The Sidon constant of sets with three elements

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Abstract

We solve an elementary extremal problem on trigonometric polynomials and obtain the exact value of the Sidon constant for sets with three elements $\{n_0, n_1, n_2\}$: it is

$$\sec\left(\pi \gcd(n_1 - n_0, n_2 - n_0)/2\max|n_i - n_j|\right).$$

1 Introduction

Let $\Lambda = \{\lambda_0, \lambda_1, \lambda_2\}$ be a set of three frequencies and $\varrho_0, \varrho_1, \varrho_2$ three positive intensities. We solve the following extremal problem:

(†) To find $\vartheta_0, \vartheta_1, \vartheta_2$ three phases such that, putting $c_j = \varrho_j \mathrm{e}^{\mathrm{i}\vartheta_j}$, the maximum $\max_t |c_0\mathrm{e}^{\mathrm{i}\lambda_0 t} + c_1\mathrm{e}^{\mathrm{i}\lambda_1 t} + c_2\mathrm{e}^{\mathrm{i}\lambda_2 t}|$ is minimal.

This enables us to generalise a result of D. J. Newman. He solved the following extremal problem for $\Lambda = \{0, 1, 2\}$:

(‡) To find $f(t) = c_0\mathrm{e}^{\mathrm{i}\lambda_0 t} + c_1\mathrm{e}^{\mathrm{i}\lambda_1 t} + c_2\mathrm{e}^{\mathrm{i}\lambda_2 t}$ with $\|f\|_\infty = \max_t |f(t)| \leqslant 1$ such that $\|\widehat{f}\|_1 = |c_0| + |c_1| + |c_2|$ is maximal.

Note that for such an f, $\|\widehat{f}\|_1$ is the Sidon constant of Λ. Newman's argument is the following (see [6, Chapter 3]): by the parallelogram law,

$$\begin{aligned}
\max_t |f(t)|^2 &= \max_t |f(t)|^2 \vee |f(t+\pi)|^2 \\
&\geqslant \max_t \big(|f(t)|^2 + |f(t+\pi)|^2\big)/2 \\
&= \max_t \big(|c_0 + c_1\mathrm{e}^{\mathrm{i}t} + c_2\mathrm{e}^{\mathrm{i}2t}|^2 + |c_0 - c_1\mathrm{e}^{\mathrm{i}t} + c_2\mathrm{e}^{\mathrm{i}2t}|^2\big)/2 \\
&= \max_t |c_0 + c_2\mathrm{e}^{\mathrm{i}2t}|^2 + |c_1|^2 = \big(|c_0| + |c_2|\big)^2 + |c_1|^2 \\
&\geqslant \big(|c_0| + |c_1| + |c_2|\big)^2/2
\end{aligned}$$

and equality holds exactly for multiples and translates of $f(t) = 1 + 2\mathrm{i}\,\mathrm{e}^{\mathrm{i}t} + \mathrm{e}^{\mathrm{i}2t}$.

Let us describe this paper briefly. We use a real-variable approach: Problem (†) reduces to studying a function of form

$$\Phi(t, \vartheta) = |1 + r\mathrm{e}^{\mathrm{i}\vartheta}\mathrm{e}^{\mathrm{i}kt} + s\mathrm{e}^{\mathrm{i}lt}|^2 \text{ for } r, s > 0,\ k \neq l \in \mathbb{Z}^*$$

and more precisely $\Phi^*(\vartheta) = \max_t \Phi(t, \vartheta)$. We obtain the variations of Φ^*: the point is that we find "by hand" a local minimum of Φ^* and that any two minima of Φ^* are separated by a maximum of Φ^*, which corresponds to an extremal point of Φ and therefore has a handy description. The solution to Problem (‡) then turns out to derive easily from this.

The initial motivation was twofold. In the first place, we wanted to decide whether sets $\Lambda = \{\lambda_n\}$ such that λ_{n+1}/λ_n is bounded by some q may have a Sidon constant arbitrarily close to 1 and to find evidence among sets with three elements. That there are such sets, arbitrarily large but finite, may in fact be proven by the method of Riesz products in [2, Appendix V, §1.II]. In the second place, we wished to show that the real and complex unconditionality constants are distinct for basic sequences of characters $\mathrm{e}^{\mathrm{i}nt}$; we prove however that they coincide in the space $\mathscr{C}(\mathbb{T})$ for sequences with three terms.

Notation. $\mathbb{T} = \{\, z \in \mathbb{C} : |z| = 1 \,\}$ and $\mathrm{e}_\lambda(z) = z^\lambda$ for $z \in \mathbb{T}$ and $\lambda \in \mathbb{Z}$.

2 Definitions

Definition 2.1. (1) Let $\Lambda \subseteq \mathbb{Z}$. Λ is a Sidon set if there is a constant C such that for all trigonometric polynomials $f(t) = \sum_{\lambda\in\Lambda} c_\lambda \mathrm{e}^{\mathrm{i}\lambda t}$ with spectrum in Λ we have

$$\|\widehat{f}\|_1 = \sum_{\lambda\in\Lambda} |c_\lambda| \leqslant C \max_t |f(t)| = \|f\|_\infty.$$

The optimal C is called the Sidon constant of Λ.

(2) Let X be a Banach space. The sequence $(x_n) \subseteq X$ is a real (vs. complex) unconditional basic sequence in X if there is a constant C such that

$$\Big\|\sum \vartheta_n c_n x_n\Big\|_X \leqslant C \Big\|\sum c_n x_n\Big\|_X$$

for every real (vs. complex) choice of signs $\vartheta_n \in \{-1,1\}$ (vs. $\vartheta_n \in \mathbb{T}$) and every finitely supported family of coefficients (c_n). The optimal C is the real (vs. complex) unconditionality constant of (x_n) in X.

Let us state the two following well known facts.

Proposition 2.2. (1) *The Sidon constant of Λ is the complex unconditionality constant of the sequence of functions $(\mathrm{e}_\lambda)_{\lambda\in\Lambda}$ in the space $\mathscr{C}(\mathbb{T})$.*

(2) *The complex unconditionality constant is at most $\pi/2$ times the real unconditionality constant.*

Proof. (1) holds because $\big\|\sum \vartheta_\lambda c_\lambda \mathrm{e}_\lambda\big\|_\infty = \sum |c_\lambda|$ for $\vartheta_\lambda = \overline{c_\lambda}/|c_\lambda|$.

(2) Because the complex unconditionality constant of the sequence (ϵ_n) of Rademacher functions in $\mathscr{C}(\{-1,1\}^\infty)$ is $\pi/2$ (see [5]),

$$\begin{aligned}\sup_{\vartheta_n\in\mathbb{T}} \Big\|\sum \vartheta_n c_n x_n\Big\|_X &= \sup_{x^*\in B_{X^*}} \sup_{\vartheta_n\in\mathbb{T}} \sup_{\epsilon_n=\pm1} \Big|\sum \vartheta_n c_n \langle x^*, x_n\rangle \epsilon_n\Big| \\ &\leqslant \pi/2 \sup_{x^*\in B_{X^*}} \sup_{\epsilon_n=\pm1} \Big|\sum c_n \langle x^*, x_n\rangle \epsilon_n\Big| \\ &= \pi/2 \sup_{\epsilon_n=\pm1} \Big\|\sum \epsilon_n c_n x_n\Big\|_X.\end{aligned}$$

Furthermore the real unconditionality constant of (ϵ_n) in $\mathscr{C}(\{-1,1\}^\infty)$ is 1: therefore the factor $\pi/2$ is optimal. ☐

Let us straighten out the expression of the Sidon constant. For

$$f(t) = c_0 \mathrm{e}^{\mathrm{i}\lambda_0 t} + c_1 \mathrm{e}^{\mathrm{i}\lambda_1 t} + c_2 \mathrm{e}^{\mathrm{i}\lambda_2 t}, \ c_j = \varrho_j \mathrm{e}^{\mathrm{i}\vartheta_j},$$

the supremum norm $\|f\|_\infty$ of f is equal to

$$\|\varrho_0 + \varrho_1 \mathrm{e}^{\mathrm{i}\vartheta} \mathrm{e}_{\lambda_1-\lambda_0} + \varrho_2 \mathrm{e}_{\lambda_2-\lambda_0}\|_\infty, \ \vartheta = \frac{\lambda_1-\lambda_2}{\lambda_2-\lambda_0}\vartheta_0 + \vartheta_1 + \frac{\lambda_0-\lambda_1}{\lambda_2-\lambda_0}\vartheta_2 \tag{1}$$

and therefore the Sidon constant C of $\Lambda = \{\lambda_0, \lambda_1, \lambda_2\}$ may be written

$$C = \max_{r,s>0,\vartheta} (1+r+s)/\|1 + r\mathrm{e}^{\mathrm{i}\vartheta}\mathrm{e}_k + s\mathrm{e}_l\|_\infty \quad \text{with } \begin{cases} k = \lambda_1 - \lambda_0 \\ l = \lambda_2 - \lambda_0. \end{cases} \tag{2}$$

By change of variables, we may suppose w.l.o.g. that k and l are coprime.

3 A solution to Extremal problem (†)

Let us first establish

Lemma 3.1. *Let* $\lambda_1,\ldots,\lambda_k\in\mathbb{Z}^*$ *and* $\varrho_1,\ldots,\varrho_k>0$. *Let*

$$f(t,\vartheta)=1+\varrho_1\mathrm{e}^{\mathrm{i}(\lambda_1t+\vartheta_1)}+\cdots+\varrho_{k-1}\mathrm{e}^{\mathrm{i}(\lambda_{k-1}t+\vartheta_{k-1})}+\varrho_k\mathrm{e}^{\mathrm{i}\lambda_kt}$$

and $\Phi(t,\vartheta)=|f(t,\vartheta)|^2$. *The critical points* (t,ϑ) *such that* $\nabla\Phi(t,\vartheta)=0$ *satisfy either* $f(t,\vartheta)=0$ *or* $\lambda_1t+\vartheta_1\equiv\cdots\equiv\lambda_{k-1}t+\vartheta_{k-1}\equiv\lambda_kt\equiv0\mod\pi$.

Proof. As $\Phi=(\Re f)^2+(\Im f)^2$, the critical points (t,ϑ) satisfy

$$\begin{cases}\Re\frac{\partial f}{\partial t}(t,\vartheta)\,\Re f(t,\vartheta)+\Im\frac{\partial f}{\partial t}(t,\vartheta)\,\Im f(t,\vartheta)=0\\-\sin(\lambda_it+\vartheta_i)\,\Re f(t,\vartheta)+\cos(\lambda_it+\vartheta_i)\,\Im f(t,\vartheta)=0\quad(1\leqslant i\leqslant k-1),\end{cases}$$

which simplifies to

$$-\sin(\lambda_it+\vartheta_i)\,\Re f(t,\vartheta)+\cos(\lambda_it+\vartheta_i)\,\Im f(t,\vartheta)=0\quad(1\leqslant i\leqslant k,\ \vartheta_k=0).$$

Suppose that $f(t,\vartheta)\neq0$: then the system above implies that

$$-\sin(\lambda_it+\vartheta_i)\cos(\lambda_jt+\vartheta_j)+\cos(\lambda_it+\vartheta_i)\sin(\lambda_jt+\vartheta_j)=0\ (1\leqslant i,j\leqslant k,\ \vartheta_k=0)$$

and it simplifies therefore to

$$\sin(\lambda_it+\vartheta_i)=0\quad(1\leqslant i\leqslant k,\ \vartheta_k=0).$$ □

The following result is the core of the paper.

Lemma 3.2. *Let* $r,s>0$, $k,l\in\mathbb{Z}^*$ *distinct and coprime. Let*

$$\begin{aligned}\Phi(t,\vartheta)&=|1+r\mathrm{e}^{\mathrm{i}\vartheta}\mathrm{e}^{\mathrm{i}kt}+s\mathrm{e}^{\mathrm{i}lt}|^2\\&=1+r^2+s^2+2r\cos(kt+\vartheta)+2s\cos lt+2rs\cos((l-k)t-\vartheta).\end{aligned}$$

Let $\Phi^*(\vartheta)=\max_t\Phi(t,\vartheta)$. *Then* Φ^* *is an even function with period* $2\pi/|l|$ *that decreases on* $[0,\pi/|l|]$. *Therefore* $\min_\vartheta\Phi^*(\vartheta)=\Phi^*(\pi/l)$.

Proof. Φ^* is continuous (see [4, Chapter 5.4]) and even, as $\Phi(t,-\vartheta)=\Phi(-t,\vartheta)$. Φ^* is $(2\pi/|l|)$-periodical: let $j\in\mathbb{Z}$ be such that $jk\equiv1$ mod. l. Then

$$\Phi(t+2j\pi/l,\vartheta)=|1+r\mathrm{e}^{\mathrm{i}(\vartheta+2\pi jk/l)}\mathrm{e}^{\mathrm{i}kt}+s\mathrm{e}^{\mathrm{i}lt}|^2=\Phi(t,\vartheta+2\pi/l).$$

Thus Φ^* attains its minimum on $[0,\pi/|l|]$. Furthermore, we have

$$\Phi(-t-2j\pi/l,\pi/l-\vartheta)=\Phi(t+2j\pi/l,-\pi/l+\vartheta)=\Phi(t,\pi/l+\vartheta),$$

so that Φ^* has an extremum at π/l. Now

$$\Phi^*(\pi/l+\vartheta)=\Phi^*(\pi/l)+|\vartheta|\max_{\Phi(t,\pi/l)=\Phi^*(\pi/l)}\left|\frac{\partial\Phi}{\partial\vartheta}(t,\pi/l)\right|+o(\vartheta).$$

Choose a t such that $\Phi(t,\pi/l)=\Phi^*(\pi/l)$. If $\partial\Phi/\partial\vartheta(t,\pi/l)\neq0$, then this shows that Φ^* has a local minimum and a cusp at π/l. Let us now suppose that $\partial\Phi/\partial\vartheta(t,\pi/l)=0$. If Φ^* had a local maximum at π/l, then $(t,\pi/l)$ would be a critical point of Φ, so that by Lemma 3.1 $\cos(kt+\pi/l)=\delta$, $\cos lt=\epsilon$, $\cos((l-k)t-\pi/l)=\delta\epsilon$ for some $\delta,\epsilon\in\{-1,1\}$. One necessarily would have $(\delta,\epsilon)\neq(1,1)$. Furthermore,

$$\begin{aligned}\frac{\partial^2\Phi}{\partial\vartheta^2}(t,\pi/l)&=-2r\delta(1+s\epsilon)\leqslant0\\\begin{vmatrix}\partial^2\Phi/\partial t^2&\partial^2\Phi/\partial t\partial\vartheta\\\partial^2\Phi/\partial\vartheta\partial t&\partial^2\Phi/\partial\vartheta^2\end{vmatrix}(t,\pi/l)&=4rsl^2(\delta\epsilon+r\epsilon+s\delta)\geqslant0,\end{aligned}$$

which would imply $\epsilon = -1$, $r = 0$, $s = 1$. Therefore Φ^* has a local minimum at π/l. Let us show that then Φ^* must decrease on $[0, \pi/|l|]$. Otherwise there are $0 \leqslant \vartheta_0 < \vartheta_1 \leqslant \pi/|l|$ such that $\Phi^*(\vartheta_1) > \Phi^*(\vartheta_0)$. As $\pi/|l|$ is a local minimum, there is a $\vartheta_0 < \vartheta^* < \pi/|l|$ such that

$$\Phi^*(\vartheta^*) = \max_{\vartheta_0 \leqslant \vartheta \leqslant \pi/|l|} \Phi^*(\vartheta) = \max_{\substack{0 \leqslant t < 2\pi \\ \vartheta_0 \leqslant \vartheta \leqslant \pi/|l|}} \Phi(t, \vartheta),$$

i.e., there further is some t^* such that Φ has a local maximum at (t^*, ϑ^*). But then $kt^* + \vartheta^* \equiv lt^* \equiv 0 \mod \pi$ and $\vartheta^* \equiv 0 \mod \pi/l$ and this is false. □

By Computation (1) and Lemma 3.2, we obtain

Theorem 3.3. *Let $\lambda_0, \lambda_1, \lambda_2 \in \mathbb{R}$ and $\varrho_0, \varrho_1, \varrho_2 > 0$. The solution to Extremal problem (†) is the following.*

- *If the smallest additive group containing $\lambda_1 - \lambda_0$ and $\lambda_2 - \lambda_0$ is dense in $\mathbb{R}$, then the maximum is independent of the phases $\vartheta_0, \vartheta_1, \vartheta_2$ and makes $\varrho_0 + \varrho_1 + \varrho_2$.*
- *Otherwise let $d = \gcd(\lambda_1 - \lambda_0, \lambda_2 - \lambda_0)$ be a generator of this group. Then the sought phases $\vartheta_0, \vartheta_1, \vartheta_2$ are given by*

$$\vartheta_0(\lambda_2 - \lambda_1) + \vartheta_1(\lambda_0 - \lambda_2) + \vartheta_2(\lambda_1 - \lambda_0) \equiv d\pi \mod 2d\pi.$$

In particular, these phases may be chosen among 0 and π.

4 A solution to Extremal problem (‡)

There are two cases where one can make explicit computations by Lemma 3.2.

Example 4.1. The real and complex unconditionality constant of $\{0, 1, 2\}$ in $\mathscr{C}(\mathbb{T})$ is $\sqrt{2}$. Indeed, a case study shows that

$$\|1 + \mathrm{i}r\mathrm{e}_1 + s\mathrm{e}_2\|_\infty = \begin{cases} r + |s-1| & \text{if } r|s-1| \geqslant 4s \\ (1+s)(1 + r^2/4s)^{1/2} & \text{if } r|s-1| \leqslant 4s \end{cases}$$

and this permits to compute the maximum (2), which is obtained for $r = 2$, $s = 1$. This yields another proof to Newman's result presented in the Introduction.

Example 4.2. The real and complex unconditionality constant of $\{0, 1, 3\}$ in $\mathscr{C}(\mathbb{T})$ is $2/\sqrt{3}$. Indeed, a case study shows that $\|1 + r\mathrm{e}^{\mathrm{i}\pi/3}\mathrm{e}_1 + s\mathrm{e}_3\|_\infty$ makes

$$\begin{cases} 1 + r - s & \text{if } s \leqslant r/(4r+9) \\ \left(\frac{2}{27}s(r^2 + 9 + 3r/s)^{3/2} - \frac{2}{27}r^3 s + \frac{2}{3}r^2 + rs + s^2 + 1\right)^{1/2} & \text{if } s \geqslant r/(4r+9) \end{cases}$$

and this permits to compute the maximum (2), which is obtained exactly at $r = 3/2$, $s = 1/2$.

These examples are particular cases of the following theorem.

Theorem 4.3. *Let $\lambda_0, \lambda_1, \lambda_2 \in \mathbb{Z}$ be distinct. Then the Sidon constant of $\Lambda = \{\lambda_0, \lambda_1, \lambda_2\}$ is $\sec(\pi/2n)$, where $n = \max|\lambda_i - \lambda_j| / \gcd(\lambda_1 - \lambda_0, \lambda_2 - \lambda_0)$.*

Proof. We may suppose $\lambda_0 < \lambda_1 < \lambda_2$. Let $k = (\lambda_1 - \lambda_0)/\gcd(\lambda_1 - \lambda_0, \lambda_2 - \lambda_0)$ and $l = (\lambda_2 - \lambda_0)/\gcd(\lambda_1 - \lambda_0, \lambda_2 - \lambda_0)$. By Lemma 3.2, the Arithmetic-Geometric Mean Inequality bounds the Sidon constant C of $\{0, k, l\}$ in the following way:

$$\begin{aligned} C = \max_{r,s>0} \frac{1 + r + s}{\|1 + r\mathrm{e}^{\mathrm{i}\pi/l}\mathrm{e}_k + s\mathrm{e}_l\|_\infty} &\leqslant \max_{r,s>0} \frac{1 + r + s}{|1 + r\mathrm{e}^{\mathrm{i}\pi/l} + s|} \\ &= \max_{r,s>0} \left(1 - \sin^2\frac{\pi}{2l}\frac{4r(1+s)}{(1+r+s)^2}\right)^{-1/2} \\ &\leqslant \left(1 - \sin^2(\pi/2l)\right)^{-1/2} = \sec(\pi/2l). \end{aligned}$$

This inequality is sharp: we have equality for $s = k/(l-k)$ and $r = 1+s$. In fact the derivative of $|1 + r\mathrm{e}^{\mathrm{i}\pi/l}\mathrm{e}^{\mathrm{i}kt} + s\mathrm{e}^{\mathrm{i}lt}|^2$ is then

$$\frac{8kl}{k-l}\cos\frac{kt+\pi/l}{2}\sin\frac{lt}{2}\cos\frac{(l-k)t-\pi/l}{2},$$

so that its critical points are

$$\frac{2j+1}{k}\pi - \frac{\pi}{kl},\ \frac{2j}{l}\pi,\ \frac{2j+1}{l-k}\pi + \frac{\pi}{l(l-k)} : j \in \mathbb{Z},$$

where it makes

$$4s^2\sin^2\frac{2j+1+l}{2k}\pi,\ 4r^2\cos^2\frac{2j+1}{2l}\pi,\ 4\cos^2\frac{2j+1+k}{2(l-k)}\pi : j \in \mathbb{Z}.$$

Therefore the maximum of $|1 + r\mathrm{e}^{\mathrm{i}\pi/l}\mathrm{e}^{\mathrm{i}kt} + s\mathrm{e}^{\mathrm{i}lt}|$ is $2r\cos(\pi/2l)$. □

This proof and (1) yield also the more precise

Proposition 4.4. *Let $\Lambda = \{\lambda_0, \lambda_1, \lambda_2\} \subseteq \mathbb{Z}$. The solution to Extremal problem (‡) is a multiple of*

$$f(t) = \epsilon_0\,|\lambda_1-\lambda_2|\,\mathrm{e}^{\mathrm{i}\lambda_0 t} + \epsilon_1\,|\lambda_0-\lambda_2|\,\mathrm{e}^{\mathrm{i}\lambda_1 t} + \epsilon_2\,|\lambda_0-\lambda_1|\,\mathrm{e}^{\mathrm{i}\lambda_2 t}$$

with $\epsilon_0, \epsilon_1, \epsilon_2 \in \{-1, 1\}$ real signs such that

- $\epsilon_0\epsilon_1 = -1$ *if* $2^j \mid \lambda_1 - \lambda_0$ *and* $2^j \nmid \lambda_2 - \lambda_0$ *for some* j*;*
- $\epsilon_0\epsilon_2 = -1$ *if* $2^j \nmid \lambda_1 - \lambda_0$ *and* $2^j \mid \lambda_2 - \lambda_0$ *for some* j*;*
- $\epsilon_1\epsilon_2 = -1$ *otherwise.*

The Sidon constant of Λ is attained for this f. Therefore the complex and real unconditionality constants of $\{\mathrm{e}_\lambda\}_{\lambda\in\Lambda}$ in $\mathscr{C}(\mathbb{T})$ coincide for sets Λ with three elements.

5 Some consequences

Let us underline the following consequences of our computation.

Corollary 5.1. (1) *The Sidon constant of sets with three elements is at most $\sqrt{2}$.*

(2) *The Sidon constant of $\{0, n, 2n\}$ is $\sqrt{2}$, while the Sidon constant of $\{0, n+1, 2n\}$ is at most $\sec(\pi/2n) = 1 + \pi^2/8n^2 + o(n^{-2})$ and thus arbitrarily close to 1.*

(3) *The Sidon constant of $\{\lambda_0 < \lambda_1 < \lambda_2\}$ does not depend on λ_1 but on the g.c.d. of $\lambda_1 - \lambda_0$ and $\lambda_2 - \lambda_0$.*

Theorem 4.3 also shows anew that no set of integers with more than two elements has Sidon constant 1 (see [6, p. 21] or [1]). Recall now that $\Lambda = \{\lambda_n\} \subseteq \mathbb{Z}$ is a Hadamard set if there is a $q > 1$ such that $|\lambda_{n+1}/\lambda_n| \geqslant q$ for all n. By [3, Cor. 9.4], the Sidon constant of Λ is at most $1 + \pi^2/(2q^2 - 2 - \pi^2)$ if $q > \sqrt{\pi^2/2 + 1} \approx 2.44$. On the other hand Theorem 4.3 shows

Corollary 5.2. (1) *If there is an integer $q \geqslant 2$ such that $\Lambda \supseteq \{\lambda, \lambda+\mu, \lambda+q\mu\}$ for some integers λ and μ, then the Sidon constant of Λ is at least*

$$\sec(\pi/2q) > 1 + \pi^2/(8q^2).$$

(2) *In particular, we have the following bounds for the Sidon constant C of the set $\Lambda = \{q^k\}$, $q \in \mathbb{Z} \setminus \{0, \pm1, \pm2\}$:*

$$1 + \frac{\pi^2}{8\max(-q, q+1)^2} < C \leqslant 1 + \frac{\pi^2}{2q^2 - 2 - \pi^2}.$$

6 Three questions

(*a*) Is there a set Λ for which the real and complex unconditionality constants of $\{e_\lambda\}_{\lambda\in\Lambda}$ in $\mathscr{C}(\mathbb{T})$ differ? The same question is open in spaces $L^p(\mathbb{T})$, $1 \leqslant p < \infty$, and even for the case of three element sets if p is not a small even integer, and especially for the set $\{0, 1, 2, 3\}$ in any space but $L^2(\mathbb{T})$.

(*b*) Let $q > 1$. Are there infinite sets $\Lambda = \{\lambda_n\}$ such that $|\lambda_{n+1}/\lambda_n| \leqslant q$ with Sidon constant arbitrarily close to 1? What about the sequence of integer parts of the powers of a transcendental number $\sigma > 1$ (see [3, Cor. 2.10, Prop. 3.2])?

(*c*) The only set with more than three elements with known Sidon constant is $\{0, 1, 2, 3, 4\}$, for which it makes 2 (see [6, Chapter 3]). Can one compute the Sidon constant of sets with four elements? I conjecture that the Sidon constant of $\{0, 1, 2, 3\}$ is $5/3$.

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