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Is mathematical truth stable?

Stefan Neuwirth

This text is an account of my contribution to a collaborative work with Erich Reck and Roy Wagner. Erich is giving a general introduction to this question, as well as a focus on how Dedekind has changed the way mathematics is being done. Roy, in a case study on the parallel postulate in Euclidean geometry, explains how this was proved in history and what such a proof means.

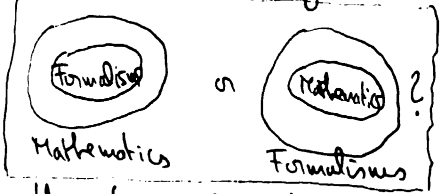
My contribution is twofold: to approach this question from a phenomenological point of view; to share how Bishop settles the question of mathematical truth.

I base truth on the experience of truth. Thus, the stability of truth is based on the stability of experience (of "Erlebnis", as Husserl would write). My talk is the sharing of an experience. I am producing a kind of theatrical performance in front of you, facilitated by the ritual of conferences: frontality, silence, attention. On my side, I am looking at you, seeking eye contact, trying to reach everybody by my presence and my speech, so that we are sharing a common strain of thoughts. Here hermeneutics come in, and in the present of my presentation, this sharing is a workout of my connection to you. We are living a shared experience: but how stable is it? What does remain after it? There will be these notes I am writing down; there will be memory. My notes are a device for returning to this past experience, to enlive it once again when reading them. In the end, its stability relies on our strategies of relating to it.

Thus, mathematical truth relies on the stability of experience, and on our strategies of sharing it. Mathematical proof, consisting in the experience of a continuous series of steps going from hypothesis to conclusion, is shared by presentations of it and by redactions of it. I see mathematics, although relying on the fragility of human experience, as an extraordinarily stable enterprise, as the result of a collective permanent effort to keep maximal faith into the possibility of returning to experiences of mathematical truth.

Constructive mathematics has been founded by Errett Bishop (1928-1983) through his book <u>Foundations of constructive analysis</u> (1967). I would like to present it as the up-to-date best presentation of informal mathematics: mathematics without any axiom, based on the intuition of natural numbers. In this, it relates to intuitionistic mathematics, and Bishop is paying due respect to L.E.J. Brouwer, but, as we shall see below, proposes an "absolute minimum of philosophical prejudice": "There are no dogmas to which we must conform".

I shall follow the presentation given by Bishop himself in 1973 in a text whose unfortunate title is "Schizophrenia in contemporary mathematics" (republished in 1985). He motivates it on the following symptoms: "rejection of common sense in favor of formalism; debasement of meaning by wilful refusal to accommodate certain aspects of reality; inappropriateness of means to ends; the esoteric quality of the communication; and fragmentation".



"Four principles [of constructivism] stand out as basis:
(A) Mathematics is common sense."

This principle posits the relationship of formalisms with respect to mathematics: formalisms, e.g. the Zermelo-Fraenkel theory of sets, or classical logic, or Martin-Löf type theory, are crucial constructions in mathematics, but their mathematical consideration depends on informal mathematics that explains its meaning. I would like to recall the following thought by Blaise Pascal: "All sciences are infinite in the range of their researches, for who can doubt that mathematics, for instance, has an infinity of infinities of propositions to expound? They are infinite also in the multiplicity and subtlety of their principles, for anyone can see that those which are supposed to be ultimate do not stand by themselves, but depend on others, which depend on others again, and thus never allow any finality" (Pensées, Sellier 230). Common sense rests on the infinite process of elaborating its principles.

"(B) Do not ask whether a statement is true until you know what it means."
Bishop posits here that two people who investigate the truth of a statement after having agreed on a meaning for this statement will also agree on its truth. A statement may have several meanings, most prominently a classical meaning, relying on the law of excluded middle, and a constructive meaning, relying on the concept of construction, and its truth may depend on the meaning considered.

"(C) A proof is any completely convincing argument."
I.e., proof relies on the experience of truth. "Clearly it is impossible to accept (C) without accepting (B), because it doesn't make sense to be convinced that something is true unless you know what it means."

"(D) Meaningful distinctions deserve to be maintained."
"Consider the number n_0, defined to be 0 if the Riemann hypothesis is true and 1 if it is false. The constructivist [...] does want the classicist to distinguish such numbers from numbers which can be 'computed,' such as the number n_1 of primes less than $10^{10^{10}}$ [...] The distinction is not observed in the systematic development of classical mathematics, nor would the tools available to the classicist permit him to observe the distinction systematically even if he were so inclined."

Let me conclude on a citation by Karl Marx: "Truth includes not only the result but also the path to it. The investigation of truth must itself be true; true investigation is developed truth, the dispersed elements of which are brought together in the result" (Comments on the latest Prussian censorship instruction, 1843, cited by Georges Perec in The things, cited by Lombardi and Quitté 2015).
I invite the interested reader to consult Henri Lombardi's website: http://hlombardi.free.fr.

Some questions and answers

* If one bases mathematics on human experience, where does the objectivity of mathematics come from?

  For me, the objectivity of mathematics comes from its intersubjectivity. We agree in particular on the meaning of computation, of construction, and we oblige ourselves to answer anyone with doubts about a step in a construction. The best ground for our trust in mathematics is the experience of truth (also with respect to computer-assisted proofs and proof certification).

* Did Bishop invent constructive mathematics?

  For me, it is justified to consider Bishop as the one who has provided constructive mathematics with a philosophical and practical foundation. In doing so, he has given a definition of informal mathematics as practiced by e.g. Gauss and more generally all mathematicians until Dedekind's inception of set theory. Note that there is also the Russian school of constructive mathematics founded by A.A. Markov and N. A. Shanin, in which the only construction allowed amounts to Turing machines. A crucial feature of Bishop's constructive mathematics is that the hypotheses of a theorem are considered as oracles that do not need to be constructive themselves.

* What is the connection of constructive mathematics with intuitionism?

  Bishop owes a great deal to Brouwer, the founder of intuitionism: in particular, he praises Brouwer as the one who has denounced the lack of meaning of mathematics in the beginning of the 20th century as introduced by a blind use of the law of excluded middle. Another crucial feature of constructive mathematics is that the only intuition of infinity upon which it bases is that of natural numbers, whereas Brouwer formulated a so-called "2nd act of intuitionism" whose aim is to get an intuition of the continuum that radically differs from the "discrete" conception of the real numbers as given e.g. by Cauchy sequences.

* What is the connection of constructive mathematics with Martin-Löf type theory?

  Constructive mathematics provide the informal background for the formal theory of types given by Martin-Löf. Its specific feature, not needed in the description of constructive mathematics, is the systematic use of the Curry-Howard correspondence. Note that for Martin-Löf, it is crucial to provide meaning explanations for every step in the construction of his type theory.

* Is there still a place for controversies in constructive mathematics?

The framework of constructive mathematics is conceived in a way that constructive proofs respect the framework of classical as well as of intuitionistic mathematics: a constructive proof is valid classically and intuitionistically (it does not necessarily correspond to a Turing machine, so that further work is needed w.r.t. the Russian school). In doing so, it is a very elegant way of escaping the controversies about the foundations of mathematics as well as of providing sound foundations.

There are however still possible controversies.

- One may consider the infinity of natural numbers as unreliable and wish to restrict to finitary or to Esenin-Volpin's ultrafinitistic mathematics
- One may consider recursive functions and Turing machines as a better ontology of mathematics.
- For Kronecker, the basis of mathematics is computation, and it may be that the meaning of a computation is more restricted than that of a construction. For example, Thierry Coquand has tried to convince H.M. Edwards of the constructive nature of an argument about Gröbner bases, without success.

Note that the concept of a construction is undefined in constructive mathematics: its meaning relies on the intersubjectivity of the mathematicians.