Two random constructions inside lacunary sets

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Résumé

Nous étudions le rapport entre la croissance d'une suite d'entiers et les propriétés harmoniques et fonctionnelles de la suite de caractères associée. Nous montrons en particulier que toute suite polynomiale, ainsi que la suite des nombres premiers, contient un ensemble $\Lambda(p)$ pour tout p qui n'est pas de Rosenthal.

Abstract

We study the relationship between the growth rate of an integer sequence and harmonic and functional properties of the corresponding sequence of characters. We show in particular that every polynomial sequence contains a set that is $\Lambda(p)$ for all p but is not a Rosenthal set. This holds also for the sequence of primes.

1 An introduction in French

1.1 Position du problème

Nous étudions le rapport entre la croissance d'une suite $\{n_k\} = E \subseteq \mathbb{Z}$ et deux de ses propriétés harmoniques et fonctionnelles éventuelles, *i. e.*

- toute fonction intégrable sur le tore à spectre dans E est en fait p-intégrable pour tout $p < \infty$: E est un ensemble $\Lambda(p)$ pour tout p;
- toute fonction mesurable bornée sur le tore à spectre dans E est continue à un ensemble de mesure nulle près: E est un ensemble de Rosenthal.

Nous sommes en mesure de dresser le tableau suivant selon la croissance

- polynomiale: $n_k \preccurlyeq k^d$ pour un $d < \infty$,
- surpolynomiale: $n_k \gg k^d$ pour tout $d \geq 1$,
- sous-exponentielle: $\log n_k \ll k$,
- géométrique: $\liminf |n_{k+1}/n_k| > 1$.

<table>
<tr><th>croissance</th><th>polynomiale</th><th>surpolynomiale et sous-exponentielle</th><th>géométrique</th></tr>
<tr><td>E $\Lambda(p)$ $\forall p$</td><td>non</td><td>presque toujours</td><td>oui</td></tr>
<tr><td>E Rosenthal</td><td colspan="2">presque jamais</td><td>oui</td></tr>
</table>

Table 1: Croissance et propriétés harmoniques ou fonctionnelles.

Li montre qu'effectivement il existe un ensemble $\Lambda(p)$ pour tout p qui n'est pas de Rosenthal. Nous traitons les deux questions suivantes.

Question 1.1 *Le schéma ci-dessus reste-t-il valable si on considère à la place de l'ensemble des sous-ensembles E de $\mathbb{Z}$ l'ensemble des sous-ensembles E d'une suite à croissance polynomiale ?*

Question 1.2 *Si E n'est pas un ensemble de Rosenthal, E contient-il un ensemble à la fois $\Lambda(p)$ pour tout p et non Rosenthal ?*

1.2 Constructions aléatoires à l'intérieur de suites lacunaires

Nous fournissons une preuve nouvelle pour une construction aléatoire d'ensembles lacunaires par Yitzhak Katznelson qui appartient au folklore de l'analyse harmonique. Nous analysons et généralisons aussi la construction aléatoire d'ensembles équidistribués par Jean Bourgain.
Cela nous permet d'établir le tableau 1.1 qui classe les propriétés de Rosenthal et $\Lambda(p)$ pour tout p selon la croissance du spectre. Nous montrons alors que la démarche probabiliste suivie par Katznelson et Bourgain pour construire ces sous-ensembles de $\mathbb{Z}$ utilise seulement la croissance "arithmétique" et l'équidistribution de la suite des entiers $\mathbb{Z}$. En fait, ces sous-ensembles peuvent être construits à l'intérieur de suites équidistribuées à croissance polynomiale. En particulier, le tableau 1.1 reste valable pour l'ensemble des sous-ensembles E d'une suite polynomiale, ainsi que de la suite des nombres premiers.
Nous fournissons une réponse partielle à la question 1.2.

Théorème 1.3 *Soit P une suite polynomiale ou la suite des nombres premiers. Alors il existe une sous-suite E de P qui est $\Lambda(p)$ pour tout p alors qu'elle ne forme pas un ensemble de Rosenthal.*

2 Introduction

The study of lacunary sets in Fourier analysis still suffers from a severe lack of examples, in particular for the purpose of distinguishing two properties. In order to bypass the individual complexity of integer sets, one frequently resorts to random constructions. In particular, Li [16] uses in his argumentation a construction due to Katznelson [13] to discriminate the following two functional properties of certain subsets $E \subseteq \mathbb{Z}$:

- A Lebesgue integrable function on the circle with Fourier frequencies in E is in fact p-integrable for all $p < \infty$. This means that all spaces $\mathrm{L}^p_E(\mathbb{T})$ coincide for $p < \infty$, *i. e.* E is a $\Lambda(p)$ set for all p in Rudin's terminology. No sequence of polynomial growth has this property [24, Th. 3.5]. By Theorem 5.7, almost every sequence of a given superpolynomial order of growth is $\Lambda(p)$ for all p.
- A bounded measurable function on the circle with Fourier frequencies in E is in fact continuous up to a set of measure 0. This means that $\mathrm{L}^\infty_E(\mathbb{T})$ and $\mathscr{C}_E(\mathbb{T})$ coincide: E is a Rosenthal set. Every sequence of exponential growth is a Sidon set and therefore has this property. By Bourgain's Theorem 3.5, almost every sequence of a given subexponential order of growth fails the Rosenthal property.

A Rosenthal set may contain arbitrarily large intervals [23] und thus fail the $\Lambda(p)$ property. This shows that these two properties cannot be characterized by some order of growth, whereas the random method is so imprecise that it ignores a range of exceptional sets. On the other hand, Li shows that some set E is $\Lambda(p)$ for all p and fails the Rosenthal property: his construction witnesses for the quantitative overlap between superpolynomial and subexponential order of growth. From a Banach space point of view, Li's set E is such that $\mathscr{C}_E(\mathbb{T})$ contains c_0 while $\mathrm{L}^1_E(\mathbb{T})$ does not contain ℓ_1.
We come back to Li [16] for two reasons: in the first place, we have been unable to locate a published proof of Katznelson's statement. We provide one for a stronger statement in Section 5. In the second place, we want to precise and supple the random construction in the following sense: can one distinguish the $\Lambda(p)$ property and the Rosenthal property among subsets of a certain given set ? That sort of questions has been investigated by Bourgain in [4]. We give the following answer (see Th. 3.8):

Main Theorem *Consider a polynomial sequence of integers, or the sequence of primes. Then some subsequence of it is $\Lambda(p)$ for all p and at the same time fails the Rosenthal property.*

This is a special case of the more general question: does every set that fails the Rosenthal property contain a subset that is $\Lambda(p)$ for all p and still fails the Rosenthal property ? We should emphasize at this point that neither of these notions has an arithmetic description. In fact, the family of Rosenthal sets is coanalytic non Borel [9] and any description would be at least as complex as their definition. This is why we study instead the following two properties for certain subsets $E \subseteq \mathbb{Z}$.

- Any integer n has at most one representation as the sum of s elements of E. This implies that E is $\Lambda(2s)$ by [24, Th. 4.5(b)].

- E is equidistributed in Hermann Weyl's sense: save for $t \equiv 0 \bmod 2\pi$, the successive means of $\{\mathrm{e}^{int}\}_{n\in E}$ tend to 0, which is the mean of e^{it} over $[0, 2\pi[$. This implies that E is not a Rosenthal set by [18, Lemma 4].

Our random construction gives no hint for explicit procedures to build such integer sets. The question whether some "natural" set of integers is $\Lambda(p)$ for all p and fails the Rosenthal property remains open. Let us describe the paper briefly. Section 3 introduces the inquired notions and gives a survey of former and new results. As the right framework for this study appears to consist in the sequences of polynomial growth, we give them a precise meaning in Section 4, and show that they are nicely distributed among the intervals of the partition of $\mathbb{Z}$ defined by $\{\pm 2^{k!}\}$. Section 5 establishes an optimal criterion for the generic subset of a set with polynomial growth to be $\Lambda(p)$ for all p. Section 6 comes back to Bourgain's proof in [3, Prop. 8.2(i)]: we simplify and strengthen it in order to investigate the generic subset of an equidistributed set.

Notation $\mathbb{T} = \{t \in \mathbb{C} : |t| = 1\}$ is the unit circle endowed with its Haar measure dm and $\mathbb{Z}$ its dual group of integers: for each $n \in \mathbb{Z}$, let $\mathrm{e}_n(t) = t^n$. The cardinal of $E = \{n_k\} \subseteq \mathbb{Z}$ is written $|E|$. We denote by $c_0(\mathbb{T})$ the space of functions on $\mathbb{T}$ which are arbitrarily small outside finite sets; such functions necessarily have countable support.
For a space of integrable functions on $\mathbb{T}$ and $E \subseteq \mathbb{Z}$, X_E denotes the space of functions with Fourier spectrum in E: $X_E = \{f \in X : \widehat{f}(n) = \int \mathrm{e}_{-n} f\, dm = 0 \text{ if } n \notin E\}$.
We shall stick to Hardy's notation: $u_n \preccurlyeq v_n$ (*vs.* $u_n \ll v_n$) if u_n/v_n is bounded (*vs.* vanishes) at infinity.

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3 Equidistributed and $\Lambda(p)$ sets

Definition 3.1 *Let $E = \{n_k\}_{k\ge1} \subseteq \mathbb{Z}$ ordered by increasing absolute value $|n_k|$.*
(i) [24, Def. 1.5] Let $p > 0$. E is a $\Lambda(p)$ set if, for some — or equivalently for any — $0 < r < p$, $\mathrm{L}^p_E(\mathbb{T})$ and $\mathrm{L}^r_E(\mathbb{T})$ coincide:

$$\exists C_r \quad \forall f \in \mathrm{L}^p_E(\mathbb{T}) \quad \|f\|_r \le \|f\|_p \le C_r \|f\|_r.$$

(ii) [26, §7] E is equidistributed if for each $t \in \mathbb{T} \setminus \{1\}$ the successive means

$$f_k(t) = \frac{1}{k}\sum_{j=1}^{k} \mathrm{e}_{n_j}(t) \xrightarrow[k\to\infty]{} 0. \tag{1}$$

Thus E is equidistributed if and only if the sequence of characters in E converges to $\mathbf{1}_{\{1\}}$ for the Cesàro summing method. If f_k tends pointwise to $f \in c_0(\mathbb{T})$, then E is weakly equidistributed.

If E is weakly equidistributed, then f defines an element of $\mathscr{C}_E(\mathbb{T})^{\perp\perp}$. By Lust-Piquard's [18, Lemma 4], $\mathscr{C}_E(\mathbb{T})$ then contains a copy of c_0 and E cannot be a Rosenthal set.
For example, $\mathbb{Z}$ and $\mathbb{N}$ are equidistributed. Arithmetic sequences are weakly equidistributed: there is a finite set on which $f_k \nrightarrow 0$. Polynomial sequences of integers ([26, Th. 9] and [25, Lemma 2.4], see [19, Ex. 2]) and the sequence of prime numbers (Vinogradov's theorem [5], see [19, Ex. 1]) are weakly equidistributed: $f_k(t)$ may not converge to 0 for rational t. There are nevertheless sequences of bounded pace that are not weakly equidistributed [8, Th. 11]. Sidon sets are $\Lambda(p)$ for all p [24, Th. 3.1], but not weakly equidistributed since they are Rosenthal sets.

Example 3.2 Consider the geometric sequence $E = \{3^k\}_{k\ge1}$ and the corresponding sequence of successive means f_k. By [8, Th. 14], the f_k do not converge to 0 on a null set of Hausdorff dimension 1. Consider

$$f_k^j = k^{-j} \sum_{1\le k_1,\dots,k_j\le k} \mathrm{e}_{3^{k_1}+\dots+3^{k_j}} = k^{-j}\Bigl(j! \sum_{1\le k_1<\dots<k_j\le k} + \sum_{\substack{1\le k_1,\dots,k_j\le k\\ \text{not all distinct}}} \Bigr) \mathrm{e}_{3^{k_1}+\dots+3^{k_j}}\,.$$

Let $j \geq 1$. Put $E^{(j)} = \{3^{k_1} + \cdots + 3^{k_j} : 1 \leq k_1 < \cdots < k_j\}$ and let $f_k^{(j)}$ be the corresponding successive means (1). Then

$$\begin{aligned}\|f_k^j - f^{(j)}_{\binom{k}{j}}\|_\infty &\leq \left(\binom{k}{j}^{-1} - \frac{j!}{k^j}\right)\binom{k}{j} + \frac{1}{k^j}\left(k^j - \frac{k!}{(k-j)!}\right) \\ &= 2\left(1 - \frac{k!}{k^j(k-j)!}\right) \xrightarrow[k]{} 0.\end{aligned}$$

Therefore the $f_k^{(j)}$ do not converge to 0 outside a countable set, and $E^{(j)}$, which is $\Lambda(p)$ for all p [20, Th. IV.3] and not Sidon, is not weakly equidistributed.

However, as Li notes, these two classes meet.

Theorem 3.3 **([16])** *There is an equidistributed sequence that is $\Lambda(p)$ for all p.*

Sketch of proof. Li uses the following random construction, discovered by Erdős [6, 7] and introduced to harmonic analysis by Katznelson and Malliavin [14, 15].

Construction 3.4 *Let $E \subseteq \mathbb{Z}$ and consider independent $\{0,1\}$-valued selectors ξ_n of mean δ_n $(n \in E)$, i. e. $\mathbb{P}[\xi_n = 1] = \delta_n$. Then the random set E' is defined by*

$$E' = \{n \in E : \ \xi_n = 1\}.$$

The first ingredient of the proof is Bourgain's following

Theorem 3.5 **([3, Prop. 8.2(i)])** *Let $E = \mathbb{N}$ in Construction 3.4. If δ_n decreases with n while $\delta_n \gg n^{-1}$, then E' is almost surely equidistributed.*

Remark 3.6 In this sense, almost every sequence of a subexponential growth given by $\{\delta_n\}$ is equidistributed: indeed, for almost every $E' \subseteq \mathbb{N}$,

$$|E' \cap [0,n]| \sim \delta_0 + \cdots + \delta_n \gg \log n$$

by the Law of Large Numbers. Note however that the set $E^{(j)}$ defined in Example 2.2 has subexponential growth: $|E^{(j)} \cap [-n,n]| \succcurlyeq (\log n)^j$, and is not equidistributed.

The second ingredient is a result announced without proof by Katznelson.

Proposition 3.7 **([13, §2])** *Put $I_k =]p_{k-1}, p_k]$ with $p_k > p_{k-1}^2$ $(k \geq 1)$. Let $E = \mathbb{N}$ in Construction 3.4. There is a choice of (ℓ_k) with $\ell_k \gg \log p_k$ such that for $\delta_n = \ell_k/|I_k|$ $(n \in I_k)$, E' is $\Lambda(p)$ for all p almost surely.*

Li suggests to apply the content of Proposition 3.7 with $p_k = 2^k$ and $\ell_k = k$: then $\delta_n \gg n^{-1}$ and Theorem 3.3 derives from Theorem 3.5. ∎

We shall generalize Katznelson's and Li's results with a new proof that permits to construct E' inside of sets E with polynomial growth (see Def. 4.1) and yields an optimal criterion on ℓ_k. We shall subsequently generalize Bourgains's Theorem 3.5 to obtain the Main Theorem via

Theorem 3.8 *Let E be equidistributed (vs. weakly) and with polynomial growth. Then there is a subset $E' \subseteq E$ equidistributed (vs. weakly) and at the same time $\Lambda(p)$ for all p.*

A precise and quantitative statement of this is Theorem 6.5.

4 Sets with polynomial growth

We start with the definition and first property of such sets.

Definition 4.1 *Let* $E = \{n_k\}_{k\geq 1} \subseteq \mathbb{Z}$ *be an infinite set ordered by increasing absolute value and* $E[t] = |E \cap [-t,t]|$ *its distribution function.*
(*i*) *E has polynomial growth if* $n_k \preccurlyeq k^d$ *for some* $1 \leq d < \infty$. *This amounts to* $E[t] \succcurlyeq t^\varepsilon$ *for* $\varepsilon = d^{-1}$.
(*ii*) *E has regular polynomial growth if there is a* $c > 1$ *such that* $|n_{\lceil ck\rceil}| \leq 2|n_k|$ *for large k. This amounts to* $E[2t] \geq cE[t]$ *for large t.*

Proof. (*i*) If $|n_k| \leq Ck^d$ for large k and $Ck^d \leq t < C(k+1)^d$, then $E[t] \geq k > (t/C)^\varepsilon - 1$. Conversely, if $E[t] \geq ct^\varepsilon$ for large t and $c(t-1)^\varepsilon < k \leq ct^\varepsilon$, then $|n_k| \leq t < (k/c)^d + 1$.
(*ii*) If $|n_{\lceil ck\rceil}| \leq 2|n_k|$ for large k and k is maximal with $|n_k| \leq t$, then $E[2t] \geq E[2|n_k|] \geq ck = cE[t]$. Conversely, if $E[2t] \geq cE[t]$ for large t, then $E[|n_k|] \in \{k, k+1\}$ and $E[2|n_k|] \geq ck$. Thus $|n_{\lceil ck\rceil}| \leq 2|n_k|$. ∎

In particular, polynomial sequences have regular polynomial growth. By the Prime Number Theorem, the sequence of primes also has. Property (*ii*) implies property (*i*): if $E[2t] \geq cE[t]$ for large t, then $E[t] \succcurlyeq t^{\log_2 c}$. The converse however is false as shows $F = \bigcup]2^{2^{2k}}, 2^{2^{2k}+1}]$, for which $F[t] \succcurlyeq t^{1/4}$ while $F[2t] = F[t]$ infinitely often.
Let us relate Definition 4.1 with certain partitions of $\mathbb{Z}$. Regular growth means in fact that E is regularly distributed on the annular dyadic partition of $\mathbb{Z}$

$$\mathscr{P} = \big\{[-p_0, p_0], I_k = [-p_k, -p_{k-1}[\, \cup\,]p_{k-1}, p_k]\big\}_{k\geq 1} \text{ where } p_k = 2^k \tag{2}$$

and F shows that there are sets with polynomial growth which are not regularly distributed on the partition defined by $p_k = 2^{2^k}$. However, the intervals of the gross partition

$$\mathscr{P} = \big\{[-p_0, p_0], I_k = [-p_k, -p_{k-1}[\, \cup\,]p_{k-1}, p_k]\big\} \text{ where } \log p_k \gg \log p_{k-1} \tag{3}$$

grow with a speed that forces regularity. Put $p_k = 2^{k!}$ for a simple explicit example. We have precisely

Proposition 4.2 *Let* $E \subseteq \mathbb{Z}$, $\mathscr{P} = \{I_k\}$ *a partition of* $\mathbb{Z}$ *and* $E_k = E \cap I_k$. *Then* $\log|E_k| \succcurlyeq \log|I_k|$ *in the two following cases:*
(*i*) *E has regular polynomial growth and* $\mathscr{P}$ *is partition* (2);
(*ii*) *E has polynomial growth and* $\mathscr{P}$ *is partition* (3).

Proof. (*i*) Choose K and $c > 1$ such that $E[2^k] \geq cE[2^{k-1}]$ for $k \geq K$. Then $E[2^k] \succcurlyeq c^k$. Thus

$$|E_k| = E[2^k] - E[2^{k-1}] \geq (1 - c^{-1})E[2^k] \succcurlyeq c^k = 2^{k\log_2 c}.$$

(*ii*) In this case $p_k^\varepsilon \gg p_{k-1}$ for each positive ε. Now there is $\varepsilon > 0$ such that

$$|E_k| = E[p_k] - E[p_{k-1}] \succcurlyeq p_k^\varepsilon \succcurlyeq |I_k|^\varepsilon. \qquad ∎$$

5 Sets that are $\Lambda(p)$ for all p

In this section, we establish an improvement (Th. 5.7) of Katznelson's statement [13, §2]. We first recall several known definitions and results.
$\Lambda(p)$ sets have a practical description in terms of unconditionality. We shall also use a combinatorial property that is more elementary than [24, 1.6(b)]: to this end, write Z_s^m for the following set of arithmetic relations.

$$\mathrm{Z}_s^m = \big\{\zeta \in \mathbb{Z}^{*m} : \zeta_1 + \cdots + \zeta_m = 0 \text{ and } |\zeta_1| + \cdots + |\zeta_m| \leq 2s\big\}.$$

Note that Z_s^1 and Z_s^m ($m > 2s$) are empty, and that every $\zeta \in \mathrm{Z}_s^2$ is of form $\zeta_1 \cdot (1, -1)$: this is the identity relation.

Definition 5.1 *Let* $1 \leq p < \infty$, $s \geq 1$ *integer and* $E \subseteq \mathbb{Z}$.
(*i*) *[12] E is an unconditional basic sequence in* $\mathrm{L}^p(\mathbb{T})$ *if*

$$\sup_{\pm}\Big\|\sum_{n\in E} \pm a_n\, \mathrm{e}_n\Big\|_p \leq C\Big\|\sum_{n\in E} a_n\, \mathrm{e}_n\Big\|_p.$$

for some C. If $C = 1$ *works, E is a 1-unconditional basic sequence in* $\mathrm{L}^p(\mathbb{T})$.
(*ii*) *E is s-independent if* $\sum_1^m \zeta_i q_i \neq 0$ *for all* $3 \leq m \leq 2s$, $\zeta \in \mathrm{Z}_s^m$ *and distinct* $q_1, \ldots, q_m \in E$.

Proposition 5.2 *Let $1 \leq p \neq 2 < \infty$, $s \geq 1$ integer and $E \subseteq \mathbb{Z}$.*
(i) [24, proof of Th. 3.1] E is a $\Lambda(\max(p,2))$ set if and only if E is an unconditional basic sequence in $\mathrm{L}^p(\mathbb{T})$.
(ii) [22, Prop. 2.5, Rem. (1)] E is a 1-unconditional basic sequence in $\mathrm{L}^{2s}(\mathbb{T})$ if and only if E is s-independent.

We need to introduce a second classical notion of unconditionality that rests on the Littlewood–Paley theory.

Definition 5.3 ([11]) *Let $\mathscr{P} = \{I_k\}$ be a partition of $\mathbb{Z}$ in finite intervals. $\mathscr{P}$ is a Littlewood–Paley partition if for each $1 < p < \infty$ there is a constant C_p such that*

$$\forall f \in \mathrm{L}^p(\mathbb{T}) \quad \sup_{\pm} \left\| \sum \pm f_k \right\|_p \leq C_p \|f\|_p \qquad \text{with } \widehat{f_k} = \begin{cases} \widehat{f} & \text{on} \\ 0 & \text{off} \end{cases} I_k. \tag{4}$$

By Khinchin's inequality, this means exactly that

$$\forall f \in \mathrm{L}^p(\mathbb{T}) \quad \|f\|_p \approx \left\| \left(\sum |f_k|^2 \right)^{1/2} \right\|_p.$$

In particular, the dyadic partition (2) and the gross partition (3) are Littlewood–Paley [17]. By Proposition 5.2 and (4), we obtain

Proposition 5.4 *Let $\{I_k\}$ be a Littlewood–Paley partition and $E_k \subseteq I_k$. If E_k is s-independent for each k, then $E = \bigcup E_k$ is an unconditional basic sequence in $\mathrm{L}^{2s}(\mathbb{T})$ and thus a $\Lambda(2s)$ set.*

We generalize now Katznelson's Proposition 3.7.

Lemma 5.5 *Let $s \geq 2$ integer, $E \subseteq \mathbb{Z}$ finite and $0 \leq \ell \leq |E|$. Put $\delta_n = \ell/|E|$ in Construction 3.4, so that all selectors ξ_n have same distribution. Then there is a constant $C(s)$ that depends only on s such that*

$$\mathbb{P}\left[E' \text{ is } s\text{-dependent}\right] \leq C(s) \frac{\ell^{2s}}{|E|}.$$

Proof. We wish to compute the probability that there are $3 \leq m \leq 2s$, $\zeta \in \mathrm{Z}_s^m$ and distinct $q_1, \ldots, q_m \in E'$ with $\sum \zeta_i q_i = 0$. As the number $C(s)$ of arithmetic relations $\zeta \in \mathrm{Z}_s^m$ ($3 \leq m \leq 2s$) is finite and depends on s only, it suffices to compute, for fixed m and $\zeta \in \mathrm{Z}_s^m$

$$\begin{aligned}
&\mathbb{P}\left[\exists q_1, \ldots, q_m \in E' \text{ distinct} : \sum \zeta_i q_i = 0\right] \\
&\quad = \mathbb{P}\left[\exists q_1, \ldots, q_{m-1} \in E' \text{ distinct} : -\zeta_m^{-1} \sum_{i=1}^{m-1} \zeta_i q_i \in E' \setminus \{q_1, \ldots, q_{m-1}\}\right] \\
&\quad = \mathbb{P}\left[\bigcup_{\substack{q_1, \ldots, q_{m-1} \\ \in E' \text{ distinct}}} \left[-\zeta_m^{-1} \sum_{i=1}^{m-1} \zeta_i q_i \in E' \setminus \{q_1, \ldots, q_{m-1}\}\right]\right] \\
&\quad = \mathbb{P}\left[\bigcup_{\substack{q_1, \ldots, q_{m-1} \\ \in E \text{ distinct}}} \left[q_m = -\zeta_m^{-1} \sum_{i=1}^{m-1} \zeta_i q_i \in E \setminus \{q_i\}_{i=1}^{m-1} \ \&\ \xi_{q_1} = \cdots = \xi_{q_m} = 1\right]\right].
\end{aligned}$$

The union in the line above runs over

$$\frac{|E|!}{(|E| - m + 1)!} \leq |E|^{m-1}$$

$(m-1)$-tuples. Further, the event in the inner brackets implies that m out of $|E|$ selectors ξ_n have value 1: its probability is bounded by $(\ell/|E|)^m$. Thus

$$\mathbb{P}\left[E' \text{ is } s\text{-dependent}\right] \leq C(s) \max_{3 \leq m \leq 2s} |E|^{m-1} \frac{\ell^m}{|E|^m} \leq C(s) \frac{\ell^{2s}}{|E|}. \qquad \blacksquare$$

The random method we shall use is the following random construction.

Construction 5.6 *Let $E \subseteq \mathbb{Z}$. Let $\{I_k\}$ be a Littlewood–Paley partition and $E_k = E \cap I_k$. Let $(\ell_k)_{k\geq 1}$ with $0 \leq \ell_k \leq |E_k|$ and put*

$$\mathbb{P}[\xi_n = 1] = \delta_n = \ell_k/|E_k| \quad (n \in E_k)$$

in Construction 3.4. Put $E'_k = E' \cap I_k$.

Theorem 5.7 *Let $E \subseteq \mathbb{Z}$ have polynomial (vs. regular) growth and $\{I_k\}$ be the gross (3) (vs. dyadic (2)) Littlewood–Paley partition. Do Construction 5.6. The following assertions are equivalent.*
(i) $\log \ell_k \ll \log |I_k|$, i. e. $\log \ell_k \ll \log p_k$ (vs. $\log \ell_k \ll k$);
(ii) E' is almost surely a $\Lambda(p)$ set for all p.

Proof. Note that by Proposition 4.2, there is a positive α such that $|E_k| > |I_k|^\alpha$ for large k.
$(i) \Rightarrow (ii)$ Let $s \geq 2$ be an arbitrary integer. By Proposition 5.5,

$$\sum_{k=1}^{\infty} \mathbb{P}[E'_k \text{ is } s\text{-dependent}] \leq C(s) \sum_{k=1}^{\infty} \frac{\ell_k^{2s}}{|E_k|}.$$

For each $\eta > 0$, $\ell_k \leq |I_k|^\eta$ for large k. Choose $\eta < \alpha/2s$. Then $\ell_k^{2s}/|E_k| \leq |I_k|^{2s\eta-\alpha}$ for large k, and the series above converges since $|I_k| \succcurlyeq 2^k$. By the Borel–Cantelli lemma, E'_k is almost surely s-independent for large k. By Proposition 5.4, E' is almost surely the union of a finite set and a $\Lambda(2s)$ set. By [24, Th. 4.4(a)], E' itself is almost surely a $\Lambda(2s)$ set.
$(ii) \Rightarrow (i)$ If E' is a $\Lambda(2s)$ set, then by [24, Th. 3.5] or simply by [4, (1.12)], there is a constant C_s such that $|E'_k| < C_s |I_k|^{1/s}$. As $|E'_k| \sim \ell_k$ almost surely by the Law of Large Numbers (cf. the following Lemma 6.1), one has $\log \ell_k \ll \log |I_k|$. ■

Remark 5.8 As one may easily construct sets that grow as slowly as one wishes and nevertheless contain arbitrarily large intervals (see also [24, Th. 3.8] for an optimal statement), one cannot remove the adverb "almost surely" in Theorem 5.7(*ii*).

Remark 5.9 The right formulation of Katznelson's Proposition 3.7 thus turns out to be the following. Let $E = \mathbb{N}$ and $I_k =]p_{k-1}, p_k]$ with $p_k > cp_{k-1}$ for some $c > 1$ in Construction 5.6. Then E' is almost surely a $\Lambda(p)$ set for all p if and only if $\log \ell_k \ll \log p_k$.

Remark 5.10 Theorem 5.7 shows that there are sets that are $\Lambda(p)$ for all p of any given superpolynomial order of growth. This is optimal since sets with distribution $E[t] \succcurlyeq t^\varepsilon$ fail the $\Lambda(p)$ property for $p > 2/\varepsilon$ by [24, Th. 3.5]. Such sets may also be constructed inductively by combinatorial means: see [10, §II, (3.52)].

6 Equidistributed sets

In this section, we shall finally state and prove our principal result. To this end, we shall first generalize Bourgain's Theorem 3.5 in order to get Theorem 6.4.
The following lemma is Bernstein's distribution inequality [2] and dates back to 1924.

Lemma 6.1 *Let $X_1, \ldots, X_n$ be complex independent random variables such that*

$$|X_i| \leq 1 \quad \text{and} \quad \mathbb{E}\, X_i = 0 \quad \text{and} \quad \mathbb{E}\,|X_1|^2 + \cdots + \mathbb{E}\,|X_n|^2 \leq \sigma. \tag{5}$$

Then, for all positive a,

$$\mathbb{P}[|X_1 + \cdots + X_n| \geq a] < 4\exp(-a^2/4(\sigma + a)). \tag{6}$$

Proof. Consider first the case of real random variables. By [1, (8b)],

$$\mathbb{P}[X_1 + \cdots + X_n \geq a] < \exp(a - (\sigma + a)\log(1 + a/\sigma));$$

as $\log(1 - u) \leq -u - u^2/2$ for $0 \leq u < 1$,

$$\mathbb{P}[X_1 + \cdots + X_n \geq a] < \exp(-a^2/2(\sigma + a)).$$

One gets (6) since for complex z

$$|z| \geq a \quad \Longrightarrow \quad \max(\Re z, -\Re z, \Im z, -\Im z) \geq a/\sqrt{2}. \qquad \blacksquare$$

The next lemma corresponds to [3, Lemma 8.8] and is the crucial step in the estimation of the successive means of $\{\mathrm{e}^{int}\}_{n\in E'}$. Note that its hypothesis is not on the individual δ_n, but on their successive sums σ_k: this is needed in order to cope with the irregularity of E.

Lemma 6.2 *Let $E = \{n_k\} \subseteq \mathbb{Z}$ be ordered by increasing absolute value. Do Construction 3.4 and put $\sigma_k = \delta_{n_1} + \cdots + \delta_{n_k}$. If $\sigma_k \gg \log|n_k|$, then almost surely*

$$\psi(k) = \left\| \frac{1}{|E' \cap \{n_1, \ldots, n_k\}|} \sum_{E'\cap\{n_1,\ldots,n_k\}} \mathrm{e}_n - \frac{1}{\sigma_k} \sum_{j=1}^{k} \delta_{n_j} \mathrm{e}_{n_j} \right\|_\infty \xrightarrow[k\to\infty]{} 0. \tag{7}$$

Proof. Note that

$$\sum_{E'\cap\{n_1,\ldots,n_k\}} \mathrm{e}_n = \sum_{j=1}^{k} \xi_{n_j} \mathrm{e}_{n_j} \quad , \quad |E' \cap \{n_1, \ldots, n_k\}| = \sum_{j=1}^{k} \xi_{n_j}.$$

Center the ξ_n by putting $f = \sum_{j=1}^{k} (\xi_{n_j} - \delta_{n_j}) \mathrm{e}_{n_j}$. Then

$$\begin{aligned} \psi(k) &\leq \left\| \left(|E' \cap \{n_1, \ldots, n_k\}|^{-1} - \sigma_k^{-1} \right) \sum_{j=1}^{k} \xi_{n_j} \mathrm{e}_{n_j} \right\|_\infty + \|\sigma_k^{-1} f\|_\infty \\ &\leq \sigma_k^{-1} \left| \frac{\delta_{n_1} + \cdots + \delta_{n_k}}{\xi_{n_1} + \cdots + \xi_{n_k}} - 1 \right| \sum_{j=1}^{k} \xi_{n_j} + \sigma_k^{-1} \|f\|_\infty \leq 2\sigma_k^{-1} \|f\|_\infty. \end{aligned}$$

Let $R = \{t \in \mathbb{T} : t^{4|n_k|} = 1\}$ and $u \in \mathbb{T}$ such that $|f(u)| = \|f\|_\infty$. Let $t \in R$ be at minimal distance of u: then $|t - u| \leq \pi/4|n_k|$. By Bernstein's theorem,

$$\|f\|_\infty - |f(t)| \leq |f(u) - f(t)| \leq |t - u| \, \|f'\|_\infty \leq \frac{4}{5} \|f\|_\infty;$$

$$\|f\|_\infty \leq 5 \sup_{t\in R} |f(t)|.$$

(For an optimal bound, cf. [21, §I.4, Lemma 8].) For each $t \in R$, the random variables $X_j = (\xi_{n_j} - \delta_{n_j}) \mathrm{e}_{n_j}(t)$ satisfy (5), so that

$$\mathbb{P}[|f(t)| \geq a] < 4\exp(-a^2/4(\sigma_k + a)).$$

As $|R| = 4|n_k|$,

$$\mathbb{P}[\|f\|_\infty \geq 5a] \leq \mathbb{P}\left[\sup_{t\in R} |f(t)| \geq a\right] < 4|n_k| \cdot 4\exp(-a^2/4(\sigma_k + a)).$$

Put $a_k = (12\sigma_k \log|n_k|)^{1/2}$. Then $a_k \ll \sigma_k$: therefore

$$\mathbb{P}[\|f\|_\infty \geq 5a_k] \preccurlyeq |n_k|^{-2}$$

and by the Borel–Cantelli lemma,

$$\sigma_k^{-1} \|f\|_\infty \preccurlyeq a_k/\sigma_k \longrightarrow 0 \text{ almost surely.} \qquad \blacksquare$$

Remark 6.3 The hypothesis in Lemma 6.2 contains implicitly a restriction on the lacunarity of E. If $\sigma_k \gg \log|n_k|$, then necessarily $\log|n_k| \ll k$ and $E[t] \gg \log t$. In particular, E cannot be a Sidon set by [24, Cor. of Th. 3.6].

We now state and prove the equidistributed counterpart of Theorem 5.7.

Theorem 6.4 *Let $E = \{n_k\} \subseteq \mathbb{Z}$ be equidistributed (vs. weakly), and ordered by increasing absolute value. Do Construction 3.4 and suppose that δ_{n_j} decreases with j. Put $\sigma_k = \delta_{n_1} + \cdots + \delta_{n_k}$. If $\sigma_k \gg \log|n_k|$, then E' is almost surely equidistributed (vs. weakly). This is in particular the case if*
(a) $\delta_{n_k} \gg (|n_k| - |n_{k-1}|)/|n_{k-1}|$;
(b) E has polynomial growth and $\delta_{n_k} \gg k^{-1}$.

Proof. Lemma 6.2 shows that almost surely (7) holds. It remains to show that

$$\lim \frac{1}{\sigma_k} \sum_{j=1}^{k} \delta_{n_j} \, \mathrm{e}_{n_j} = \lim \frac{1}{k} \sum_{j=1}^{k} \mathrm{e}_{n_j},$$

i. e. that the matrix summing method $(a_{k,j})$ given by

$$a_{k,j} = \begin{cases} \delta_{n_j}/\sigma_k & \text{if } j \le k \\ 0 & \text{if not} \end{cases}$$

is regular and stronger than the Cesàro method C_1 by arithmetic means. This is the case because $a_{k,j} \ge 0$, $\sum_j a_{k,j} = 1$ and (cf. [27, §52, Th. I])

$$\forall k \quad \sum_j j|a_{k,j} - a_{k,j+1}| = \sum_j j(a_{k,j} - a_{k,j+1}) = 1 < \infty$$

since $a_{k,j}$ decreases with j for each k.
(a) In this case $\delta_{n_k} \gg \log|n_k| - \log|n_{k-1}|$ and thus $\sigma_k \gg \log|n_k|$.
(b) In this case, $\sigma_k \gg \log k \succcurlyeq \log|n_k|$. ∎

In conclusion, we obtain, by combining Theorems 5.7 and 6.4, our principal result.

Theorem 6.5 *Let $E \subseteq \mathbb{Z}$ be equidistributed (vs. weakly) and do Construction 5.6. Then E' is almost surely $\Lambda(p)$ for all p and at the same time equidistributed (vs. weakly) in the two following cases:*
(i) E is a set of regular polynomial growth, $\{I_j\}$ is the dyadic Littlewood–Paley partition (2), $1 \ll \log \ell_j \ll j$ and $\ell_j/|E_j|$ decreases eventually.
(ii) E is a set of polynomial growth, $\{I_j\}$ is the gross Littlewood–Paley partition (3), $\ell_j/|E_j|$ decreases eventually and $\ell_j \gg \log p_{j+1}$ while $\log \ell_j \ll \log p_j$. This is the case if we put $p_j = 2^{j!}$ and $\ell_j = \min((j+2)!, |E_j|)$.

Proof. In each case $\log \ell_j \ll \log|I_j|$. Let us show that the hypothesis of Theorem 6.4 is verified. If $n_k \in E_j \subseteq I_j$, then $|n_k| \le p_j$ and

$$\sigma_k \ge \sum_{i=1}^{j-1} \sum_{n \in E_i} \delta_n = \ell_1 + \cdots + \ell_{j-1}$$

and in each case $\ell_{j-1} \gg \log p_j - \log p_{j-1}$.
Let us make sure in *(ii)* that our choice for p_j and ℓ_j is accurate. Indeed, there is an $\varepsilon > 0$ such that $|E_j| \succcurlyeq 2^{\varepsilon j!}$. Thus $(j+2)! \ll |E_j|$ and $\ell_j = (j+2)!$ for large j. Note further that $(j+2)! \gg (j+1)!$ while $\log(j+2)! \ll j!$. Finally

$$\frac{\ell_{j+1}}{|E_{j+1}|} \preccurlyeq \frac{(j+3)!}{2^{\varepsilon(j+1)!}} \preccurlyeq \frac{j\ell_j}{2^{\varepsilon(j+1)!}} \ll \frac{\ell_j}{2^{j!}} \preccurlyeq \frac{\ell_j}{|E_j|},$$

so that $\ell_j/|E_j|$ decreases eventually. ∎

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