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Thierry Coquand and Paul Lorenzen.

Lorenzen: "My algebraic works [...] were concerned with a problem that had *formally* the same structure as the problem of freedom from contradiction of the classical calculus of logic".

About seven years ago, Henri Lombardi told me that his colleague Thierry Coquand was wishing for a translation of an article of a mathematician that was yet unknown to me, Paul Lorenzen. At the time, my skills for that purpose were linguistic (the German of the article) and my training in the history and epistemology of the subject of mathematics as a whole, but focused on Greek mathematics. Even though I had been knowing about Hilbert's Programme and read some of Hilbert's relevant articles, the expertise I have acquired in the last seven years gives me the feeling that I was a complete ignorant at the time. E.g., I did not know about natural deduction. It took me some time to free myself from duties (an article about Zeno's paradoxa of multiplicity), and then I did the duty Henri had asked me.

Lorenzen's article was *Algebraische und logistische Untersuchungen über freie Verbände* (1951). Part I of it explains how to construct a certain number of free objects over a preordered set: the semilattice, the distributive lattice, the pseudocomplemented semilattice, the countably complete Boolean algebra. In each case, the set is conservatively embedded into the free object, i.e. no new order relations are being added between elements of the original set. If one applies this to the set of formulas about numbers, preordered by material implication, one gets the consistency of arithmetic and even of ramified type theory without axiom of reducibility. Part II provides a proof of this that does this in the language of logic without resorting to the free constructions. As such, it appears as Gentzen's Hauptsatz for ω-logic, proved as in Gentzen's dissertation (for elementary number theory without axiom of complete induction), the new idea being the "ω-rule", i.e. $\forall x\, A(x)$ holds if $A(n)$ holds for each numeral.

Thierry's interest in Lorenzen dates back to at least the beginning of the 90s. There is a reference to him in Thierry's Semantics of evidence for game semantics; in fact, the crucial idea of backtracking appears in Lorenzen's Logik und Agon (1960) and even in his Operative mathematics and computers (1959). As for the 1951 article, there is a reference in Thierry's answer to Solomon Feferman's 1999 inquiry about the future of proof theory: he was calling for a textbook exposition of cut-elimination for ω-logic as in Martin-Löf's book or one of Lorenzen's books.

Only after my translation did I first meet Thierry in Göteborg, for the first of a series of week-long working sessions together with Henri. I have been impressed by Thierry's generosity from the beginning: his complete engagement for the whole week, his readiness in explaining (once more...) an argument, his eager listening. The nights have a peculiar effect on him. Everyday, when meeting in the morning, yesterday's hesitating thoughts became insights, sometimes as lasting results, sometimes as steps towards a more profound encounter with the mistery. We noticed a first coincidence of investigations between Lorenzen and Thierry: his fundamental theorem together with Jan Cederquist (2001), stating that an entailment relation $\vdash$ for a preordered set $(M, \leq)$, i.e. a relation between finite subsets of M satisfying $\frac{a \leq b}{a \vdash b}$, $\frac{A \vdash B}{A' \vdash B'}$ if $\begin{smallmatrix} A' \supseteq A \\ B' \supseteq B \end{smallmatrix}$, $\frac{A \vdash B, x \quad A, x \vdash B}{A \vdash B}$ generates the free distributive lattice over M, for which holds $a_1 \wedge \ldots \wedge a_m \leq b_1 \vee \ldots \vee b_n$ if and only if $a_1, \ldots, a_m \vdash b_1, \ldots, b_n$, i.e. conservativity, is in fact already contained in Lorenzen 1951.

In the second working session, at Oberwolfach, I understood that the second sentence of the 1951 paper, "It has turned out lately that the essential property of Dedekind's system of ideals lies in that ideals form a semilattice" is in fact an implicit reference to Lorenzen's habilitation (published in 1950), so we worked on Lorenzen's papers in algebra (1950, 1952, 1953). A second coincidence strikes Thierry and Henri: Lorenzen's statement that in the multiplicative group of the field of fractions of an integral domain I, the relation of integral dependence of an element b on the ideal $\mathfrak{a}$ generated by $a_1, \dots, a_m$, given by $b^k + c_1 b^{k-1} + \dots + c_k = 0$ with $c_\ell \in \mathfrak{a}^\ell$ may be extended into the entailment relation $a_1, \dots, a_m \vdash b_1, \dots, b_n$ defined by $1 \in \sum_{\ell=1}^{\infty} (a_1 b_1^{-1}, \dots, a_1 b_n^{-1}, \dots, a_m b_1^{-1}, \dots, a_m b_n^{-1})^\ell$. We therefore have begun working on an account of Lorenzen's work in algebra, which turns out to be on entailment relations on a group that are equivariant w.r.t. its order.

It is in understanding this work that we have realised a third coincidence: the heart of Lorenzen's method is the observation that b is integrally dependent on $\mathfrak{a}$ if and only if one can find elements $z_1, \dots, z_p$ such that $b \in \mathfrak{a} I[z_1^{\pm 1}, \dots, z_p^{\pm 1}]$ for each combination of signs, i.e. if b is in $\mathfrak{a}$ when doing as if it was totally preordered. To do so is the computational content of the representation of an integrally closed integral domain as the intersection of valuation rings. In this sense, Lorenzen may be credited for having given the first paradigmatic example of dynamic algebra, which consists in doing as if the ideal objects existed.

Thierry's interest in the 1951 article comes also with his intuition that the emphasis on analysing the ordinal strength of axiomatic theories becomes a detour when trying to understand e.g. Π^1_1-CA: a great deal of work is necessary for establishing that the ordinal notation system used for transfinite induction is really well-founded, and necessitates generalised inductive definitions, so an approach using directly generalised inductive definitions is desirable. The 1951 article is such an approach for the "easy case" of elementary number theory.

Thierry has also very high esteem for Lorenzen's constructive proof of the Cantor-Bendixson theorem through a generalised inductive definition for the derivative of a closed set (stated for the complement sets), which has enlarged the realm of constructive mathematics

I have not spoken about Lorenzen further work in logic: operative logic, dialogic logic, because I have little expertise on this. I would like to emphasise a point: After his proof of consistency, Lorenzen wished to formalise the means of his proof, but his efforts led to the experience of an infinite regress: the foundation of mathematics cannot result from formalisation because there is no end to it. He therefore tried to understand how things, sentences get their meaning and arrived at an analysis of language as rooted in life. In fact, he had been a student of Josef König in Göttingen, the main exponent of Lebensphilosophie [philosophy of life], and Lorenzen cites ~~Wilhelm~~ Dilthey: "Knowledge cannot recede behind life".

I would like to make a last parallel between Lorenzen and Thierry on the basis of a citation of Blaise Pascal (appearing in an article by Lorenzen on the occasion of Pascal's tercentenary in 1962): «Thus we see that all sciences are infinite in the range of their researches, for who can doubt that mathematics, for instance, has an infinity of infinities of propositions to expound? They are infinite also in the multiplicity and subtlety of their principles, for anyone can see that those which are supposed to be ultimate do not stand by themselves, but depend on others, which depend on others again, and thus never allow any finality» (Thoughts, Sellier 230.) Both are interested in both infinities, and experience them.

References: Thierry Coquand: Lorenzen and constructive mathematics, 2021,
A semantics of evidence for classical arithmetic, 1995

Solomon Feferman: Proof theory on the eve of year 2000, 1999

Thierry Coquand and Stefan Neuwirth: Lorenzen's proof of consistency for elementary number theory, 2020
An introduction to Lorenzen's Algebraic and logistic investigations on free lattices (1951), forthcoming

Thierry Coquand, Henri Lombardi and Stefan Neuwirth: Lattice-ordered groups generated by an ordered group and regular systems of ideals, 2019
Regular entailment relations, 2021

Stefan Neuwirth: Lorenzen's reshaping of Krull's Fundamentalsatz for integral domains (1938-1953), 2021
«C'est comme oeufs»: Paul Lorenzen et l'effondrement des preuves de non-contradiction, 2022