The philosophy of dynamical algebra: days 2 and 3

We have had a thorough look into Krull's theory of valuations and Lorenzen's analysis of it. Lorenzen shows that

$$a \in B \text{ for every valuation ring } B \supseteq I \iff \begin{array}{l}\text{there are } z_1,\dots,z_n \in G \text{ such that} \\ a \in I[z_1^{\pm 1},\dots,z_n^{\pm 1}] \text{ for each choice of signs}\end{array}$$

$$\iff a \text{ is integrally dependent on } I$$

Note also the following: if a is integrally dependent on $I[z]$ and $I[z^{-1}]$, then a is integrally dependent on I: one has $1 \in \langle a^{-1}, a^{-1}z, \dots, a^{-1}z^{n}\rangle_{I[a^{-1}]}$ and $1 \in \langle a^{-1}, a^{-1}z^{-1}, \dots, a^{-1}z^{-n}\rangle_{I[a^{-1}]}$, i.e. $\langle a^{-1}z^{-n}, \dots, a^{-1}, \dots, a^{-1}z^{n}\rangle_{I[a^{-1}]} \ni z^k$ for $k=-n,\dots,n$, i.e. there is a matrix with coefficients in $a^{-1}I[a^{-1}]$ such that $M\begin{pmatrix} z^{-n} \\ \vdots \\ z^{n}\end{pmatrix} = \begin{pmatrix} z^{-n} \\ \vdots \\ z^{n}\end{pmatrix}$, i.e. $\underbrace{(\mathrm{Id}-M)}_{1 \in a^{-1}I[a^{-1}]}\begin{pmatrix} z^{-n} \\ \vdots \\ z^{n}\end{pmatrix} = 0$. Multiply $\mathrm{Id}-M$ by the matrix of its cofactors and expand: you get 0, i.e. a relation of integral dependence for a ("determinant trick")

What do these two results mean?

The first means that the computational content of being in the intersection of valuation superrings of I is exactly that one can do as if it was linearly ordered: you can at any point for any given pair of elements u, v of G make the case distinction $u \mid v$ / $v \mid u$; if you obtain a conclusion in each case, it will hold altogether. This "as if" not only targets that it is not really linearly ordered; it targets also that you are not really making case distinctions: the second result shows how the case distinction may be removed by gluing together computations made in each case into a computation without case distinction.

The dynamical method in algebra proposes to make this precise by a description of the computations, so that the nature of the theory, typically its being geometric, grants the possibility of case distinctions and of gluing the computations made in each case together: Lorenzen's analysis does all this by hand and arrives at an exposition of the theory of divisibility without valuations: in this sense, it goes further than dynamical algebra, which gives a constructive meaning to classical reasoning and has the aim of revealing the constructive nature of classical algebra.