o.e.} \quad \text{if } p \text{ is prime}$$

$$\frac{\Gamma, \varphi \vdash}{\Gamma \vdash \bar{\varphi}}\ \vdash {}^{-}$$

$$\frac{\Gamma \vdash \varphi \quad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi}\ \vdash \wedge$$

The extended calculus comes with a corresponding theorem induction: if a claim holds

- for each *prime* theorem, that is, for each application of the `left` $\vdash$`-axiom` or of the `right` $\vdash$`-axiom` rule,
- for the conclusion of each rule whose premisses are theorems if for these premisses,

then it holds for each theorem.

Note that the two last rules produce a formula in their conclusion that cannot be produced by any of the other rules, respectively a negation and a conjunction on the right.

In particular, this implies that each of the rules $\vdash {}^{-}$ and $\vdash \wedge$ is reversible, as shows a theorem induction in the same way as for the left side calculus.

Furthermore, the following fact will be important in the proofs below: if q is a prime formula and $\Gamma \vdash q$ is a theorem, then Γ has the form p, Γ' with $p \vdash q$ a theorem, or $\Gamma \vdash$ is a theorem.

Let us now derive two rules.

$$\frac{\Gamma \vdash \varphi}{\Gamma, \bar{\varphi} \vdash}\ {}^{-}\vdash$$

$$\frac{\Gamma \vdash \varphi \quad \varphi, \Delta \vdash \psi}{\Gamma, \Delta \vdash \psi}\ \vdash\texttt{-cut}$$

Left negation. Proof by formula induction. If φ is a prime formula q, let us do a theorem induction. If $\Gamma \vdash q$ is a prime theorem, then Γ has the form p, Γ' with p a prime formula and $p \leqslant q$. Then holds

$$\frac{}{p, \bar{q}, \Gamma' \vdash}\ \texttt{left}\ \vdash\texttt{-axiom}$$

If $\Gamma \vdash q$ is conclusion of a rule, it is conclusion of rule $\vdash$`-p.o.e.` with premiss $\Gamma \vdash$; then holds $\Gamma, \bar{q} \vdash$ by `left thinning`.

If φ is a negation $\bar{\psi}$, then

$$\cfrac{\cfrac{\Gamma \vdash \bar{\psi}}{\Gamma, \psi \vdash}\ \texttt{reverse}\ \vdash {}^{-}}{\Gamma, \bar{\bar{\psi}} \vdash}\ {}^{=}\vdash$$

Now suppose that φ is a conjunction $\psi \wedge \chi$ and suppose that ${}^{-}\vdash$ holds for each of the formulas ψ and χ. Then holds

$$\cfrac{\cfrac{\cfrac{\Gamma \vdash \psi \wedge \chi}{\Gamma \vdash \psi}\ \texttt{reverse}\ \vdash \wedge}{\Gamma, \bar{\psi} \vdash}\ \text{claim} \qquad \cfrac{\cfrac{\Gamma \vdash \psi \wedge \chi}{\Gamma \vdash \chi}\ \texttt{reverse}\ \vdash \wedge}{\Gamma, \bar{\chi} \vdash}\ \text{claim}}{\Gamma, \overline{\psi \wedge \chi} \vdash}\ \overline{\wedge}\vdash$$

Cut. Proof by formula induction on ψ. If ψ is a prime formula q, let us do a theorem induction on $\varphi, \Delta \vdash q$. If $\varphi, \Delta \vdash q$ is a prime theorem, then there are two cases

- Δ has the form p, Δ' with p a prime formula such that $p \leqslant q$. Then holds

$$\dfrac{}{p, \Delta', \Gamma \vdash q}\ \texttt{right } \vdash\texttt{-axiom}$$

- φ is a prime formula p and $p \leqslant q$ holds. Let us do a theorem induction on $\Gamma \vdash p$. If $\Gamma \vdash p$ is a prime theorem, then Γ has the form p', Γ' with p' a prime formula such that $p' \leqslant p$, so that $p' \leqslant q$ and holds

$$\dfrac{}{p', \Gamma', \Delta \vdash q}\ \texttt{right } \vdash\texttt{-axiom}$$

  If $\Gamma \vdash p$ is conclusion of a rule, it is conclusion of rule $\vdash$-`p.o.e.` with premiss $\Gamma \vdash$; then holds

$$\dfrac{\dfrac{\Gamma \vdash}{\Gamma, \Delta \vdash}\ \texttt{left thinning}}{\Gamma, \Delta \vdash q}\ \vdash\texttt{-p.o.e.}$$

If $\varphi, \Delta \vdash q$ is conclusion of a rule, it is conclusion of rule $\vdash$-`p.o.e.` with premiss $\varphi, \Delta \vdash$; then holds

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash \varphi}{\Gamma, \bar{\varphi} \vdash}\ {}^{-}\vdash \qquad \varphi, \Delta \vdash}{\Gamma, \Delta \vdash}\ \texttt{left } \vdash\texttt{-cut}}{\Gamma, \Delta \vdash q}\ \vdash\texttt{-p.o.e.}$$

If ψ is a negation $\bar{\chi}$, then holds

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash \varphi}{\Gamma, \bar{\varphi} \vdash}\ {}^{-}\vdash \qquad \dfrac{\varphi, \Delta \vdash \bar{\chi}}{\varphi, \Delta, \chi \vdash}\ \texttt{reverse } \vdash{}^{-}}{\Gamma, \Delta, \chi \vdash}\ \texttt{left } \vdash\texttt{-cut}}{\Gamma, \Delta \vdash \bar{\chi}}\ \vdash{}^{-}$$

Now suppose that cut holds for formulas ψ, χ. Then holds

$$\dfrac{\dfrac{\Gamma \vdash \varphi \qquad \dfrac{\varphi, \Delta \vdash \psi \wedge \chi}{\varphi, \Delta \vdash \psi}\ \texttt{reverse } \vdash \wedge}{\Gamma, \Delta \vdash \psi}\ \vdash\texttt{-cut} \qquad \dfrac{\Gamma \vdash \varphi \qquad \dfrac{\varphi, \Delta \vdash \psi \wedge \chi}{\varphi, \Delta \vdash \chi}\ \texttt{reverse } \vdash \wedge}{\Gamma, \Delta \vdash \chi}\ \vdash\texttt{-cut}}{\Gamma, \Delta \vdash \psi \wedge \chi}\ \vdash \wedge$$

2.3 Further derived rules

The two following rules also hold.

$$\dfrac{\Gamma \vdash \varphi}{\Gamma, \Delta \vdash \varphi}\ \vdash\texttt{-thinning}$$

$$\dfrac{\Gamma, \Delta, \Delta \vdash \psi}{\Gamma, \Delta \vdash \psi}\ \vdash\texttt{-contraction}$$

Thinning. This is proved exactly as before.

Contraction. It suffices to prove

$$\frac{\varphi, \varphi, \Gamma \vdash \psi}{\varphi, \Gamma \vdash \psi}\ \texttt{$\vdash$-contraction}$$

Proof by formula induction on ψ. If ψ is a prime formula q, let us do a theorem induction on $\varphi, \varphi, \Gamma \vdash q$. If $\varphi, \varphi, \Gamma \vdash q$ is a prime theorem, there are two cases.

- Γ has the form p, Γ' with p a prime formula and $p \leqslant q$. Then holds

$$\frac{}{p, \Gamma', \varphi \vdash q}\ \texttt{right $\vdash$-axiom}$$

- φ is a prime formula p and $p \leqslant q$. Then holds

$$\frac{}{p, \Gamma \vdash q}\ \texttt{right $\vdash$-axiom}$$

If $\varphi, \varphi, \Gamma \vdash q$ is conclusion of a rule, it is conclusion of rule $\vdash$`-p.o.e.` with premiss $\varphi, \varphi, \Gamma \vdash$; then holds

$$\frac{\dfrac{\varphi, \varphi, \Gamma \vdash}{\varphi, \Gamma \vdash}\ \texttt{left $\vdash$-contraction}}{\varphi, \Gamma \vdash q}\ \texttt{$\vdash$-p.o.e.}$$

If ψ is a negation $\bar{\chi}$, then holds

$$\frac{\dfrac{\dfrac{\varphi, \varphi, \Gamma \vdash \bar{\chi}}{\varphi, \varphi, \Gamma, \chi \vdash}\ \texttt{reverse $\vdash^-$}}{\varphi, \Gamma, \chi \vdash}\ \texttt{left $\vdash$-contraction}}{\varphi, \Gamma \vdash \bar{\chi}}\ \vdash^-$$

If contraction holds for two formulas ψ, χ, then holds

$$\frac{\dfrac{\dfrac{\varphi, \varphi, \Gamma \vdash \psi \wedge \chi}{\varphi, \varphi, \Gamma \vdash \psi}\ \texttt{reverse $\vdash \wedge$}}{\varphi, \Gamma \vdash \psi}\ \text{claim} \qquad \dfrac{\dfrac{\varphi, \varphi, \Gamma \vdash \psi \wedge \chi}{\varphi, \varphi, \Gamma \vdash \chi}\ \texttt{reverse $\vdash \wedge$}}{\varphi, \Gamma \vdash \chi}\ \text{claim}}{\varphi, \Gamma \vdash \psi \wedge \chi}\ \vdash \wedge$$

2.4 Definition of the free Lorenzen algebra

Let H_∞ be the union of the H_i and let H be the union of H_∞ with two further elements: 0 and 1. We define an order on H by the following truth table:

$\leqslant$	0	ψ	1
0	true	true	true
φ	$\varphi \vdash$	$\varphi \vdash \psi$	true
1	false	false	true