Krull's fundamentalsatz for integral domains.

Let I be an integral domain, K its field of fractions, $G=K^*$ the multiplicative group.

think of $I=\mathbb{Z}$, $K=\mathbb{Q}$, $G=\mathbb{Q}^*$. The main question is: when is $\alpha\in G$ divisible by $\beta\in G$, i.e. $\alpha\cdot\beta^{-1}\in I$? For $\mathbb{Z}$ we have the answer that this holds iff for every prime number p the value of α at p is at least as great as that of β, where the value at p is the exponent of p in the prime factor decomposition.

- Valuations(1) and valuation rings(2) are the concepts ideated by Krull to play the role of prime numbers for general I. Characterisation: A proper subring B of K is a valuation ring if its noninvertible elements form an ideal (i.e. it is a local ring) such that every proper superring B' (i.e. $B\subsetneq B'\subseteq K$) contains the inverse of a noninvertible of B. What is hoped for is that I is the intersection of all valuation rings containing I
  - For $\mathbb{Z}$, the valuation ring corresponding to the prime number p is the ring of those fractions that can be written with a denominator not divisible by p.
  - In general this intersection is greater*. Krull's fundamental satz states that it is equal to I if and only if I is integrally closed. This means that every element $a\in K$ satisfying the relation of integral dependence $a^n+a_1a^{n-1}+\dots+a_n=0$ with $a_i\in I$ is actually in I. This follows from:

    * a valuation ring is integrally closed: the superring of integrally dependent elements must coincide with the valuation ring because if it did contain the inverse of an element a of B, we would have $a^{-n}+a_1a^{-n+1}+\dots+a_n=0$ and $a(-a_1-a_2a-\dots-a_na^{n-1})=1$ and a would be invertible in B.

    ** if $a\in G\setminus I$ with I integrally closed, then a^{-1} is a noninvertible of $I[a^{-1}]$. By Zorn's lemma, consider a maximal subring B of K containing $I[a^{-1}]$ but not a. This is a valuation ring.

In Lorenzen 1950, he proposes the following new construction: "We ask ourselves under what circumstance $a\in B$ holds for every valuation superring B of I. For arbitrary $z\in G$ and any valuation superring B of I, it always holds that $z\in B$ or $z^{-1}\in B$. Therefore, whenever $a\in I[z]$ or $a\in I[z^{-1}]$ holds, so does $a\in B$. Likewise follows: if there are elements $z_1,\dots,z_n\in G$ for which $a\in I[z_1^{\pm1},\dots,z_n^{\pm1}]$ holds for each of the 2^n combinations of signs, then, for every valuation superring B of I, it holds that $a\in B$. The condition is also necessary. Namely, if it is not fulfilled for any $z_1,\dots,z_n$, then a simple well-ordering argument shows that there is a maximal superring $\bar{I}$ of I with the property that, for all $z_1,\dots,z_n\in G$, $a\notin\bar{I}[z_1^{\pm1},\dots,z_n^{\pm1}]$ holds for at least one combination of signs. Therefore in particular $a\notin\bar{I}$. If this $\bar{I}$ were not a valuation ring, then there would be a $z\in G$ with $z\notin\bar{I}$ and $z^{-1}\notin\bar{I}$, therefore there would be elements $x_1,\dots,x_m,y_1,\dots,y_n$ with $a\in\bar{I}[z][x_1^{\pm1},\dots,x_m^{\pm1}]$, $a\in\bar{I}[z^{-1}][y_1^{\pm1},\dots,y_n^{\pm1}]$ for each combination of signs. But from this, $a\in\bar{I}[z^{\pm1},x_1^{\pm1},\dots,x_m^{\pm1},y_1^{\pm1},\dots,y_n^{\pm1}]$ would follow, i.e. a contradiction. Therefore $\bar{I}$ is a valuation ring.

Krull's fundamentalsatz becomes a characterisation of an intersection of valuation rings as an integral domain in which behaving as if it were linearly preordered, i.e. assuming that certain pairs of elements $\{a_1, a'_1\}, \dots, \{a_i, a'_i\}$ enjoy a ~~relation of divisibility [illegible]~~ add new relations of divisibility. The formulation of this "as if" goes as follows: it is the simultaneous consideration of 2^i different relations of divisibility among i pairs of elements

(1) a valuation w of K is defined as a function of G into an abelian linearly preordered group Γ (2) such that it is multiplicative and $w(a+b) \geq \min(w(a), w(b))$. The corresponding valuation ring is the ring formed by 0 together with the elements that have valuation ≥ 0. A valuation ring is characterised by the fact that its field of fractions is K and that it is linearly ordered by divisibility. One has for every $z \in G$ $\underset{z \in B}{w(z) \geq 0}$ or $\underset{z^{-1} \in B}{w(z) \leq 0}$

of K, each generating the integral domain in which these relations are forced, and the statement that the intersection of these integral domains coincides with the integral domain we have started with.

A citation from Blaise Pascal's Thoughts (Sellier 230) by Lorenzen in [illegible].

« Thus we see that all sciences are infinite in the range of their researches, for who can doubt that mathematics, for instance, has an infinity of infinities of propositions to expound? They are infinite also in the multiplicity and subtlety of their principles, for anyone can see that those which are supposed to be ultimate do not stand by themselves, but depend on others again, and thus never allow any finality. »