The maximum modulus of a trigonometric trinomial

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Abstract

Let Λ be a set of three integers and let C_Λ be the space of 2π-periodic functions with spectrum in Λ endowed with the maximum modulus norm. We isolate the maximum modulus points x of trigonometric trinomials $T \in \mathrm{C}_\Lambda$ and prove that x is unique unless $|T|$ has an axis of symmetry. This enables us to compute the exposed and the extreme points of the unit ball of C_Λ, to describe how the maximum modulus of T varies with respect to the arguments of its Fourier coefficients and to compute the norm of unimodular relative Fourier multipliers on C_Λ. We obtain in particular the Sidon constant of Λ.

1 Introduction

Let λ_1, λ_2 and λ_3 be three pairwise distinct integers. Let r_1, r_2 and r_3 be three positive real numbers. Given three real numbers t_1, t_2 and t_3, let us consider the *trigonometric trinomial*

$$T(x) = r_1 \mathrm{e}^{\mathrm{i}(t_1+\lambda_1 x)} + r_2 \mathrm{e}^{\mathrm{i}(t_2+\lambda_2 x)} + r_3 \mathrm{e}^{\mathrm{i}(t_3+\lambda_3 x)} \tag{1}$$

for $x \in \mathbb{R}$. The λ's are the *frequencies* of the trigonometric trinomial T, the r's are the *moduli* or *intensities* and the t's the *arguments* or *phases* of its *Fourier coefficients* $r_1\mathrm{e}^{\mathrm{i}t_1}$, $r_2\mathrm{e}^{\mathrm{i}t_2}$ and $r_3\mathrm{e}^{\mathrm{i}t_3}$.

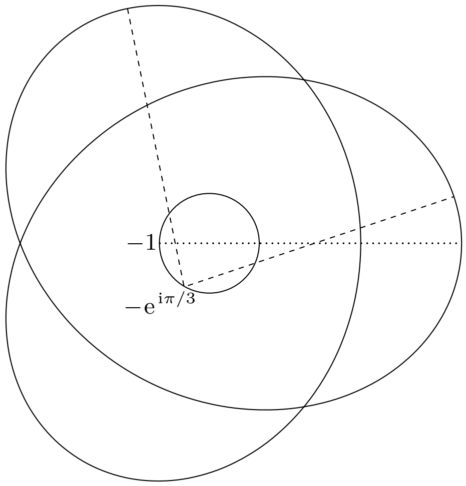

Figure 1: The unit circle, the hypotrochoid H with equation $z = 4\mathrm{e}^{-\mathrm{i}2x} + \mathrm{e}^{\mathrm{i}x}$, the segment from -1 to the unique point on H at maximum distance and the segments from $-\mathrm{e}^{\mathrm{i}\pi/3}$ to the two points on H at maximum distance.

The maximum modulus of a trigonometric trinomial has an interpretation in plane geometry. Without loss of generality, we may assume that λ_2 is between λ_1 and λ_3. Let H be the curve with complex equation

$$z = r_1 \mathrm{e}^{\mathrm{i}(t_1-(\lambda_2-\lambda_1)x)} + r_3 \mathrm{e}^{\mathrm{i}(t_3+(\lambda_3-\lambda_2)x)} \quad (-\pi < x \leqslant \pi). \tag{2}$$

H is a *hypotrochoid*: it is drawn by a point at distance r_3 to the centre of a circle with radius $r_1|\lambda_2-\lambda_1|/|\lambda_3-\lambda_2|$ that rolls inside another circle with radius $r_1|\lambda_3-\lambda_1|/|\lambda_3-\lambda_2|$. The maximum modulus of (1) is the maximum distance of points $z \in H$ to a given point $-r_2\mathrm{e}^{\mathrm{i}t_2}$ of the complex plane. If $\lambda_3-\lambda_2=\lambda_2-\lambda_1$, then H is the ellipse with centre 0, half major axis r_1+r_3 and half minor axis $|r_1-r_3|$. Note that an epitrochoid (Ptolemy's epicycle) amounts also to a hypotrochoid. Figure 1 illustrates the particular case $T(x)=4\,\mathrm{e}^{-\mathrm{i}2x}+\mathrm{e}^{\mathrm{i}t}+\mathrm{e}^{\mathrm{i}x}$.

We deduce an interval on which T attains its maximum modulus independently of the moduli of its Fourier coefficients (see Theorem 7.1 (a) for a detailed answer.) We prove in particular the following result.

Theorem 1.1. *Let $d=\gcd(\lambda_2-\lambda_1,\lambda_3-\lambda_2)$ and let τ be the distance of*

$$\frac{\lambda_2-\lambda_3}{d}t_1+\frac{\lambda_3-\lambda_1}{d}t_2+\frac{\lambda_1-\lambda_2}{d}t_3 \tag{3}$$

to $2\pi\mathbb{Z}$. The trigonometric trinomial T attains its maximum modulus at a unique point modulo $2\pi/d$, with multiplicity 2, unless $\tau=\pi$.

Theorem 1.1 shows that if there are two points of the hypotrochoid H at maximum distance to $-r_2\mathrm{e}^{\mathrm{i}t_2}$, it is so only because $-r_2\mathrm{e}^{\mathrm{i}t_2}$ lies on an axis of symmetry of H.

We obtain a precise description of those trigonometric trinomials that attain their maximum modulus twice modulo $2\pi/d$: see Theorem 7.1 (c). Their rôle becomes clear by the following result in convex geometry: they yield the exposed points of the unit ball of the ambient normed space. Let us first put up the proper functional analytic framework. Let $\varLambda=\{\lambda_1,\lambda_2,\lambda_3\}$ be the *spectrum* of the trigonometric trinomial T and write $\mathrm{e}_\lambda\colon x\mapsto \mathrm{e}^{\mathrm{i}\lambda x}$. Let $\mathrm{C}_\varLambda$ be the space of functions spanned by $\{\mathrm{e}_\lambda:\lambda\in\varLambda\}$, endowed with the maximum modulus norm. Recall that a point P of a convex set K is *exposed* if there is a hyperplane that meets K only in P; P is *extreme* if it is not the midpoint of any two other points of K.

Theorem 1.2. *Let K be the unit ball of the space $\mathrm{C}_\varLambda$ and let $P\in K$.*

- (a) *The point P is an exposed point of K if and only if P is either a trigonometric monomial $\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}_\lambda$ with $\alpha\in\mathbb{R}$ and $\lambda\in\varLambda$ or a trigonometric trinomial that attains its maximum modulus, 1, at two points modulo $2\pi/d$. Every linear functional on $\mathrm{C}_\varLambda$ attains its norm on an exposed point of K.*
- (b) *The point P is an extreme point of K if and only if P is either a trigonometric monomial $\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}_\lambda$ with $\alpha\in\mathbb{R}$ and $\lambda\in\varLambda$ or a trigonometric trinomial such that $1-|P|^2$ has four zeroes modulo $2\pi/d$, counted with multiplicities.*

We describe the dependence of the maximum modulus of the trigonometric trinomial T on the arguments. The general issue has been studied for a long time; [14, 23] are two early references. In particular, the following problem has been addressed: see [6, page 2 and Supplement].

Extremal problem 1.3 (Complex Mandelʹshtam problem)**.** To find the minimum of the maximum modulus of a trigonometric polynomial with given Fourier coefficient moduli.

It appeared originally in electrical circuit theory: "L. I. Mandelʹshtam communicated to me a problem on the phase choice of electric currents with different frequencies such that the capacity of the resulting current to blow [the circuit] is minimal", tells N. G. Chebotarëv in [5, p. 396], where he discusses applications of a formula given in Section 9 that we would like to advertise.

Our main theorem solves an elementary case of the complex Mandelʹshtam problem.

Theorem 1.4. *The maximum modulus of T as defined in (1) is a strictly decreasing function of τ as defined in Theorem 1.1. In particular,*

$$\min_{t_1,t_2,t_3}\max_x\left|r_1\mathrm{e}^{\mathrm{i}(t_1+\lambda_1x)}+r_2\mathrm{e}^{\mathrm{i}(t_2+\lambda_2x)}+r_3\mathrm{e}^{\mathrm{i}(t_3+\lambda_3x)}\right|=\max_x\left|\epsilon_1r_1\mathrm{e}^{\mathrm{i}\lambda_1x}+\epsilon_2r_2\mathrm{e}^{\mathrm{i}\lambda_2x}+\epsilon_3r_3\mathrm{e}^{\mathrm{i}\lambda_3x}\right|$$

if ϵ_1, ϵ_2 and ϵ_3 are real signs $+1$ or -1 such that $\epsilon_i\epsilon_j=-1$, where i,j,k is a permutation of $1,2,3$ such that the power of 2 in $\lambda_i-\lambda_j$ is greater than the power of 2 in $\lambda_i-\lambda_k$ and in $\lambda_k-\lambda_j$.

This shows that the maximum modulus is minimal when the phases are chosen in opposition, independently of the intensities r_1, r_2 and r_3.

The decrease of the maximum modulus of (1) may be bounded as shown in the next result.

Theorem 1.5. *Let d and τ be defined as in Theorem 1.1. Suppose that λ_2 is between λ_1 and λ_3. The quotient of the maximum modulus of T by $\left|r_1 + r_2 \mathrm{e}^{\mathrm{i}\tau d/|\lambda_3-\lambda_1|} + r_3\right|$ is a strictly increasing function of τ unless $r_1 : r_3 = |\lambda_3 - \lambda_2| : |\lambda_2 - \lambda_1|$, in which case it is constantly equal to 1.*

When $r_1 : r_3 = |\lambda_3 - \lambda_2| : |\lambda_2 - \lambda_1|$, the hypotrochoid H with equation (2) is a *hypocycloid* with $|\lambda_3 - \lambda_1|/d$ cusps: the rolling point is *on* the rolling circle. Note that an epicycloid amounts also to an hypocycloid. Figure 2 illustrates the particular case $T(x) = (1/3)\mathrm{e}^{-\mathrm{i}2x} + \mathrm{e}^{\mathrm{i}t} + (2/3)\mathrm{e}^{\mathrm{i}x}$.

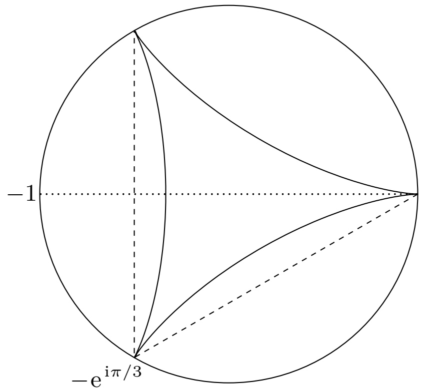

Figure 2: The unit circle, the deltoid H with equation $z = (1/3)\mathrm{e}^{-\mathrm{i}2x} + (2/3)\mathrm{e}^{\mathrm{i}x}$, the segment from -1 to the unique point on H at maximum distance and the segments from $-\mathrm{e}^{\mathrm{i}\pi/3}$ to the two points on H at maximum distance.

We may deduce from Theorem 1.5 a less precise but handier inequality.

Theorem 1.6. *Let d and τ be defined as in Theorem 1.1. Let*

$$D = \frac{\max(|\lambda_2 - \lambda_1|, |\lambda_3 - \lambda_2|, |\lambda_3 - \lambda_1|)}{\gcd(\lambda_2 - \lambda_1, \lambda_3 - \lambda_1)} \tag{4}$$

be the quotient of the diameter of Λ by d. Let t_1', t_2' and t_3' be another three real numbers and define correspondingly τ'. If $\tau > \tau'$, then

$$\max_x \left|r_1 \mathrm{e}^{\mathrm{i}(t_1+\lambda_1 x)} + r_2 \mathrm{e}^{\mathrm{i}(t_2+\lambda_2 x)} + r_3 \mathrm{e}^{\mathrm{i}(t_3+\lambda_3 x)}\right| \geqslant \frac{\cos(\tau/2D)}{\cos(\tau'/2D)} \max_x \left|r_1 \mathrm{e}^{\mathrm{i}(t_1'+\lambda_1 x)} + r_2 \mathrm{e}^{\mathrm{i}(t_2'+\lambda_2 x)} + r_3 \mathrm{e}^{\mathrm{i}(t_3'+\lambda_3 x)}\right|$$

with equality if and only if $r_1 : r_2 : r_3 = |\lambda_3 - \lambda_2| : |\lambda_3 - \lambda_1| : |\lambda_2 - \lambda_1|$.

Figure 3 illustrates the inequalities obtained in Theorems 1.5 and 1.6 for the particular case $T(x) = 4\mathrm{e}^{-\mathrm{i}2x} + \mathrm{e}^{\mathrm{i}t} + \mathrm{e}^{\mathrm{i}x}$, as in Figure 1.

If we choose $\tau' = 0$ in Theorem 1.6, we get the solution to an elementary case of the following extremal problem.

Extremal problem 1.7. To find the minimum of the maximum modulus of a trigonometric polynomial with given spectrum, Fourier coefficient arguments and moduli sum.

Theorem 1.8. *Let τ be defined as in Theorem 1.1 and D be given by (4). Then*

$$\frac{\max_x \left|r_1 \mathrm{e}^{\mathrm{i}(t_1+\lambda_1 x)} + r_2 \mathrm{e}^{\mathrm{i}(t_2+\lambda_2 x)} + r_3 \mathrm{e}^{\mathrm{i}(t_3+\lambda_3 x)}\right|}{r_1 + r_2 + r_3} \geqslant \cos(\tau/2D)$$

with equality if and only if $\tau = 0$ or $r_1 : r_2 : r_3 = |\lambda_3 - \lambda_2| : |\lambda_3 - \lambda_1| : |\lambda_2 - \lambda_1|$.

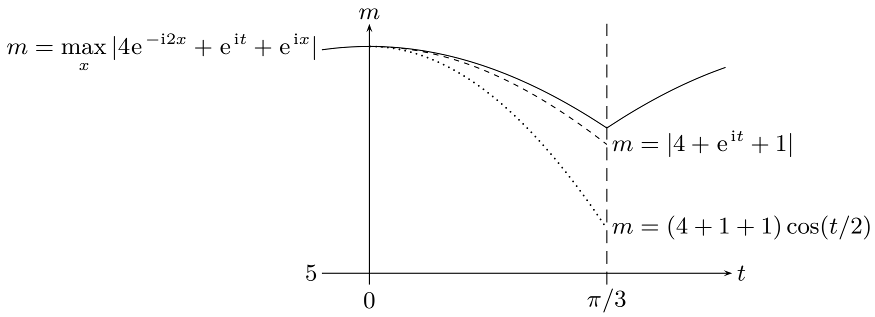

Figure 3: Let H be the hypotrochoid with equation $z = 4\mathrm{e}^{-\mathrm{i}2x} + \mathrm{e}^{\mathrm{i}x}$. This plot shows the maximum distance m of points $z \in H$ to the point $-\mathrm{e}^{\mathrm{i}t}$ and the two estimates of this maximum distance provided by Theorems 1.5 and 1.6 for $t \in [0, \pi/3]$.

The dependence of the maximum modulus of (1) on the arguments may also be expressed as properties of relative multipliers. Given three real numbers t_1, t_2 and t_3, the linear operator on C_Λ defined by $\mathrm{e}_{\lambda_j} \mapsto \mathrm{e}^{\mathrm{i}t_j}\mathrm{e}_{\lambda_j}$ is a *unimodular relative Fourier multiplier*: it multiplies each Fourier coefficient of elements of C_Λ by a fixed unimodular number; let us denote it by (t_1, t_2, t_3). Consult [9] for general background on relative multipliers.

Theorem 1.9. *The unimodular relative Fourier multiplier* (t_1, t_2, t_3) *has norm*

$$\cos\bigl((\pi-\tau)/2D\bigr) \big/ \cos(\pi/2D),$$

where τ *is defined as in Theorem 1.1 and* D *is given by* (4)*, and attains its norm exactly at functions of the form*

$$r_1\mathrm{e}^{\mathrm{i}(u_1+\lambda_1 x)} + r_2\mathrm{e}^{\mathrm{i}(u_2+\lambda_2 x)} + r_3\mathrm{e}^{\mathrm{i}(u_3+\lambda_3 x)}$$

with $r_1 : r_2 : r_3 = |\lambda_3 - \lambda_2| : |\lambda_3 - \lambda_1| : |\lambda_2 - \lambda_1|$ *and*

$$\frac{\lambda_2-\lambda_3}{d}u_1 + \frac{\lambda_3-\lambda_1}{d}u_2 + \frac{\lambda_1-\lambda_2}{d}u_3 = \pi \mod 2\pi.$$

The maximum of the norm of unimodular relative Fourier multipliers is the *complex unconditional constant* of the canonical basis $(\mathrm{e}_{\lambda_1}, \mathrm{e}_{\lambda_2}, \mathrm{e}_{\lambda_3})$ of C_Λ. As

$$r_1 + r_2 + r_3 = \max_x \bigl|r_1\mathrm{e}^{\mathrm{i}\lambda_1 x} + r_2\mathrm{e}^{\mathrm{i}\lambda_2 x} + r_3\mathrm{e}^{\mathrm{i}\lambda_3 x}\bigr|,$$

this constant is the minimal constant C such that

$$r_1 + r_2 + r_3 \leqslant C \max_x \bigl|r_1\mathrm{e}^{\mathrm{i}(u_1+\lambda_1 x)} + r_2\mathrm{e}^{\mathrm{i}(u_2+\lambda_2 x)} + r_3\mathrm{e}^{\mathrm{i}(u_3+\lambda_3 x)}\bigr|;$$

it is therefore the *Sidon constant* of Λ. It is also the solution to the following extremal problem.

Extremal problem 1.10 (Sidon constant problem)**.** To find the minimum of the maximum modulus of a trigonometric polynomial with given spectrum and Fourier coefficient moduli sum.

Setting $\tau = \pi$ in Theorem 1.9, we obtain the following result.

Corollary 1.11. *The Sidon constant of* Λ *is* $\sec(\pi/2D)$*, where* D *is given by* (4)*. It is attained exactly at functions of the form given in Theorem 1.9.*

Finally, we would like to stress that each of the above results gives rise to open questions if the set Λ is replaced by *any* set of four integers.

Let us now give a brief description of this article. In Sections 2 and 3, we use carefully the invariance of the maximum modulus under rotation, translation and conjugation to reduce the arguments t_1, t_2 and t_3 of the Fourier coefficients of the trigonometric trinomial T to the variable τ. Section 4 shows how to further reduce this study to the trigonometric trinomial

$$r_1\mathrm{e}^{-\mathrm{i}kx} + r_2\mathrm{e}^{\mathrm{i}t} + r_3\mathrm{e}^{\mathrm{i}lx} \tag{5}$$

with k and l positive coprime integers and $t \in [0, \pi/(k+l)]$. In Section 5, we prove that (5) attains its maximum modulus for $x \in [-t/k, t/l]$. Section 6 studies the variations of the modulus of (5) for $x \in [-t/k, t/l]$: it turns out that it attains its absolute maximum only once on that interval. This yields Theorem 1.1. Section 7 restates the results of the two previous sections for a general trigonometric trinomial T. Section 8 is dedicated to the proof of Theorem 1.2. In Section 9, we compute the directional derivative of the maximum modulus of (5) with respect to the argument t and prove Theorems 1.4, 1.5, 1.6 and 1.8. In Section 10, we prove Theorem 1.9 and show how to lift unimodular relative Fourier multipliers to operators of convolution with a linear combination of two Dirac measures. Section 11 replaces our computation of the Sidon constant in a general context; it describes the initial motivation for this research.

Part of these results appeared previously, with a different proof, in [19, Chapter II.10] and in [17].

Notation. Throughout this article, λ_1, λ_2 and λ_3 are three pairwise distinct integers, Λ is the set $\{\lambda_1, \lambda_2, \lambda_3\}$ and $d = \gcd(\lambda_2 - \lambda_1, \lambda_3 - \lambda_2)$. If λ is an integer, e_λ is the function $x \mapsto \mathrm{e}^{\mathrm{i}\lambda x}$ of the real variable x. A *trigonometric polynomial* is a linear combination of functions e_λ; it is a *monomial*, *binomial* or *trinomial* if this linear combination has one, two or three nonzero coefficients, respectively. The normed space C_Λ is the three-dimensional space of complex functions spanned by e_λ with $\lambda \in \Lambda$, endowed with the maximum modulus norm. The Dirac measure δ_x is the linear functional $T \mapsto T(x)$ of evaluation at x on the space of continuous functions. Given three real numbers t_1, t_2 and t_3, the linear operator on C_Λ defined by $\mathrm{e}_{\lambda_j} \mapsto \mathrm{e}^{\mathrm{i}t_j}\mathrm{e}_{\lambda_j}$ is a *unimodular relative Fourier multiplier* denoted by (t_1, t_2, t_3).

2 Isometric relative Fourier multipliers

The rôle of Quantity (3) is explained by the following lemma.

Lemma 2.1. *Let t_1, t_2 and t_3 be real numbers. The unimodular relative Fourier multiplier $M = (t_1, t_2, t_3)$ is an isometry on C_Λ if and only if*

$$\frac{\lambda_2 - \lambda_3}{d} t_1 + \frac{\lambda_3 - \lambda_1}{d} t_2 + \frac{\lambda_1 - \lambda_2}{d} t_3 \in 2\pi\mathbb{Z}. \tag{6}$$

Then it is a unimodular multiple of a translation: there are real numbers α and v such that $Mf(x) = \mathrm{e}^{\mathrm{i}\alpha} f(x - v)$ for all $f \in \mathrm{C}_\Lambda$ and all $x \in \mathbb{R}$.

Proof. If M is a unimodular multiple of a translation by a real number v, then

$$\left| r_1 \mathrm{e}^{\mathrm{i}(t_1 + \lambda_1 v)} + r_2 \mathrm{e}^{\mathrm{i}(t_2 + \lambda_2 v)} + r_3 \mathrm{e}^{\mathrm{i}(t_3 + \lambda_3 v)} \right| = r_1 + r_2 + r_3,$$

which holds if and only if

$$t_1 + \lambda_1 v = t_2 + \lambda_2 v = t_3 + \lambda_3 v \quad \text{modulo } 2\pi. \tag{7}$$

There is a v satisfying (7) if and only if Equation (6) holds as (7) means that there exist integers a_1 and a_3 such that

$$v = \frac{t_2 - t_1 + 2\pi a_1}{\lambda_1 - \lambda_2} = \frac{t_2 - t_3 + 2\pi a_3}{\lambda_3 - \lambda_2}.$$

If t_1, t_2 and t_3 are three real numbers satisfying (6), let v be such that (7) holds. Then

$$\begin{aligned} &r_1 \mathrm{e}^{\mathrm{i}(t_1 + u_1 + \lambda_1 x)} + r_2 \mathrm{e}^{\mathrm{i}(t_2 + u_2 + \lambda_2 x)} + r_3 \mathrm{e}^{\mathrm{i}(t_3 + u_3 + \lambda_3 x)} \\ &\qquad = \mathrm{e}^{\mathrm{i}(t_2 + \lambda_2 v)} \big(r_1 \mathrm{e}^{\mathrm{i}(u_1 + \lambda_1 (x - v))} + r_2 \mathrm{e}^{\mathrm{i}(u_2 + \lambda_2 (x - v))} + r_3 \mathrm{e}^{\mathrm{i}(u_3 + \lambda_3 (x - v))} \big) \end{aligned}$$

for all real numbers u_1, u_2, u_3 and x. $\square$

3 The arguments of the Fourier coefficients of a trigonometric trinomial

We have used a translation and a rotation to reduce the three arguments of the Fourier coefficients of a trigonometric trinomial to just one variable. Use of the involution $\overline{f(-x)}$ of C_Λ allows us to restrict even further the domain of that variable.

Lemma 3.1. *Let t_1, t_2 and t_3 be real numbers and let $\tilde{t}_2$ be the representative of*

$$\frac{\lambda_2-\lambda_3}{\lambda_3-\lambda_1}t_1+t_2+\frac{\lambda_1-\lambda_2}{\lambda_3-\lambda_1}t_3 \tag{8}$$

modulo $2\pi/|\lambda_3-\lambda_1|$ in $\left[-\pi d/|\lambda_3-\lambda_1|,\pi d/|\lambda_3-\lambda_1|\right[$.

(*a*) *There are real numbers α and v such that*

$$r_1\mathrm{e}^{\mathrm{i}(t_1+\lambda_1x)}+r_2\mathrm{e}^{\mathrm{i}(t_2+\lambda_2x)}+r_3\mathrm{e}^{\mathrm{i}(t_3+\lambda_3x)}=\mathrm{e}^{\mathrm{i}\alpha}\left(r_1\mathrm{e}^{\mathrm{i}\lambda_1(x-v)}+r_2\mathrm{e}^{\mathrm{i}(\tilde{t}_2+\lambda_2(x-v))}+r_3\mathrm{e}^{\mathrm{i}\lambda_3(x-v))}\right) \tag{9}$$

for all x.

(*b*) *Let $t=|\tilde{t}_2|$ be the distance of (8) to $(2\pi d/|\lambda_3-\lambda_1|)\mathbb{Z}$. There is a sign $\varepsilon\in\{+1,-1\}$ such that*

$$\left|r_1\mathrm{e}^{\mathrm{i}(t_1+\lambda_1x)}+r_2\mathrm{e}^{\mathrm{i}(t_2+\lambda_2x)}+r_3\mathrm{e}^{\mathrm{i}(t_3+\lambda_3x)}\right|=\left|r_1\mathrm{e}^{\mathrm{i}\lambda_1\varepsilon(x-v)}+r_2\mathrm{e}^{\mathrm{i}(t+\lambda_2\varepsilon(x-v))}+r_3\mathrm{e}^{\mathrm{i}\lambda_3\varepsilon(x-v))}\right|$$

for all x.

Proof. (*a*). The argument $\tilde{t}_2$ is chosen so that the relative multiplier $(t_1,t_2-\tilde{t}_2,t_3)$ is an isometry.
(*b*). If $\tilde{t}_2$ is negative, take the conjugate under the modulus of the right hand side in (9). ☐

Remark 3.2. This proves the following periodicity formula:

$$\left|r_1\mathrm{e}^{\mathrm{i}\lambda_1x}+r_2\mathrm{e}^{\mathrm{i}(t+2\pi d/(\lambda_3-\lambda_1)+\lambda_2x)}+r_3\mathrm{e}^{\mathrm{i}\lambda_3x}\right|=\left|r_1\mathrm{e}^{\mathrm{i}\lambda_1(x-v)}+r_2\mathrm{e}^{\mathrm{i}(t+\lambda_2(x-v))}+r_3\mathrm{e}^{\mathrm{i}\lambda_3(x-v)}\right|$$

for all x and t, where v satisfies $\lambda_1v=2\pi d/(\lambda_3-\lambda_1)+\lambda_2v=\lambda_3v$ modulo 2π, that is

$$v=\frac{2m\pi}{\lambda_3-\lambda_1}\quad\text{with } m \text{ an inverse of } \frac{\lambda_3-\lambda_2}{d} \text{ modulo } \frac{\lambda_3-\lambda_1}{d}.$$

4 The frequencies of a trigonometric trinomial

We may suppose without loss of generality that $\lambda_1<\lambda_2<\lambda_3$. Let $k=(\lambda_2-\lambda_1)/d$ and $l=(\lambda_3-\lambda_2)/d$. Then

$$r_1\mathrm{e}^{\mathrm{i}(t_1+\lambda_1x)}+r_2\mathrm{e}^{\mathrm{i}(t_2+\lambda_2x)}+r_3\mathrm{e}^{\mathrm{i}(t_3+\lambda_3x)}=\mathrm{e}^{\mathrm{i}\lambda_2x}(r_1\mathrm{e}^{\mathrm{i}(t_1-k(dx))}+r_2\mathrm{e}^{\mathrm{i}t_2}+r_3\mathrm{e}^{\mathrm{i}(t_3+l(dx))})$$

This defines an isometry between C_Λ and $\mathrm{C}_{\{-k,0,l\}}$ and shows that C_Λ is normed by the maximum modulus norm on $[0,2\pi/d[$. With Lemma 3.1 (*b*), this shows that a homothety by d^{-1} allows us to restrict our study to the function

$$f(t,x)=\left|r_1\mathrm{e}^{-\mathrm{i}kx}+r_2\mathrm{e}^{\mathrm{i}t}+r_3\mathrm{e}^{\mathrm{i}lx}\right|^2$$

for $x\in\mathbb{R}$ with k and l two positive coprime numbers and $t\in\left[0,\pi/(k+l)\right]$. We have

$$f(-t,x)=f(t,-x) \tag{10}$$

and Remark 3.2 shows that

$$f\left(t+2\pi/(k+l),x\right)=f(t,x-2m\pi/(k+l)) \tag{11}$$

for all x and t, where m is the inverse of l modulo $k+l$. In particular, if $t=\pi/(k+l)$, we have the symmetry relation

$$f\left(\pi/(k+l),x\right)=f\left(\pi/(k+l),2m\pi/(k+l)-x\right). \tag{12}$$

5 Location of the maximum point

The purpose of our first proposition is to deduce the existence of a small interval on which a trigonometric trinomial attains its maximum modulus. Note that a trigonometric binomial attains its maximum modulus at a point that depends only on the phase of its coefficients:

- $\left|r_1 \mathrm{e}^{-\mathrm{i}kx} + r_2 \mathrm{e}^{\mathrm{i}t}\right|$ attains its maximum at $-t/k$ independently of r_1 and r_2,
- $\left|r_1 \mathrm{e}^{-\mathrm{i}kx} + r_3 \mathrm{e}^{\mathrm{i}lx}\right|$ attains its maximum at 0 independently of r_1 and r_3,
- $\left|r_2 \mathrm{e}^{\mathrm{i}t} + r_3 \mathrm{e}^{\mathrm{i}lx}\right|$ attains its maximum at t/l independently of r_2 and r_3.

The next proposition shows that if the point at which a trigonometric trinomial attains its maximum modulus changes with the modulus of its coefficients, it changes very little; we get bounds for this point that are independent of the intensities.

Proposition 5.1. *Let k and l be positive coprime integers. Let r_1, r_2 and r_3 be three positive real numbers. Let $t \in \left[0, \pi/(k+l)\right]$. Let*

$$f(x) = \left|r_1 \mathrm{e}^{-\mathrm{i}kx} + r_2 \mathrm{e}^{\mathrm{i}t} + r_3 \mathrm{e}^{\mathrm{i}lx}\right|^2$$

for $x \in \mathbb{R}$.

(*a*) *The function f attains its absolute maximum in the interval $[-t/k, t/l]$.*

(*b*) *If f attains its absolute maximum at a point y outside of $[-t/k, t/l]$ modulo 2π, then $t = \pi/(k+l)$ and $2m\pi/(k+l) - y$ lies in $[-t/k, t/l]$ modulo 2π, where m is the inverse of l modulo $k+l$.*

Proof. (*a*). We have

$$f(x) = r_1^2 + r_2^2 + r_3^2 + 2 \cdot \left(r_1 r_2 \cos(t+kx) + r_1 r_3 \cos\big((k+l)x\big) + r_2 r_3 \cos(t-lx)\right). \tag{13}$$

Let us prove that f attains its absolute maximum on $[-t/k, t/l]$. Let y be outside of $[-t/k, t/l]$ modulo 2π. Let I be the set of all $x \in [-t/k, t/l]$ such that

$$\begin{cases} \cos(t+kx) \geqslant \cos(t+ky) \\ \cos\big((k+l)x\big) \geqslant \cos\big((k+l)y\big) \\ \cos(t-lx) \geqslant \cos(t-ly). \end{cases}$$

Note that if $x \in [-t/k, t/l]$, then

$$\begin{cases} t+kx \in [0, (k+l)t/l] \\ (k+l)x \in [-(k+l)t/k, (k+l)t/l] \\ t-lx \in [0, (k+l)t/k], \end{cases}$$

and that $(k+l)t/k$, $(k+l)t/l \in [0, \pi]$. Let

- α be the distance of $t/k + y$ to $(2\pi/k)\mathbb{Z}$,
- β be the distance of y to $\big(2\pi/(k+l)\big)\mathbb{Z}$,
- γ be the distance of $t/l - y$ to $(2\pi/l)\mathbb{Z}$.

Then

$$I = [-t/k, t/l] \cap [-t/k - \alpha, -t/k + \alpha] \cap [-\beta, \beta] \cap [t/l - \gamma, t/l + \gamma]. \tag{14}$$

Let us check that I is the nonempty interval

$$I = \left[\max(-t/k, -\beta, t/l - \gamma), \min(t/l, -t/k + \alpha, \beta)\right]. \tag{15}$$

In fact, we have the following triangular inequalities:

- $-\beta \leqslant -t/k + \alpha$ because t/k is the distance of $(t/k + y) - y$ to $(2\pi/k(k+l))\mathbb{Z}$;

- $t/l-\gamma \leqslant -t/k+\alpha$ because $t/l+t/k$ is the distance of $(t/k+y)+(t/l-y)$ to $(2\pi/kl)\mathbb{Z}$;
- $t/l-\gamma \leqslant \beta$ because t/l is the distance of $(t/l-y)+y$ to $(2\pi/l(k+l))\mathbb{Z}$.

The other six inequalities that are necessary to deduce (15) from (14) are obvious.

(b). We have proved in (a) that there is an $x \in [-t/k, t/l]$ such that $\cos(t+kx) \geqslant \cos(t+ky)$, $\cos\bigl((k+l)x\bigr) \geqslant \cos\bigl((k+l)y\bigr)$ and $\cos(t-lx) \geqslant \cos(t-ly)$. In fact, at least one of these inequalities is strict unless there are signs $\delta, \varepsilon, \eta \in \{-1,1\}$ such that $t+kx = \delta(t+ky)$, $t-lx = \varepsilon(t-ly)$ and $(k+l)x = \eta(k+l)y$ modulo 2π. Two out of these three signs are equal and the corresponding two equations imply the third one with the same sign. This system is therefore equivalent to

$$\begin{cases} k(x-y)=0 \\ l(x-y)=0 \end{cases} \quad \text{or} \quad \begin{cases} k(x+y)=-2t \\ l(x+y)=2t \end{cases}$$

modulo 2π. The first pair of equations yields $x=y$ modulo 2π because k and l are coprime. Let m be an inverse of l modulo $k+l$; then the second pair of equations is equivalent to

$$\begin{cases} 2(k+l)t=0 \\ \quad x+y=2mt \end{cases}$$

modulo 2π. Therefore g does not attain its absolute maximum at y unless $t=\pi/(k+l)$ and $2m\pi/(k+l)-y \in [-t/k, t/l]$. □

Remark 5.2. This proposition is a complex counterpart to Lemma 2.1. i) in [22], where cosine trinomials are investigated.

6 Uniqueness of the maximum point

Note that

$$\begin{aligned} r_1\mathrm{e}^{-\mathrm{i}kx}+r_2\mathrm{e}^{\mathrm{i}t}+r_3\mathrm{e}^{\mathrm{i}lx} &= r_3\mathrm{e}^{-\mathrm{i}l(-x)}+r_2\mathrm{e}^{\mathrm{i}t}+r_1\mathrm{e}^{\mathrm{i}k(-x)} \\ &= r_1'\mathrm{e}^{-\mathrm{i}k'x'}+r_2\mathrm{e}^{\mathrm{i}t}+r_3'\mathrm{e}^{\mathrm{i}l'x'} \end{aligned}$$

with $r_1'=r_3$, $r_3'=r_1$, $k'=l$, $l'=k$ and $x'=-x$. We may therefore suppose without loss of generality that $kr_1 \leqslant lr_3$.

Our second proposition studies the points at which a trigonometric trinomial attains its maximum modulus. Note that if $k=l=1$, the derivative of $|f|^2$ has at most 4 zeroes, so that the modulus of f has at most two maxima and attains its absolute maximum in at most two points. Proposition 6.1 shows that this is true in general, and that if it may attain its absolute maximum in two points, it is so only because of the symmetry given by (12).

Proposition 6.1. *Let k and l be positive coprime integers. Let r_1, r_2 and r_3 be three positive real numbers such that $kr_1 \leqslant lr_3$. Let $t \in \,]0, \pi/(k+l)]$. Let*

$$f(x) = \bigl|r_1\mathrm{e}^{-\mathrm{i}kx}+r_2\mathrm{e}^{\mathrm{i}t}+r_3\mathrm{e}^{\mathrm{i}lx}\bigr|^2$$

for $x \in [-t/k, t/l]$.

(a) *There is a point $x^* \in [0, t/l]$ such that $\mathrm{d}f/\mathrm{d}x > 0$ on $]-t/k, x^*[$ and $\mathrm{d}f/\mathrm{d}x < 0$ on $]x^*, t/l[$.*

(b) *There are three cases:*

1. *f attains its absolute maximum at 0 if and only if $kr_1 = lr_3$;*
2. *f attains its absolute maximum at t/l if and only if $l=1$, $t=\pi/(k+1)$ and $k^2r_1r_2+(k+1)^2r_1r_3-r_2r_3 \leqslant 0$;*
3. *otherwise, f attains its absolute maximum in $]0, t/l[$.*

(c) *The function f attains its absolute maximum with multiplicity 2 unless $l=1$, $t=\pi/(k+1)$ and $k^2r_1r_2+(k+1)^2r_1r_3-r_2r_3=0$, in which case it attains its absolute maximum at $\pi/(k+1)$ with multiplicity 4.*

Proof. (a). By Proposition 5.1, the derivative of f has a zero in $[-t/k, t/l]$. Let us study the sign of this derivative. Equation (13) yields

$$\frac{1}{2}\frac{\mathrm{d}f}{\mathrm{d}x}(x) = -kr_1r_2\sin(t+kx) - (k+l)r_1r_3\sin\big((k+l)x\big) + lr_2r_3\sin(t-lx). \tag{16}$$

We wish to compare $\sin(t+kx)$ with $\sin(t-lx)$: note that

$$\sin(t+kx) - \sin(t-lx) = 2\sin\big((k+l)x/2\big)\cos\big(t+(k-l)x/2\big),$$

and that if $x \in [-t/k, t/l]$, then

$$-\pi/2 \leqslant -\pi/2k \leqslant -(k+l)t/2k \leqslant (k+l)x/2 \leqslant (k+l)t/2l \leqslant \pi/2l \leqslant \pi/2$$

$$0 \leqslant t+(k-l)x/2 \leqslant \begin{cases} t+(l-k)t/2k = (k+l)t/2k & \text{if } k \leqslant l \\ t+(k-l)t/2l = (k+l)t/2l & \text{if } l \leqslant k \end{cases} \leqslant \pi/2.$$

Suppose that $x \in [-t/k, 0[$: then it follows that $\sin(t+kx) \leqslant \sin(t-lx)$ and also $\sin\big((k+l)x\big) \leqslant 0$, with equality if and only if $k=1$ and $-x = t = \pi/(1+l)$. This yields with $kr_1 \leqslant lr_3$ that

$$\frac{1}{2}\frac{\mathrm{d}f}{\mathrm{d}x}(x) \geqslant -(k+l)r_1r_3\sin\big((k+l)x\big) \geqslant 0 \tag{17}$$

with equality if and only if $k=1$ and $-x = t = \pi/(1+l)$.

Suppose that $x \in [0, t/l]$. If $l \geqslant 2$, then

$$\begin{cases} t+kx \in [t, (k+l)t/l] \subset [t, \pi/2] \\ (k+l)x \in [0, (k+l)t/l] \subset [t, \pi/2] \\ t-lx \in [0, t] \subset [0, \pi/3], \end{cases}$$

so that the second derivative of f is strictly negative on $[0, t/l]$: its derivative is strictly decreasing on this interval and (a) is proved. If $l=1$, let $g(x) = f(t-x)$ for $x \in [0,t]$: we have to prove that there is a point x^* such that $\mathrm{d}g/\mathrm{d}x > 0$ on $]0, x^*[$ and $\mathrm{d}g/\mathrm{d}x < 0$ on $]x^*, t[$. We already know that $(\mathrm{d}g/\mathrm{d}x)(0) \geqslant 0$ and that $\mathrm{d}g/\mathrm{d}x$ has a zero on $[0,t]$. Put $\alpha = (k+1)t$: then

$$\frac{1}{2}\frac{\mathrm{d}g}{\mathrm{d}x}(x) = kr_1r_2\sin(\alpha-kx) + (k+1)r_1r_3\sin\big(\alpha-(k+1)x\big) - r_2r_3\sin x$$

and it suffices to prove that

$$\frac{1}{2\sin x}\frac{\mathrm{d}g}{\mathrm{d}x}(x) = kr_1r_2\frac{\sin(\alpha-kx)}{\sin x} + (k+1)r_1r_3\frac{\sin\big(\alpha-(k+1)x\big)}{\sin x} - r_2r_3 \tag{18}$$

is a strictly decreasing function of x on $]0, \alpha/(k+1)]$. Let us study the sign of

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin(\alpha-kx)}{\sin x} = \frac{-k\cos(\alpha-kx)\sin x - \sin(\alpha-kx)\cos x}{\sin^2 x}$$

for $\alpha \in]0,\pi]$ and $x \in]0, \alpha/k]$. If $k=1$, then

$$-k\cos(\alpha-kx)\sin x - \sin(\alpha-kx)\cos x = -\sin\alpha \leqslant 0$$

and the inequality is strict unless $\alpha = \pi$. Let us prove by induction on k that

$$k\cos(\alpha-kx)\sin x + \sin(\alpha-kx)\cos x > 0$$

for all $k \geqslant 2$, $\alpha \in]0,\pi]$ and $x \in]0, \alpha/k]$. This will complete the proof of (a). Let $k \geqslant 1$ and $x \in]0, \alpha/(k+1)]$. Then

$$\begin{aligned}(k+1)\cos\big(\alpha-(k+1)x\big)\sin x + \sin\big(\alpha-(k+1)x\big)\cos x \\ &= (k+1)\cos(\alpha-kx)\cos x\sin x + (k+1)\sin(\alpha-kx)\sin^2 x \\ &\qquad + \sin(\alpha-kx)\cos^2 x - \cos(\alpha-kx)\sin x\cos x \\ &= \big(k\cos(\alpha-kx)\sin x + \sin(\alpha-kx)\cos x\big)\cos x \\ &\qquad + (k+1)\sin(\alpha-kx)\sin^2 x \\ &\geqslant (k+1)\sin(\alpha-kx)\sin^2 x > 0\end{aligned}$$

(b). 1. By Proposition 5.1 and (a), f attains its absolute maximum at 0 if and only if 0 is a critical point for f. We have

$$\frac{1}{2}\frac{\mathrm{d}f}{\mathrm{d}x}(0) = (lr_3 - kr_1)r_2 \sin t \geqslant 0$$

and equality holds if and only if $kr_1 = lr_3$.

2. We have

$$\frac{1}{2}\frac{\mathrm{d}f}{\mathrm{d}x}(t/l) = \big(-kr_1r_2 - (k+l)r_1r_3\big)\sin\big((k+l)t/l\big) \leqslant 0$$

and equality holds if and only if $l = 1$ and $t = \pi/(k+1)$. Let $l = 1$ and $t = \pi/(k+1)$ and let us use the notation introduced in the last part of the proof of (a): we need to characterise the case that g has a maximum at 0. As $\alpha = \pi$ and

$$\frac{1}{2\sin x}\frac{\mathrm{d}g}{\mathrm{d}x}(x) = k^2r_1r_2 + (k+1)^2r_1r_3 - r_2r_3 + o(x) \tag{19}$$

is a strictly decreasing function of x on $]0, \pi/(k+1)]$, g has a maximum at 0 if and only if $k^2r_1r_2 + (k+1)^2r_1r_3 - r_2r_3 \leqslant 0$.

(c). If $l \geqslant 2$, then the second derivative of f is strictly negative on $[0, t/l]$. If $l = 1$, then the derivative of (18) is strictly negative on $]0, \alpha/(k+1)]$: this yields that the second derivative of g can only vanish at 0. By (b) 2., g has a maximum at 0 only if $t = \pi/(k+1)$; then

$$\frac{1}{2}\frac{\mathrm{d}^2g}{\mathrm{d}x^2}(0) = k^2r_1r_2 + (k+1)^2r_1r_3 - r_2r_3 \tag{20}$$

$$\frac{1}{2}\frac{\mathrm{d}^4g}{\mathrm{d}x^4}(0) = -k^4r_1r_2 - (k+1)^4r_1r_3 + r_2r_3 \tag{21}$$

If (20) vanishes, then the sum of (21) with (20) yields

$$\frac{1}{2}\frac{\mathrm{d}^4g}{\mathrm{d}x^4}(0) = -k(k+1)r_1\big((k-1)kr_2 + (k+1)(k+2)r_3\big) < 0. \qquad \square$$

Remark 6.2. We were able to prove directly that the system

$$\begin{cases} f(x) = f(y) \\ \dfrac{\mathrm{d}f}{\mathrm{d}x}(x) = \dfrac{\mathrm{d}f}{\mathrm{d}x}(y) = 0 \\ \dfrac{\mathrm{d}^2f}{\mathrm{d}x^2}(x), \dfrac{\mathrm{d}^2f}{\mathrm{d}x^2}(y) \leqslant 0 \end{cases}$$

implies $x = y$ modulo 2π or $t = \pi/(k+l)$ and $x + y = 2m\pi/(k+l)$, but our computations are very involved and opaque.

Remark 6.3. This proposition is a complex counterpart to [22, Lemma 2.1. ii)].

Remark 6.4. Suppose that $l = k = 1$. If $t \in]0, \pi/2[$, it is necessary to solve a generally irreducible quartic equation in order to compute the maximum of f. If $t = \pi/2$, it suffices to solve a linear equation and one gets the following expression for $\max_x \left|r_1\mathrm{e}^{-\mathrm{i}x} + \mathrm{i}r_2 + r_3\mathrm{e}^{\mathrm{i}x}\right|$:

$$\begin{cases} (r_1 + r_3)\sqrt{1 + r_2^2/4r_1r_3} & \text{if } \left|r_1^{-1} - r_3^{-1}\right| < 4r_2^{-1} \\ r_2 + |r_3 - r_1| & \text{otherwise.} \end{cases}$$

This formula appears in [1, (3.1)]. In the first case, the maximum is attained at the two points x^* such that $\sin x^* = r_2(r_3 - r_1)/4r_1r_3$.

Remark 6.5. Suppose that $l = 1$ and $k = 2$. If $t \in]0, \pi/3[$, it is necessary to solve a generally irreducible sextic equation in order to compute the maximum of f. If $t = \pi/3$, it suffices to solve a quadratic equation and one gets the following expression for $\max_x \left|r_1\mathrm{e}^{-\mathrm{i}2x} + r_2\mathrm{e}^{\mathrm{i}\pi/3} + r_3\mathrm{e}^{\mathrm{i}x}\right|$: if $r_1^{-1} - 4r_3^{-1} < 9r_2^{-1}$, then its square makes

$$r_1^2 + \frac{2}{3}r_2^2 + r_3^2 + r_1r_2 + 2r_1r_3\left[\left(\left(\frac{r_2}{3r_3}\right)^2 + \frac{r_2}{3r_1} + 1\right)^{3/2} - \left(\frac{r_2}{3r_3}\right)^3\right]$$

and the maximum is attained at the two points x^* such that

$$2\cos(\pi/3 - x^*) = \left(\left(\frac{r_2}{3r_3}\right)^2 + \frac{r_2}{3r_1} + 1\right)^{1/2} - \frac{r_2}{3r_3};$$

otherwise, it makes $-r_1 + r_2 + r_3$.

7 The maximum modulus points of a trigonometric trinomial

If we undo all the reductions made in Sections 3 and 4 and at the beginning of Section 6, we get the following theorem.

Theorem 7.1. *Let λ_1, λ_2 and λ_3 be three pairwise distinct integers such that λ_2 is between λ_1 and λ_3. Let r_1, r_2 and r_3 be three positive real numbers. Given three real numbers t_1, t_2 and t_3, consider the trigonometric trinomial*

$$T(x) = r_1 \mathrm{e}^{\mathrm{i}(t_1+\lambda_1 x)} + r_2 \mathrm{e}^{\mathrm{i}(t_2+\lambda_2 x)} + r_3 \mathrm{e}^{\mathrm{i}(t_3+\lambda_3 x)}$$

for $x \in \mathbb{R}$. Let $d = \gcd(\lambda_2 - \lambda_1, \lambda_3 - \lambda_2)$ and choose integers a_1 and a_3 such that

$$\tau = \frac{\lambda_2 - \lambda_3}{d}(t_1 - 2\pi a_1) + \frac{\lambda_3 - \lambda_1}{d} t_2 + \frac{\lambda_1 - \lambda_2}{d}(t_3 - 2\pi a_3),$$

satisfies $|\tau| \leqslant \pi$. Let $\tilde{t}_1 = t_1 - 2\pi a_1$ and $\tilde{t}_3 = t_3 - 2\pi a_3$.

(*a*) *The trigonometric trinomial T attains its maximum modulus at a unique point of the interval bounded by $(\tilde{t}_1 - t_2)/(\lambda_2 - \lambda_1)$ and $(t_2 - \tilde{t}_3)/(\lambda_3 - \lambda_2)$. More precisely,*

- *if $r_1|\lambda_2 - \lambda_1| \leqslant r_3|\lambda_3 - \lambda_2|$, then this point is between $(\tilde{t}_1 - \tilde{t}_3)/(\lambda_3 - \lambda_1)$ and $(t_2 - \tilde{t}_3)/(\lambda_3 - \lambda_2)$;*
- *if $r_1|\lambda_2 - \lambda_1| \geqslant r_3|\lambda_3 - \lambda_2|$, then this point is between $(\tilde{t}_1 - \tilde{t}_3)/(\lambda_3 - \lambda_1)$ and $(\tilde{t}_1 - t_2)/(\lambda_2 - \lambda_1)$;*
- *T attains its maximum modulus at $(\tilde{t}_1 - \tilde{t}_3)/(\lambda_3 - \lambda_1)$ if and only if $r_1|\lambda_2 - \lambda_1| = r_3|\lambda_3 - \lambda_2|$ or $\tau = 0$.*

(*b*) *The function T attains its maximum modulus at a unique point modulo $2\pi/d$, and with multiplicity 2, unless $|\tau| = \pi$.*

(*c*) *Suppose that $|\tau| = \pi$, i.e.,*

$$\frac{\lambda_2 - \lambda_3}{d} t_1 + \frac{\lambda_3 - \lambda_1}{d} t_2 + \frac{\lambda_1 - \lambda_2}{d} t_3 = \pi \mod 2\pi. \tag{22}$$

Let s be a solution to $2t_1 + \lambda_1 s = 2t_2 + \lambda_2 s = 2t_3 + \lambda_3 s$ modulo 2π: s is unique modulo $2\pi/d$. Then $T(s - x) = \mathrm{e}^{\mathrm{i}(2t_2+\lambda_2 s)}\overline{T(x)}$ for all x. Suppose that $|\lambda_3 - \lambda_2| \leqslant |\lambda_2 - \lambda_1|$. There are three cases.

1. *If $\lambda_2 - \lambda_1 = k(\lambda_3 - \lambda_2)$ with $k \geqslant 2$ integer and*
$$r_1^{-1} - k^2 r_3^{-1} \geqslant (k+1)^2 r_2^{-1},$$
then T attains its maximum modulus, $-r_1 + r_2 + r_3$, only at $x = (t_2 - \tilde{t}_3)/(\lambda_3 - \lambda_2)$ modulo $2\pi/d$, with multiplicity 2 if the inequality is strict and with multiplicity 4 if there is equality;
2. *if $\lambda_2 - \lambda_1 = \lambda_3 - \lambda_2$ and*
$$\left|r_1^{-1} - r_3^{-1}\right| \geqslant 4r_2^{-1},$$
then T attains its maximum modulus, $r_2 + |r_3 - r_1|$, at a unique point x modulo $2\pi/d$, with multiplicity 2 if the inequality is strict and with multiplicity 4 if there is equality. This point is $(t_2 - \tilde{t}_3)/(\lambda_3 - \lambda_2)$ if $r_1 < r_3$, and $(\tilde{t}_1 - t_2)/(\lambda_2 - \lambda_1)$ if $r_3 < r_1$;
3. *otherwise T attains its maximum modulus at exactly two points x and y modulo $2\pi/d$, with multiplicity 2, where x is strictly between $(\tilde{t}_1 - t_2)/(\lambda_2 - \lambda_1)$ and $(t_2 - \tilde{t}_3)/(\lambda_3 - \lambda_2)$, and $x + y = s$ modulo $2\pi/d$.*

Note that $s - x = x$ modulo $2\pi/d$ in Cases 1 and 2.

8 Exposed and extreme points of the unit ball of C_Λ

The characterisation of the maximum modulus points of a trigonometric trinomial enables us to compute the exposed and the extreme points of the unit ball of C_Λ. We begin with a lemma.

Lemma 8.1. (*a*) *A trigonometric trinomial with a given spectrum that attains its maximum modulus at two given points modulo $2\pi/d$ is determined by its value at these points.*

(*b*) *The trigonometric trinomials with a given spectrum that attain their maximum modulus with multiplicity 4 at a given point and have a given value at this point lie on a parabola.*

Proof. We will use the notation of Theorem 7.1. Without loss of generality, we may suppose that $\lambda_1 = -k$, $\lambda_2 = 0$ and $\lambda_3 = l$ with k and l positive coprime integers. Let x and y be two real numbers that are different modulo $2\pi/d$, let ϑ and ζ be real numbers and let ϱ be a positive real number.

(*a*). Let us prove that at most one trigonometric trinomial T attains its maximum modulus at x and y and satisfies $T(x) = \varrho \mathrm{e}^{\mathrm{i}\vartheta}$ and $T(y) = \varrho \mathrm{e}^{\mathrm{i}\zeta}$. Let us translate T by $(x+y)/2$: we may suppose that $x + y = 0$. Let us divide T by $\mathrm{e}^{\mathrm{i}(\vartheta+\zeta)/2}$: we may suppose that $\vartheta + \zeta = 0$. As T attains its maximum modulus at the two points x and y, we have $s = x + y = 0$ and $2t_1 - ks = 2t_2 = 2t_3 + ls = \vartheta + \zeta = 0$ modulo 2π. Therefore $t_1 = t_2 = t_3 = 0$ modulo π. Let $p_j = \mathrm{e}^{\mathrm{i}t_j} r_j$: the p_j are nonzero real numbers. We have

$$T(x) = p_1 \mathrm{e}^{-\mathrm{i}kx} + p_2 + p_3 \mathrm{e}^{\mathrm{i}lx} = \varrho \mathrm{e}^{\mathrm{i}\vartheta},$$

so that, multiplying by $\mathrm{e}^{-\mathrm{i}\vartheta}$ and taking real and imaginary parts,

$$p_1 \cos(\vartheta + kx) + p_3 \cos(\vartheta - lx) = \varrho - p_2 \cos\vartheta \tag{23}$$

$$p_1 \sin(\vartheta + kx) + p_3 \sin(\vartheta - lx) = -p_2 \sin\vartheta. \tag{24}$$

The computation

$$\frac{1}{2}\frac{\mathrm{d}|T|^2}{\mathrm{d}x}(x) = \Re\Big(\overline{T(x)}\frac{\mathrm{d}T}{\mathrm{d}x}(x)\Big) = \Re\Big(\overline{T(x)}\big(-\mathrm{i}kp_1\mathrm{e}^{-\mathrm{i}kx} + \mathrm{i}lp_3\mathrm{e}^{\mathrm{i}lx}\big)\Big),$$

yields

$$kp_1 \sin(\vartheta + kx) - lp_3 \sin(\vartheta - lx) = 0. \tag{25}$$

Equations (24) and (25) yield p_1 and p_3 as linear functions of p_2 because $\sin(\vartheta + kx)\sin(\vartheta - lx) \neq 0$: otherwise both factors would vanish, so that $\vartheta = x = 0$ modulo π and $x = y$ modulo 2π. As $\varrho \neq 0$, Equation (23) has at most one solution in p_2.

(*b*). We are necessarily in Case 1 or 2 of Theorem 7.1(*c*), so that we may suppose that $l = 1$. Let us determine all trigonometric trinomials T that attain their maximum modulus at x with multiplicity 4 and satisfy $T(x) = \varrho \mathrm{e}^{\mathrm{i}\vartheta}$. Let us translate T by x: we may suppose that $x = 0$. Let us divide T by $\mathrm{e}^{\mathrm{i}\vartheta}$: we may suppose that $\vartheta = 0$. As T attains its maximum modulus at 0 with multiplicity 4, we have $s - 0 = 0$ and $2t_1 - ks = 2t_2 = 2t_3 + s = 2\vartheta = 0$ modulo 2π. Therefore $t_1 = t_2 = t_3 = 0$ modulo π. Let $p_j = \mathrm{e}^{\mathrm{i}t_j} r_j$: the p_j are nonzero real numbers and satisfy the system

$$\begin{cases} p_1 + p_2 + p_3 = \varrho \\ k^2 p_1 p_2 + (k+1)^2 p_1 p_3 + p_2 p_3 = 0, \end{cases}$$

that is

$$\begin{cases} p_2 = \varrho - p_1 - p_3 \\ (kp_1 - p_3)^2 = \varrho(k^2 p_1 + p_3). \end{cases}$$

This is the equation of a parabola. □

Remark 8.2. The equality

$$\max_x \big| r_1 \mathrm{e}^{\mathrm{i}(t_1 + \lambda_1 x)} + r_2 \mathrm{e}^{\mathrm{i}(t_2 + \lambda_2 x)} \big| = r_1 + r_2 \tag{26}$$

shows that the exposed points of the unit ball of the space $\mathrm{C}_{\{\lambda_1,\lambda_2\}}$ are the trigonometric monomials $\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}_{\lambda_1}$ and $\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}_{\lambda_2}$ with $\alpha \in \mathbb{R}$ and that no trigonometric binomial is an extreme point of the unit ball of C_Λ.

Proof of Theorem 1.2 (a). *Necessity.* Let $f \in K$. Let us make four remarks:

- if f is an exposed point, then $\|f\| = 1$;
- if l is a nonzero linear functional on C_Λ and z is a complex number such that $l^{-1}(z) \cap K = \{f\}$, then $|l(f)| = |z| = \|l\|$, so that l attains its norm at f;
- f is exposed if and only if $\|f\| = 1$ and there is a nonzero linear functional l on C_Λ that attains its norm only at multiples of f;
- if l is a linear functional on C_Λ that attains its norm at f and μ is the measure on $[0, 2\pi/d[$ identified with the Hahn-Banach extension of l to the space of continuous functions on $[0, 2\pi/d[$, then the support of μ must be a subset of the maximum modulus points of f on $[0, 2\pi/d[$.

Theorem 7.1 (b, c) tells that a trigonometric trinomial attains its maximum modulus at one or two points modulo $2\pi/d$. We have therefore to show that trigonometric binomials and trigonometric trinomials with only one maximum modulus point modulo $2\pi/d$ are not exposed.

- A linear functional that attains its modulus at a trigonometric binomial attains its norm at a trigonometric monomial because this trigonometric binomial is a convex combination of two trigonometric monomials with same norm by Equation (26).
- If f attains its maximum modulus at a unique point $x \in [0, 2\pi/d[$, then l must be a multiple of the Dirac measure δ_x at x, so that l attains also its norm at the monomials in C_Λ.

These arguments show also that every linear functional on C_Λ attains its norm at monomials or at trigonometric trinomials with two maximum modulus points.

Sufficiency. Conversely, the trigonometric monomial $\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}_\lambda$ is exposed to the linear form

$$P \mapsto \frac{1}{2\pi}\int_0^{2\pi} P(x)\mathrm{e}^{-\mathrm{i}(\alpha+\lambda x)}\,\mathrm{d}x.$$

A trigonometric trinomial T that attains its maximum modulus, 1, at two points x_1^* and x_2^* modulo $2\pi/d$ is exposed, by Lemma 8.1 (a), to any nontrivial convex combination of the unimodular multiples of Dirac measures $\overline{T(x_1^*)}\delta_{x_1^*}$ and $\overline{T(x_2^*)}\delta_{x_2^*}$. ☐

Remark 8.3. This is a complex counterpart to Lemma 2.3 in [22], dealing with the exposed points of the unit ball of the three-dimensional space spanned by the functions 1, $\cos x$ and $\cos kx$ in the space of continuous functions.

Proof of Theorem 1.2 (b). Let K be the unit ball of C_Λ. Straszewicz's Theorem [27] tells that the exposed points of K are dense in the set of its extreme points. Let P be a limit point of exposed points of K. If P is a trigonometric monomial, P is exposed. If P is a trigonometric binomial, P is not an extreme point of K by Remark 8.2. If P is a trigonometric trinomial, it is the limit point of trigonometric trinomials that attain their maximum modulus twice modulo $2\pi/d$, so that either P also attains its maximum modulus twice modulo $2\pi/d$ or, by Rolle's Theorem, P attains its maximum modulus with multiplicity 4. Let us prove that if a trigonometric trinomial T attains its maximum modulus with multiplicity 4 at a point x, then T is an extreme point of K. Suppose that T is the midpoint of two points A and B in K. Then $|A(x)| \leqslant 1$, $|B(x)| \leqslant 1$ and $\bigl(A(x) + B(x)\bigr)/2 = T(x)$, so that $A(x) = B(x) = T(x)$. Furthermore

$$|T(x+h)| \leqslant \frac{|A(x+h)| + |B(x+h)|}{2} = 1 + \frac{h^2}{4}\left(\frac{\mathrm{d}^2|A|}{\mathrm{d}x^2}(x) + \frac{\mathrm{d}^2|B|}{\mathrm{d}x^2}(x)\right) + o(h^2)$$

so that, as $|T(x+h)| = 1 + o(h^3)$,

$$\frac{\mathrm{d}^2|A|}{\mathrm{d}x^2}(x) + \frac{\mathrm{d}^2|B|}{\mathrm{d}x^2}(x) \geqslant 0 \text{ while } \frac{\mathrm{d}^2|A|}{\mathrm{d}x^2}(x), \frac{\mathrm{d}^2|B|}{\mathrm{d}x^2}(x) \leqslant 0$$

and therefore A and B also attain their maximum modulus with multiplicity 4 at x. As this implies that A and B are trigonometric trinomials, Lemma 8.1 (b) yields that T, A and B lie on a parabola: this implies $A = B = T$. ☐

Remark 8.4. The set of extreme points of the unit ball of C_Λ is not closed: for example, if λ_2 is between λ_1 and λ_3, every absolutely convex combination of e_{λ_1} and e_{λ_3} is a limit point of exposed points.

Remark 8.5. If λ_2 is between λ_1 and λ_3, and $\lambda_1-\lambda_2$ is not a multiple of $\lambda_3-\lambda_2$ nor vice versa, then we obtain that every extreme point of the unit ball of C_Λ is exposed.

Remark 8.6. In particular, compare our description of the extreme points of the unit ball of $\mathrm{C}_{\{0,1,2\}}$ with the characterisation given by K. M. Dyakonov in [7, Theorem 1]. He shows in his Example 1 that it is false that "in order to recognize the extreme points", "one only needs to know 'how often' [the modulus of a trigonometric polynomial] takes the extremal value 1". We show that *with the exception of trigonometric binomials*, the extreme points of the unit ball of $\mathrm{C}_{\{0,1,2\}}$ are characterised by the number of zeroes of $1-|P|^2$.

Remark 8.7. This is a complex counterpart to [16], dealing with the extreme points of the unit ball of the three-dimensional space spanned by the functions 1, x^n and x^m in the space of real valued continuous functions on $[-1,1]$.

9 Dependence of the maximum modulus on the arguments

We wish to study how the maximum modulus of a trigonometric trinomial depends on the phase of its coefficients. We shall use the following formula that gives an expression for the directional derivative of a maximum function. It was established in [4]. Elementary properties of maximum functions are addressed in [20, Part Two, Problems 223–226].

N. G. Chebotarëv's formula ([6, Theorem VI.3.2, (3.6)])**.** *Let $I\subset\mathbb{R}$ be an open interval and let K be a compact space. Let $f(t,x)$ be a function on $I\times K$ that is continuous along with $\frac{\partial f}{\partial t}(t,x)$. Let*

$$f^*(t)=\max_{x\in K} f(t,x).$$

Then $f^*(t)$ admits the following expansion at every $t\in I$:

$$f^*(t+h)=f^*(t)+\max_{f(t,x)=f^*(t)}\Big(h\frac{\partial f}{\partial t}(t,x)\Big)+o(h). \tag{27}$$

Proposition 9.1. *Let k and l be positive coprime integers. Let r_1, r_2 and r_3 be three positive real numbers. Then*

$$\max_x\left|r_1\mathrm{e}^{-\mathrm{i}kx}+r_2\mathrm{e}^{\mathrm{i}t}+r_3\mathrm{e}^{\mathrm{i}lx}\right|$$

is an even $2\pi/(k+l)$-periodic function of t that decreases strictly on $[0,\pi/(k+l)]$: in particular

$$\min_t\max_x\left|r_1\mathrm{e}^{-\mathrm{i}kx}+r_2\mathrm{e}^{\mathrm{i}t}+r_3\mathrm{e}^{\mathrm{i}lx}\right|=\max_x\left|r_1\mathrm{e}^{-\mathrm{i}kx}+r_2\mathrm{e}^{\mathrm{i}\pi/(k+l)}+r_3\mathrm{e}^{\mathrm{i}lx}\right|.$$

Proof. Let

$$f(t,x)=\left|r_1\mathrm{e}^{-\mathrm{i}kx}+r_2\mathrm{e}^{\mathrm{i}t}+r_3\mathrm{e}^{\mathrm{i}lx}\right|^2. \tag{28}$$

By (10) and (11), f^* is an even $2\pi/(k+l)$-periodic function.

Let $t\in\,]0,\pi/(k+l)[$ and choose x^* such that $f(t,x^*)=f^*(t)$: then $x^*\in[-t/k,t/l]$ by Proposition 5.1, so that

$$\frac{1}{2r_2}\frac{\partial f}{\partial t}(t,x^*)=-r_1\sin(t+kx^*)-r_3\sin(t-lx^*)<0$$

because $t+kx^*\in[0,(k+l)t/l]$ and $t-lx^*\in[0,(k+l)t/k]$ do not vanish simultaneously. By Formula (27), f^* decreases strictly on $[0,\pi/(k+l)]$. ◻

Proposition 9.2. *Let k and l be positive coprime integers. Let r_1, r_2 and r_3 be three positive real numbers. Then*

$$\frac{\max_x\left|r_1\mathrm{e}^{-\mathrm{i}kx}+r_2\mathrm{e}^{\mathrm{i}t}+r_3\mathrm{e}^{\mathrm{i}lx}\right|}{\left|r_1+r_2\mathrm{e}^{\mathrm{i}t}+r_3\right|} \tag{29}$$

is an increasing function of $t\in[0,\pi/(k+l)]$. If $kr_1=lr_3$, it is constantly equal to 1; otherwise it is strictly increasing.

Proof. Let $f(t,x)$ be as in (28): then the expression (29) is $g^*(t)^{1/2}$ with

$$g(t,x)=\frac{f(t,x)}{f(t,0)}.$$

If $kr_1=lr_3$, then $f(t,0)=f^*(t)$, so that $g^*(t)=1$. As shown in the beginning of Section 6, we may suppose without loss of generality that $kr_1<lr_3$. Let $t\in\,]0,\pi/(k+l)[$ and choose x^* such that $f(t,x^*)=f^*(t)$: then $x^*\in\,]0,t/l[$ by Propositions 5.1 and 6.1 and

$$\begin{aligned}\frac{f(t,0)^2}{2r_2}\frac{\partial g}{\partial t}(t,x^*)&=\frac{1}{2r_2}\left(\frac{\partial f}{\partial t}(t,x^*)f(t,0)-f(t,x^*)\frac{\partial f}{\partial t}(t,0)\right)\\&=\big(-r_1\sin(t+kx^*)-r_3\sin(t-lx^*)\big)f(t,0)+f^*(t)(r_1+r_3)\sin t\\&=h(0)f^*(t)-h(x^*)f(t,0)\end{aligned}$$

with

$$h(x)=r_1\sin(t+kx)+r_3\sin(t-lx).$$

Let us show that h is strictly decreasing on $[0,t/l]$: in fact, if $x\in\,]0,t/l[$,

$$\frac{\mathrm{d}h}{\mathrm{d}x}(x)=kr_1\cos(t+kx)-lr_3\cos(t-lx)<(kr_1-lr_3)\cos(t-lx)<0.$$

As $f^*(t)>f(t,0)$ and $h(0)>h(x^*)$, $(\partial g/\partial t)(t,x^*)>0$. By N. G. Chebotarëv's formula, g^* increases strictly on $[0,\pi/(k+l)]$. ☐

It is possible to describe the decrease of the maximum modulus of a trigonometric trinomial independently of the r's as follows.

Proposition 9.3. *Let k and l be two positive coprime integers. Let r_1, r_2 and r_3 be three positive real numbers. Let $0\leqslant t'<t\leqslant\pi/(k+l)$. Then*

$$\max_x\left|r_1\mathrm{e}^{-\mathrm{i}kx}+r_2\mathrm{e}^{\mathrm{i}t'}+r_3\mathrm{e}^{\mathrm{i}lx}\right|\leqslant\frac{\cos(t'/2)}{\cos(t/2)}\max_x\left|r_1\mathrm{e}^{-\mathrm{i}kx}+r_2\mathrm{e}^{\mathrm{i}t}+r_3\mathrm{e}^{\mathrm{i}lx}\right|\tag{30}$$

with equality if and only if $r_1:r_2:r_3=l:k+l:k$.

Proof. Let us apply Proposition 9.2. We have

$$\begin{aligned}\frac{\left|r_1+r_2\mathrm{e}^{\mathrm{i}t'}+r_3\right|^2}{\left|r_1+r_2\mathrm{e}^{\mathrm{i}t}+r_3\right|^2}&=1+\frac{2r_2(r_1+r_3)(\cos t'-\cos t)}{(r_1+r_3)^2+2r_2(r_1+r_3)\cos t+r_2^2}\\&=1+\frac{\cos t'-\cos t}{\cos t+\big(r_2^2+(r_1+r_3)^2\big)/2r_2(r_1+r_3)}\\&\leqslant1+\frac{\cos t'-\cos t}{\cos t+1}=\frac{\cos t'+1}{\cos t+1}\end{aligned}$$

by the arithmetic-geometric inequality, with equality if and only if $r_2=r_1+r_3$. Therefore Inequality (30) holds, with equality if and only if $kr_1=lr_3$ and $r_2=r_1+r_3$. ☐

We may now find the minimum of the maximum modulus of a trigonometric trinomial with given spectrum, Fourier coefficient arguments and moduli sum. Proposition 9.3 yields with $t'=0$

Corollary 9.4. *Let k and l be two positive coprime integers. Let r_1, r_2 and r_3 be three positive real numbers. Let $t\in\,]0,\pi/(k+l)]$. Then*

$$\frac{\max_x\left|r_1\mathrm{e}^{-\mathrm{i}kx}+r_2\mathrm{e}^{\mathrm{i}t}+r_3\mathrm{e}^{\mathrm{i}lx}\right|}{r_1+r_2+r_3}\geqslant\cos(t/2)$$

with equality if and only if $r_1:r_2:r_3=l:k+l:k$.

Remark 9.5. There is a shortcut proof to Corollary 9.4:

$$\begin{aligned}\frac{\max_x\left|r_1\mathrm{e}^{-\mathrm{i}kx}+r_2\mathrm{e}^{\mathrm{i}t}+r_3\mathrm{e}^{\mathrm{i}lx}\right|}{r_1+r_2+r_3} &\geqslant \frac{\left|r_1+r_2\mathrm{e}^{\mathrm{i}t}+r_3\right|}{r_1+r_2+r_3}\\ &= \sqrt{1-\frac{4(r_1+r_3)r_2}{(r_1+r_2+r_3)^2}\sin^2(t/2)}\\ &\geqslant \sqrt{1-\sin^2(t/2)}=\cos(t/2)\end{aligned}$$

and equality holds if and only if $\left|r_1\mathrm{e}^{-\mathrm{i}kx}+r_2\mathrm{e}^{\mathrm{i}t}+r_3\mathrm{e}^{\mathrm{i}lx}\right|$ is maximal for $x=0$ and $r_1+r_3=r_2$.

10 The norm of unimodular relative Fourier multipliers

We may now compute the norm of unimodular relative Fourier multipliers.

Corollary 10.1. *Let k and l be two positive coprime integers. Let $t\in[0,\pi/(k+l)]$. Let M be the relative Fourier multiplier $(0,t,0)$ that maps the element*

$$r_1\mathrm{e}^{\mathrm{i}u_1}\mathrm{e}_{-k}+r_2\mathrm{e}^{\mathrm{i}u_2}\mathrm{e}_0+r_3\mathrm{e}^{\mathrm{i}u_3}\mathrm{e}_l \tag{31}$$

of the normed space $\mathrm{C}_{\{-k,0,l\}}$ on

$$r_1\mathrm{e}^{\mathrm{i}u_1}\mathrm{e}_{-k}+r_2\mathrm{e}^{\mathrm{i}(t+u_2)}\mathrm{e}_0+r_3\mathrm{e}^{\mathrm{i}u_3}\mathrm{e}_l.$$

Then M has norm $\cos\bigl(\pi/2(k+l)-t/2\bigr)/\cos\bigl(\pi/2(k+l)\bigr)$ and attains its norm exactly at elements of form (31) with $r_1:r_2:r_3=l:k+l:k$ and

$$-lu_1+(k+l)u_2-ku_3=\pi \mod 2\pi.$$

Proof. This follows from Proposition 9.3 and the concavity of cos on $[0,\pi/2]$. ☐

Remark 10.2. This corollary enables us to guess how to lift M to an operator that acts by convolution with a measure μ. Note that μ is a Hahn-Banach extension of the linear form $f\mapsto Mf(0)$. The relative multiplier M is an isometry if and only if $t=0$ and μ is the Dirac measure in 0. Otherwise, $t\neq 0$; the proof of Theorem 1.2 (*a*) in Section 8 shows that μ is a linear combination $\alpha\delta_y+\beta\delta_w$ of two Dirac measures such that the norm of M is $|\alpha|+|\beta|$. Let $f(x)=l\mathrm{e}^{-\mathrm{i}kx}+(k+l)\mathrm{e}^{\mathrm{i}\pi/(k+l)}+k\mathrm{e}^{\mathrm{i}lx}$: M attains its norm at f, f attains its maximum modulus at 0 and $2m\pi/(k+l)$, and Mf attains its maximum modulus at $2m\pi/(k+l)$, where m is the inverse of l modulo $k+l$. As

$$\begin{aligned}(|\alpha|+|\beta|)\max_x|f(x)| &= \max_x|Mf(x)|\\ &= \left|\mu * f\bigl(2m\pi/(k+l)\bigr)\right|\\ &= \left|\alpha f\bigl(2m\pi/(k+l)-y\bigr)+\beta f\bigl(2m\pi/(k+l)-w\bigr)\right|,\end{aligned}$$

we must choose $\{y,w\}=\{0,2m\pi/(k+l)\}$. A computation yields then

$$\mu=\mathrm{e}^{\mathrm{i}t/2}\frac{\sin\bigl(\pi/(k+l)-t/2\bigr)}{\sin\bigl(\pi/(k+l)\bigr)}\delta_0+\mathrm{e}^{\mathrm{i}(t/2+\pi/(k+l))}\frac{\sin(t/2)}{\sin\bigl(\pi/(k+l)\bigr)}\delta_{2m\pi/(k+l)}.$$

If $k=l=1$, this is a special case of the formula appearing in [10, proof of Prop. 1]. Consult [26] on this issue.

11 The Sidon constant of integer sets

Let us study the maximum modulus of a trigonometric trinomial with given spectrum and Fourier coefficient moduli sum. We get the following result as an immediate consequence of Corollary 9.4.

Proposition 11.1. *Let k and l be two positive coprime integers. Let r_1, r_2 and r_3 be three positive real numbers. Let $t \in [0, \pi/(k+l)]$. Then*

$$\max_x \left| r_1 \mathrm{e}^{-\mathrm{i}kx} + r_2 \mathrm{e}^{\mathrm{i}t} + r_3 \mathrm{e}^{\mathrm{i}lx} \right| \geqslant \cos\bigl(\pi/2(k+l)\bigr) \cdot (r_1 + r_2 + r_3)$$

with equality if and only if $r_1 : r_2 : r_3 = l : k+l : k$ and $t = \pi/(k+l)$.

This means that the Sidon constant of $\{-k, 0, l\}$ equals $\sec\bigl(\pi/2(k+l)\bigr)$.
The Sidon constant of integer sets was previously known only in the following three instances:

- The equality
$$\max_x \left| r_1 \mathrm{e}^{\mathrm{i}(t_1+\lambda_1 x)} + r_2 \mathrm{e}^{\mathrm{i}(t_2+\lambda_2 x)} \right| = r_1 + r_2$$
shows that the Sidon constant of sets with one or two elements is 1.

- The Sidon constant of $\{-1, 0, 1\}$ is $\sqrt{2}$ and it is attained for $\mathrm{e}_{-1} + 2\mathrm{i} + \mathrm{e}_1$. Let us give the original argument: if $f(x) = \left| r_1 \mathrm{e}^{-\mathrm{i}x} + r_2 \mathrm{e}^{\mathrm{i}t} + r_3 \mathrm{e}^{\mathrm{i}x} \right|^2$, the parallelogram identity and the arithmetic-quadratic inequality yield
$$\begin{aligned}\max_x f(x) &\geqslant \max_x \frac{f(x) + f(x+\pi)}{2}\\ &= \max_x \frac{\left| r_1 \mathrm{e}^{-\mathrm{i}x} + r_3 \mathrm{e}^{\mathrm{i}x} + r_2 \mathrm{e}^{\mathrm{i}t} \right|^2 + \left| r_1 \mathrm{e}^{-\mathrm{i}x} + r_3 \mathrm{e}^{\mathrm{i}x} - r_2 \mathrm{e}^{\mathrm{i}t} \right|^2}{2}\\ &= \max_x \left| r_1 \mathrm{e}^{-\mathrm{i}x} + r_3 \mathrm{e}^{\mathrm{i}x} \right|^2 + \left| r_2 \mathrm{e}^{\mathrm{i}t} \right|^2\\ &= (r_1 + r_3)^2 + r_2^2 \geqslant \frac{(r_1 + r_2 + r_3)^2}{2}.\end{aligned}$$

- The Sidon constant of $\{0, 1, 2, 3, 4\}$ is 2 and it is attained for $1 + 2\mathrm{e}_1 + 2\mathrm{e}_2 - 2\mathrm{e}_3 + \mathrm{e}_4$.

These results were obtained by D. J. Newman (see [25].) The fact that the Sidon constant of sets of three integers cannot be 1 had been noted with pairwise different proofs in [25, 3, 13].

Remark 11.2. The real algebraic counterpart is better understood: the maximal absolute value of a real algebraic polynomial of degree at most n with given coefficient absolute value sum is minimal for multiples of the nth Chebyshev polynomial T_n (look up the last paragraph of [8], and [21, Theorem 16.3.3] for a proof). As the sum of the absolute values of T_n's coefficients is the integer t_n nearest to $(1+\sqrt{2})^n/2$, we have for real $a_0, a_1, \dots, a_n$

$$\max_{x \in [-1,1]} \left| a_0 + a_1 x + \dots + a_n x^n \right| \geqslant t_n^{-1} (|a_0| + |a_1| + \dots + |a_n|).$$

The following estimates for the Sidon constant of large integer sets are known.

- E. Beller and D. J. Newman [2] showed that the Sidon constant of $\{0, 1, \dots, n\}$ is equivalent to $\sqrt{n}$.

- (Hadamard sets.) Let $q > 1$ and suppose that the sequence $(\lambda_j)_{j \geqslant 1}$ grows with geometric ratio q: $|\lambda_{j+1}| \geqslant q |\lambda_j|$ for every j. Then the Sidon constant of $\{\lambda_1, \lambda_2, \dots\}$ is finite; it is at most 4.27 if $q \geqslant 2$ (see [11]), at most 2 if $q \geqslant 3$ (see [15]), and at most $1 + \pi^2/\left(2q^2 - 2 - \pi^2\right)$ if $q > \sqrt{1 + \pi^2/2}$ (see [18, Corollary 9.4] or the updated [19, Corollary 10.2.1].)

Our computations show that the last estimate of the Sidon constant has the right order in q^{-1} for geometric progressions.

Proposition 11.3. *Let C be the Sidon constant of the geometric progression $\{1, q, q^2, \dots\}$, where $q \geqslant 3$ is an integer. Then*

$$1 + \pi^2/8(q+1)^2 \leqslant \sec\bigl(\pi/2(q+1)\bigr) \leqslant C \leqslant 1 + \pi^2/\left(2q^2 - 2 - \pi^2\right).$$

One initial motivation for this work was to decide whether there are sets $\{\lambda_j\}_{j\geqslant 1}$ with $|\lambda_{j+1}| \geqslant q|\lambda_j|$ whose Sidon constant is arbitrarily close to 1 and to find evidence among sets with three elements. That there are such sets, arbitrarily large albeit finite, may in fact be proved by the method of Riesz products in [12, Appendix V, §1.II]; see also [19, Proposition 13.1.3]. The case of infinite sets remains open.

A second motivation was to show that the real and complex unconditional constants of the basis $(\mathrm{e}_{\lambda_1}, \mathrm{e}_{\lambda_2}, \mathrm{e}_{\lambda_3})$ of C_Λ are different; we prove however that they coincide, and it remains an open question whether they may be different for larger sets. The *real unconditional constant* of $(\mathrm{e}_{\lambda_1}, \mathrm{e}_{\lambda_2}, \mathrm{e}_{\lambda_3})$ is the maximum of the norm of the eight unimodular relative Fourier multipliers (t_1, t_2, t_3) such that $t_k = 0$ modulo π. Let i, j, k be a permutation of $1, 2, 3$ such that the power of 2 in $\lambda_i - \lambda_k$ and in $\lambda_j - \lambda_k$ are equal. Lemma 2.1 shows that the four relative multipliers satisfying $t_i = t_j$ modulo 2π are isometries and that the norm of any of the four others, satisfying $t_i \neq t_j$ modulo 2π, gives the real unconditional constant. In general, the complex unconditional constant is bounded by $\pi/2$ times the real unconditional constant, as proved in [24]; in our case, they are equal.

Corollary 11.4. *The complex unconditional constant of the basis* $(\mathrm{e}_{\lambda_1}, \mathrm{e}_{\lambda_2}, \mathrm{e}_{\lambda_3})$ *of* C_Λ *is equal to its real unconditional constant.*

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