Ideal objects

Dynamical algebra arises as a reflection on the computational content of ideal objects. There is a huge philosophical reflection on ideal objects that I do not wish to enter. I rather see this course as a particularly perspicuous analysis of two ideal objects that should trigger people to think about their conception of ideal objects in general. The very reason for this is again philosophical: it is a matter of fact that people have very different views on the principles of mathematics, i.e on their beginnings, roots in our understanding of things. For me they are rooted in life itself, and for me it is important that these roots themselves are alive, that the fluids necessary for blooming are circulating from them and back to them. But there are much more pragmatic views. The question is how much of a private issue this is, and how much one ought to speak about it or rather mute it out.

For me, to speak about it is definitely part of the mathematical business, and it is a part I like to call philosophy and of which I wish to stress the proficiency.

Let us get back to ideal objects. The most simple object that has come to my mind is the following: the minimal element of a set of integers (≥ 0). Why is it ideal? In the sense that there is no effective way of getting it if I am given a set of integers that is infinite. One may think that this means that it is meaningless, just by expressing that a meaning is a way of effectively accessing it. Are there weaker forms of meaning than this one? The one we shall be particularly interested in is that we know how to disprove that an element is minimal: by exhibiting a still smaller one. The dynamical point of view is then to make sense out of a potentially minimal element, i.e. an element that is regarded as minimal not in an effective way (with a proof that all elements are bigger than this one) but in a provisional way (with an expression of the situation in which it is considered as minimal and with an expression of the situation in which it turns out that it is not minimal, or an expression of how our previous knowledge is affected by the discovery of a still smaller element).

I have said enough to ring many bells in the head of certain people present here, and these bells have names like Kripke semantics, backtracking, game semantics, sheaf semantics, toposes. Those people are able to spell out everything I say in such terms, and it is the ambition of notes to come to do that, but it is not the aim of this course, which tries to put in motion philosophical reflections about simple historically relevant situation. Before doing so, I want to give an application of this simplest example, well known to many. You may have seen the definition of the gcd of the integers a and b as the minimal element $d>0$ among those that have the form $ua+vb$ with u and v integers, and here there is absent an effective way of getting it, because the plain meaning of this is to consider the elements >0 among $a, b, a+b, a-b, -a+b, -a-b, 2a, 2b, -2a, -2b, 2a+b, a+2b, 2a-b, a-2b, -2a+b, -a+2b, -2a-b, -a-2b$, etc. and to pretend that one of those is d minimal. Let us look at the proof that d is a gcd: suppose that d does not divide a: then the remainder of the euclidean division of a by d has the form $a-qd$, i.e. $a-q(ua+bv)$, i.e. $(1-qu)a-qvb$, and is smaller than d. The same argument works if d does not divide b. We have here an example of a typical scheme: the use of an ideal object in an argument by contradiction. The ideality of the element is spelled out by the argument by contradiction; its feature is that its ideality makes it win beforehand against every competitor. The dynamical counterpart to this ideal object is to start with any $d>0$ of the form $ua+vb$, e.g. b. If it divides a and b, we are done because every divisor of both a and b divides d. Otherwise, the argument above shows how to obtain a new d which is smaller than the former one, which is still of the form $ua+vb$. This change of d can only happen a finite number of times because a decreasing sequence of integers stops. This means that one gets a d such that $d \mid a$; $d \mid b$; $d=ua+vb$. Some more reflections on the proof: it is a remarkable fact to have found the form $ua+vb$ as the important invariant when turning from a provisional minimum to a new provisional minimum. This is in fact the main point in the ideal object we have considered, and the dynamical method explains its relevance and connects it with the proof of correction of the algorithm.